Problem Set # 1

M392C: Morse Theory

You should do these problems and write them up carefully, but do not hand them in. Rather, form informal groups to exchange and discuss solutions and lectures.

1. Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$ be two coordinate systems on an open subset U of affine space $\mathbb{A}^n$. You may want to set $n = 1$, then $n = 2$, to explore this problem.

   (a) Recall the transformation law which expresses $\partial/\partial x^i$ in terms of the partial derivatives $\partial/\partial y^\alpha$. Derive an expression for $\partial^2/\partial x^i \partial x^j$ in terms of partial derivatives in the y-coordinate system.

   (b) Can you make sense of an *intrinsic* second derivative of a function $f \colon U \to \mathbb{R}$, i.e., one which is independent of changes of coordinates?

   (c) Now suppose M is a Riemannian manifold and $f \colon M \to \mathbb{R}$ a smooth function. Can you define a second derivative which is a symmetric bilinear form?

2. Let V be a finite dimensional real vector space and $B \colon V \times V \to \mathbb{R}$ a symmetric bilinear form. Let U be the subset of the appropriate Grassmannian consisting of maximal subspaces of V on which B is positive definite. Prove that U is contractible.

3. Let V be a finite dimensional real or complex inner product space and $T \colon V \to V$ a (Hermitian) symmetric operator with one-dimensional eigenspaces. Show that the function

$$f \colon \mathbb{P}(V) \longrightarrow \mathbb{R}$$
$$L \longmapsto \frac{\langle \xi, T\xi \rangle}{\langle \xi, \xi \rangle}$$

is Morse, where $\xi \in L$ is a nonzero vector in the line $L \subset V$. More specifically, show that the critical points of f are the eigenlines. What is the Hessian of f at those points? What are the indices of the critical points? What happens if T has eigenspaces of dimension greater than one?

4. Prove the Cartan formula $\mathcal{L}_\xi \omega = (d\iota_\xi + \iota_\xi d)\omega$ for ξ a vector field and ω a differential form. Recall the definition of the Lie derivative using the flow generated by ξ.

   (a) First check the case in which ω is a function (0-form).

   (b) Prove that each side of the equation is a derivation in ω.

   (c) Show it suffices to work locally. Conclude.

5. Let M be a connected manifold, $p, q \in M$. Prove that there exists a diffeomorphism $\varphi \colon M \to M$ such that $\varphi(p) = q$.

6. (a) Use the Moser technique to prove the Darboux lemma. Namely, suppose (M^{2m}, ω) is a symplectic manifold. Prove that about any $p \in M$ there exist local coordinates $x^1, \ldots, x^{2m}$ such that
$$\omega = dx^1 \wedge dx^2 + \cdots + dx^{2m-1} \wedge dx^{2m}$$
in that coordinate system.

(b) Formulate and prove a normal form for volume forms on a manifold.