Problem Set # 11

M392C: Morse Theory

1. Let M be a Riemannian manifold and N_0, N_1 two submanifolds. Suppose $\gamma\colon [0,1] \to M$ is a smooth curve such that $\gamma(0) \in N_0$, $\gamma(1) \in N_1$, and γ is length minimizing among all curves satisfying these conditions. Prove that $\dot{\gamma}(0) \perp N_0$ and $\dot{\gamma}(1) \perp N_1$.

2. Recall the formula for the Riemann curvature tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

   (a) Verify that R is a tensor by checking that it is linear over functions in all three variables.

   (b) Let $x^1, \ldots, x^n$ be a local coordinate system. Define functions $R^i_{jk\ell}$ by the formula

$$R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell})\frac{\partial}{\partial x^j} = R^i_{jk\ell}\frac{\partial}{\partial x^i}.$$

   Compute $R^i_{jk\ell}$ in terms of the metric

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle.$$