Problem Set # 10

M392C: Morse Theory

1. Let M be a Riemannian manifold of dimension n and $\pi\colon \mathcal{B}_O(M) \to M$ its principal O_n-bundle of orthonormal frames. Let ∂_k, $k = 1, \ldots, n$, be the horizontal vector field whose value at a frame is the horizontal lift of the k^{th} vector in the frame. Prove that integral curves of ∂_k project to geodesics on M.

2. Is it possible for a geodesic to intersect itself? Example or counter-proof.

3. Let (M, g) be a connected Riemannian manifold. Define the associated distance function $d\colon M \times M \to \mathbb{R}^{\geq 0}$ as in lecture. Prove that (M, d) is a metric space. Prove that the metric topology is equivalent to the original topology on M.

4. Suppose (M, g) is a connected Riemannian manifold and the associated metric space (M, d) is complete. Prove that for all $p \in M$ and $\xi \in T_pM$, the maximal domain of the geodesic $\gamma\colon I \to M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = \xi$ is $I = \mathbb{R}$.