Lectures 9 & 10: Fredholm operators

Let X be a compact Hausdorff space. Recall (Definition 3.10) that $K(X)$ is defined as the group completion of the commutative monoid of equivalence classes of complex (finite rank) vector bundles on X. So a general element is represented as the formal difference of two vector bundles. In this lecture we develop another model for K-theory which is more flexible and useful in geometric applications. For example, it allows us to include Example 1.11, but it will allow quite a bit more.

The basic idea is as follows. Suppose H^0, H^1 are complex vector spaces and $T\colon H^0 \to H^1$ a linear map. Then there is an exact sequence

$$0 \longrightarrow \ker T \longrightarrow H^0 \xrightarrow{T} H^1 \longrightarrow \operatorname{coker} T \longrightarrow 0 \tag{9.1}$$

The exactness means that, choosing splittings, there is an isomorphism

$$H^0 \oplus \operatorname{coker} T \cong H^1 \oplus \ker T, \tag{9.2}$$

and so formally we identify the difference $H^1 - H^0$ with $\ker T - \operatorname{coker} T$ in $K(\mathrm{pt}) \cong \mathbb{Z}$. This equality of dimensions is a basic theorem in linear algebra. Now imagine that we have a continuous family of linear operators parametrized by X, and perhaps the vector spaces H^0, H^1 also vary in a locally trivial way and so form vector bundles over X. Then their formal difference defines an element of $K(X)$. The kernels and cokernels of T, however, in general are not locally trivial. In fact their dimensions typically jump. (As a simple example take $X = \mathbb{R}$, $H^0 = H^1 = \mathbb{C}$, and T_x the linear map multiplication by $x \in \mathbb{R}$.) In some sense we control the jumping by considering not the kernel and cokernel, but the entire vector spaces H^0, H^1. The new idea is to allow H^0, H^1 to be infinite dimensional while requiring that $\ker T, \operatorname{coker} T$ be finite dimensional. An operator with this property is called *Fredholm*. In continuous families there is jumping of kernels and cokernels, but that is controlled by by considering finite dimensional subspaces containing the kernels where there is no jumping. In this way we make sense of the formal difference between kernels and cokernels. A canonical open cover of the space of Fredholm operators implements this idea universally.

The infinite dimensional vector spaces H^0, H^1 have a topology, and there are many species of infinite dimensional topological vector spaces. We use Hilbert spaces: the topology is induced from a Hermitian metric, and this retains the usual Euclidean notions of length and angle. The theory is equally smooth for Banach spaces [Pa1, §VII]. We also need to topologize the space of continuous linear maps $\operatorname{Hom}(H^0, H^1)$. In this lecture we use the *norm topology*, which makes this a Banach space. However, other choices are possible and we will see later that the *compact-open topology* is a more flexible and applicable choice [ASe1], [FM, Appendix B]. We remark that the Hilbert spaces H^0, H^1 needn't be infinite dimensional, though much of the theory becomes trivial if not.

Some functional analysis

We remind of some basics. Let H^0, H^1 be Hilbert spaces. A linear map $T\colon H^0 \to H^1$ is continuous if and only if it is *bounded*, i.e., there exists $C > 0$ such that

$$\|T\xi\| \leqslant C\|\xi\|, \qquad \xi \in H^0. \tag{9.3}$$

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In that case the infimum over all C which satisfy (9.3) is the *operator norm* $\|T\|$. Let $\operatorname{Hom}(H^0, H^1)$ denote the linear space of continuous linear maps. The operator norm is complete and makes $\operatorname{Hom}(H^0, H^1)$ a Banach space. The operator norm satisfies

$$\|T_2 \circ T_1\| \leqslant \|T_1\| \, \|T_2\| \tag{9.4}$$

whenever the composition makes sense. Let $\operatorname{Hom}(H^0, H^1)^\times \subset \operatorname{Hom}(H^0, H^1)$ denote the subspace of invertible linear operators.

Theorem 9.5.
(i) *If* $T\colon H^0 \to H^1$ *is continuous and bijective, then* T^{-1} *is continuous.*
(ii) $\operatorname{Hom}(H^0, H^1)^\times \subset \operatorname{Hom}(H^0, H^1)$ *is an open subspace.*
(iii) $\operatorname{Hom}(H^0, H^1)^\times$ *is contractible in the norm topology.*

(i) is the open mapping theorem. (ii) is proved by constructing a ball of invertible operators around any given invertible using the power series for $1/(1+x)$. (iii) is a theorem of Kuiper [Ku] which we prove soon.

We remark that if $V \subset H$ is a finite dimensional subspace of a Hilbert space H, then V is closed and $V^\perp$ is a closed complement. For any closed subspace $V \subset H$ the quotient H/V inherits a Hilbert space structure by identifying it with $V^\perp$ via the quotient map $V^\perp \hookrightarrow H \twoheadrightarrow H/V$.

Fredholm operators

Definition 9.6. Let H^0, H^1 be Hilbert spaces. A continuous linear map $T\colon H^0 \to H^1$ is *Fredholm* if its range $T(H^0) \subset H^1$ is closed and if $\ker T, \operatorname{coker} T$ are finite dimensional. Let $\operatorname{Fred}(H^0, H^1) \subset \operatorname{Hom}(H^0, H^1)$ denote the subset of Fredholm operators, topologized with the norm topology.

The closed range condition is redundant [Pa1, §VII], but as $\operatorname{coker} T$ is not Hausdorff if T is not closed range it seems sensible to include it as part of the definition. The *numerical index* of a Fredholm operator is defined as

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T. \tag{9.7}$$

Remark 9.8. From some point of view this definition has the wrong sign! For if H^0, H^1 are finite dimensional we identify $T\colon H^0 \to H^1$ as an element of $H^1 \otimes (H^0)^*$. It is the domain which is dualized, not the codomain, so we expect the minus sign in (9.7) on the subspace $\ker T$ of the domain. This sign mistake causes minor headaches in certain parts of index theory; for example, see [Q, §2].

We give several examples.

Example 9.9. If H is separable it has a countable basis $e_1, e_2, \dots$ in the sense that any element of H can be written as $\sum_n a_n e_n$ where the complex coefficients satisfy $\sum_n |a_n|^2 < \infty$. For each $k \in \mathbb{Z}$ define the shift operator T_k which on the basis is

$$T_k(e_j) = \begin{cases} e_{j-k}, & j > k; \\ 0, & j \leqslant k. \end{cases} \tag{9.10}$$

Then T_k is Fredholm of index k. This shows there exist Fredholm operators of any index.

Example 9.11. The differential operator d/dx is Fredholm acting on complex-valued functions on the circle S^1 with coordinate x. We use Hilbert space completions—Sobolev spaces—of the space of smooth functions:

$$\frac{d}{dx}\colon L^2_1(S^1) \longrightarrow L^2(S^1) \tag{9.12}$$

The index is 0: the kernel and cokernel are each 1-dimensional. This example generalizes to elliptic operators on compact manifolds.

The canonical open cover

Recall that a linear map $T\colon H^0 \to H^1$ is *transverse* to a subspace $W \subset H^1$, written $T \pitchfork W$, if $T(H^0) + W = H^1$. Fix Hilbert spaces H^0, H^1. For each *finite dimensional* subspace $W \subset H^1$ define

$$\mathcal{O}_W = \{T \in \mathrm{Fred}(H^0, H^1) : T \pitchfork W\}. \tag{9.13}$$

Observe that $\mathcal{O}_W \subset \mathcal{O}_{W'}$ if $W \subset W'$.

Proposition 9.14.
(*i*) $\mathcal{O}_W \subset \mathrm{Hom}(H^0, H^1)$ *is open.* $\mathrm{Fred}(H^0, H^1) \subset \mathrm{Hom}(H^0, H^1)$ *is open.*
(*ii*) $\{\mathcal{O}_W\}_W$ *is an open cover of* $\mathrm{Fred}(H^0, H^1)$.
(*iii*) *If X is compact and $T\colon X \to \mathrm{Fred}(H^0, H^1)$ continuous, then $T(X) \subset \mathcal{O}_W$ for some finite dimensional $W \subset H^1$.*

Proof. Fix $T_0 \in \mathcal{O}_W$. Observe that $T \pitchfork W$ if and only if $H^0 \xrightarrow{T} H^1 \twoheadrightarrow H^1/W$ is surjective, and the latter is true if the composition

$$(T_0^{-1}W)^{\perp} \hookrightarrow H^0 \xrightarrow{T} H^1 \twoheadrightarrow H^1/W \tag{9.15}$$

is an isomorphism. That is true for $T = T_0$, and since (Theorem 9.5(ii)) isomorphisms are open in $\mathrm{Hom}\big(T_0^{-1}W, H^1/W\big)$ and the map $\mathrm{Hom}(H^0, H^1) \to \mathrm{Hom}\big((T_0^{-1}W, H^1/W\big)$ is continuous, the space of transverse maps (9.13) is open as well. This proves (i). That every Fredholm operator is transverse to a finite dimensional subspace follows directly from the finite dimensionality of its cokernel, which proves (ii). For (iii) the cover $\{T^{-1}\mathcal{O}_W\}_W$ of X has a finite subcover indexed by finite dimensional subspaces $W_1, W_2, \dots, W_N$. Define $W = W_1 + W_1 + \cdots + W_N$. $\square$

On O_W we have the parametrized family of vector spaces $K_W \to \mathcal{O}_W$ whose fiber at $T \in \mathcal{O}_W$ is the finite dimensional subspace $T^{-1}W \subset H^0$.

Lemma 9.16. $K_W \to \mathcal{O}_W$ *is a locally trivial vector bundle.*

Proof. Fix $T_0 \in \mathcal{O}_W$ and let $p\colon H^0 \to T_0^{-1}W$ be orthogonal projection. On the open set U of $T \in \mathcal{O}_W$ for which (9.15) is an isomorphism the restriction of p to $T^{-1}W$ is an isomorphism $T^{-1}W \to T_0^{-1}W$, as is easily verified. Topologize K_W as a subspace of $\mathrm{Hom}(H^0, H^1) \times H^0$. Since the map $(T, \xi) \mapsto (T, p(\xi))$ is continuous, we have a local trivialization of the restriction of $K_W \to \mathcal{O}_W$ to U with the constant vector bundle with fiber $T_0^{-1}W$. $\square$

Corollary 9.17. *The function*

$$\begin{aligned} \mathrm{ind}\colon\ \mathrm{Fred}(H^0, H^1) &\longrightarrow \mathbb{Z} \\ T &\longmapsto \dim\ker T - \dim\operatorname{coker} T \end{aligned} \tag{9.18}$$

is locally constant.

Proof. Observe that for $T \in \mathcal{O}_W$ the sequence

$$0 \longrightarrow \ker T \longrightarrow T^{-1}W \xrightarrow{\ T\ } W \longrightarrow \operatorname{coker} T \longrightarrow 0 \tag{9.19}$$

is exact, from which

$$\mathrm{ind}\, T = \dim\ker T - \dim\operatorname{coker} T = \dim T^{-1}W - \dim W. \tag{9.20}$$

The right hand side is locally constant on $\mathcal{O}_W$, by Lemma 9.16. $\square$

Fredholms and the K-theory of a compact space

As a preliminary we prove that the composition of Fredholms is Fredholm and that the numerical index behaves well under composition. For convenience we now consider Fredholm operators on a fixed Hilbert space H.

Lemma 9.21. *If* $T_1, T_2 \in \mathrm{Fred}(H)$, *then* $T_2 \circ T_1 \in \mathrm{Fred}(H)$ *and* $\mathrm{ind}\, T_2 \circ T_1 = \mathrm{ind}\, T_1 + \mathrm{ind}\, T_2$.

Proof. If $T_2T_1 \pitchfork W$, then $T_2 \pitchfork W$ and $T_1 \pitchfork T_2^{-1}W$. Thus

$$\begin{aligned} \mathrm{ind}\, T_2T_1 &= \dim T_1^{-1}T_2^{-1}W - \dim W \\ &= \dim T_1^{-1}T_2^{-1}W - \dim T_2^{-1}W + \dim T_2^{-1}W + \dim W \\ &= \mathrm{ind}\, T_1 + \mathrm{ind}\, T_2. \end{aligned} \tag{9.22}$$

$\square$

Suppose X is compact Hausdorff and $T\colon X \to \mathrm{Fred}(H)$ is continuous. By Proposition 9.14(iii) there exists a finite dimensional subspace $W \subset H$ such that $T_x \pitchfork W$ for all $x \in X$. Then $T^*K_W \to X$ is a vector bundle, and we define

$$[T^*K_W] - [\underline{W}] \in K(X). \tag{9.23}$$

Theorem 9.24 (Atiyah-Jänich)**.** *Assume X is compact Hausdorff. Then $T \mapsto [T^*K_W] - [\underline{W}]$ is a well-defined map*

$$i\colon [X, \mathrm{Fred}(H)] \longrightarrow K(X) \tag{9.25}$$

which is an isomorphism of abelian groups.

The map i sends a family of Fredholm operators to its *index* in K-theory.

Corollary 9.26. *The numerical index*

$$\mathrm{ind}\colon \pi_0 \,\mathrm{Fred}(H) \longrightarrow \mathbb{Z} \tag{9.27}$$

is an isomorphism.

This follows from Theorem 9.24 by taking $X = \mathrm{pt}$.

Proof. We first prove that i is well-defined. If W_1, W_2 are finite dimensional subspaces for which $T_x \pitchfork W_i$ for all $x \in X$, then the same holds for $W_1 + W_2$, so it suffices to check well-definedness of (9.23) for subspaces $W \subset W'$. In that case there is a short exact sequence

$$0 \longrightarrow T^*K_W \longrightarrow T^*K_{W'} \longrightarrow \underline{W'}/\underline{W} \longrightarrow 0 \tag{9.28}$$

of vector bundles over X. Choosing a splitting we construct an isomorphism $\underline{W'}/\underline{W} \oplus T^*K_W \cong T^*K_{W'}$ and then, adding $\underline{W}$ to both sides, we obtain an isomorphism $\underline{W'} \oplus T^*K_W \cong \underline{W} \oplus T^*K_{W'}$. It follows that $[T^*K_{W'}] - [\underline{W'}] = [T^*K_W] - [\underline{W}] \in K(X)$.

To check that the index (9.25) is invariant under homotopy, suppose that $H\colon [0,1] \times X \to \mathrm{Fred}(H)$ is a continuous map. Choose $W \subset H$ such that $H_{(t,x)} \pitchfork W$ for all (t,x). Then by Theorem 2.1 the restrictions of $H^*K_W \to [0,1] \times X$ to the ends of $[0,1] \times X$ are isomorphic, and it follows that the K-theory classes (9.23) on the two ends agree.

The domain $[X, \mathrm{Fred}(H)]$ of (9.25) is a monoid by pointwise composition. To see that i is a homomorphism of monoids, begin as in the proof of Lemma 9.21 by choosing $W \subset H$ such that $(T_2T_1)_x \pitchfork W$ for all $x \in X$. Let $E'_x = (T_2)_x^{-1}(W)$. For each $x_0 \in X$ there is an open neighborhood of $x \in X$ such that the orthogonal projection of E'_x to E'_{x_0} is an isomorphism and $(T_1)_x \pitchfork E'_{x_0}$. Cover X by a finite set of such neighborhoods and let $V \subset H$ be a subspace containing the sum of the corresponding E'_{x_0} such that $(T_1)_x \pitchfork V$ for all $x \in X$. Let E_x denote the orthogonal projection of E'_x to V and define $F \to X$ by $F_x = T_1^{-1}E_x$. Note that orthogonal projection is an isomorphism $E' \xrightarrow{\cong} E$. Compute

$$\begin{aligned} i(T_2T_1) &= [(T_2T_1)^*K_W] - [\underline{W}] \\ &= [(T_2T_1)^*K_W] - [T_2^*K_W] + i(T_2) \\ &= [F] - [E] + i(T_2) \\ &= [T_1^*K_V] - [\underline{V}] + i(T_2) \\ &= i(T_1) + i(T_2). \end{aligned} \tag{9.29}$$

In the penultimate step we use the exact sequence $0 \to F \to T_1^* K_V \to \underline{V}/E \to 0$, analogous to (9.28).

To prove that i is surjective observe from Proposition 3.15 that any element of $K(X)$ has the form $[E] - N$ for some vector bundle $E \to X$ and $N \in \mathbb{Z}^{\geqslant 0}$. By Example 9.9 there is a constant family of Fredholm operators whose index is N. Embed $E \hookrightarrow \underline{\mathbb{E}}$ in a trivial bundle (Proposition 3.15 again) and define $p_x \in \operatorname{End} \mathbb{E}$ as orthogonal projection with kernel E_x. Finally, embed $\mathbb{E} \hookrightarrow H$ and extend p_x to be the identity on $\mathbb{E}^\perp$.

To prove that i is injective, if $i(T) = 0$ for $T\colon X \to \operatorname{Fred}(H)$, then for some finite dimensional vector space $\mathbb{E}$ there exists an isomorphism

$$T^* K_W \oplus \underline{\mathbb{E}} \xrightarrow{\varphi} \underline{W} \oplus \underline{\mathbb{E}} \tag{9.30}$$

of vector bundles over X. The fiber of $T^* K_W$ at $x \in X$ is $T_x^{-1} W$. Add to (9.30) the isomorphism $T_x\colon (T_x^{-1}W)^\perp \to H \twoheadrightarrow W^\perp$ to obtain the family of isomorphisms

$$H \oplus \mathbb{E} \xrightarrow{\Phi_x = T_x + \varphi_x} H \oplus \mathbb{E}. \tag{9.31}$$

Then $t \mapsto T + t\varphi$ is a homotopy from T to this family of invertibles. By Kuiper's Theorem 9.5(iii) the latter is homotopically trivial. (To obtain operators on H rather than $H \oplus \mathbb{E}$ conjugate by an isomorphism $H \to H \oplus \mathbb{E}$.)

Since i is a bijective homomorphism of monoids, and $K(X)$ is an abelian group, it follows that $[X, \operatorname{Fred}(H)]$ is also an abelian group and i is an isomorphism of abelian groups. □

Further remarks

We will have more to say about Fredholm operators in future lectures. For now a few comments will suffice.

(9.32) *Invertibles as a fat basepoint.* In homotopy theory we work with pointed topological spaces, that is, topological spaces with a distinguished basepoint. For sure $\operatorname{Fred}(H)$ has one—the identity operator—though if $H^0 \neq H^1$ then $\operatorname{Fred}(H^0, H^1)$ does not have a distinguished basepoint. In both cases there is a natural contractible subspace, the subspace of invertible operators. So we can work with the pair $(\operatorname{Fred}(H), \operatorname{Fred}(H)^\times)$ in lieu of a pointed space.

(9.33) *Relative K-theory.* In this spirit if (X, A) is a pair of spaces and $T\colon X \to \operatorname{Fred}(H)$ such that T_a is invertible for all $a \in A$, then T defines an element in the *relative K-theory* group $K(X, A)$. (Take this as the definition of relative K-theory.) The *support* of a Fredholm family is the set of points at which the Fredholm operator fails to be invertible. The family of linear operators in Example 1.11 is trivially Fredholm, since they act on a finite dimensional space, and the support is the set of eigenvalues of the given operator.

(9.34) *Topology of* $\mathrm{Fred}(H)$. By Corollary 9.26 we can write

$$\mathrm{Fred}(H) = \coprod_n \mathrm{Fred}^{(n)}(H) \tag{9.35}$$

as the disjoint union of connected spaces of Fredholms of a fixed index. The components are all homeomorphic, and the underlying homotopy type is that of BU, the classifying space of the group described in Remark 3.32.

Half of a cohomology theory (Bonus material)

In lecture we covered these ideas following [A1, Ha] in the context of compact Hausdorff spaces. At this point we have defined a map

$$X \longmapsto K(X) = [X, \mathrm{Fred}(H)] \tag{9.36}$$

which attaches an abelian group to every space X. For compact Hausdorff spaces Theorem 9.24 asserts that this is the same as Definition 3.10. It is conventional to restrict to a category of "nice" spaces, such as CW complexes or compactly generated spaces. We will see that for free we obtain half of a cohomology theory through suspensions and loopings, but we need new ideas to recover the other half. For K-theory that idea is Bott periodicity (Theorem 3.33, Theorem 3.35) In the next several lectures we give a proof of Bott periodicity in the context of Fredholm operators.

A reference for this section is [A1].

(9.37) *Pointed spaces.* Let $\mathcal{S}$ denote a convenient category of spaces (CW complexes, compactly generated spaces), $\mathcal{S}_*$ the category of pointed spaces—spaces with a (nondegenerate) basepoint—and $\mathcal{S}_2$ the category of (excisive) pairs. There are functors

$$\begin{aligned} \mathcal{S}_2 &\longrightarrow \mathcal{S}_* \\ (X, A) &\longmapsto X/A \\ (X, \{x_0\}) &\longleftarrow\!\shortmid X \end{aligned} \tag{9.38}$$

where in the last formula X has basepoint x_0. Note that unpointed spaces $\mathcal{S}$ map to $\mathcal{S}_2$ via $X \mapsto (X, \varnothing)$, and

$$X/\varnothing = X_+ = X \amalg \{*\} \tag{9.39}$$

is the union of X with a disjoint basepoint. Recall that the *suspension* ΣX of a pointed space is the smash product $S^1 \wedge X$.

Definition 9.40. A *cohomology theory* is a sequence of abelian group-valued functors

$$\widetilde{K}^n\colon \mathcal{S}_*^{\mathrm{op}} \longrightarrow \mathrm{AbGp}, \tag{9.41}$$

one for each $n \in \mathbb{Z}$, and a sequence of natural transformations

$$\widetilde{K}^n(X) \longrightarrow \widetilde{K}^{n+1}(\Sigma X), \qquad X \in \mathcal{S}_*, \tag{9.42}$$

such that

(i) if $f_t\colon X \to Y$ is a homotopy, then

$$f_0^* = f_1^*\colon \widetilde{K}^n(Y) \longrightarrow \widetilde{K}^n(X); \tag{9.43}$$

(ii) for $f\colon X \to Y$ with mapping cone C_f and $j\colon Y \hookrightarrow C_f$ the inclusion, the sequence

$$\widetilde{K}^n(C_f) \xrightarrow{j^*} \widetilde{K}^n(Y) \xrightarrow{f^*} \widetilde{K}^n(X) \tag{9.44}$$

is exact;

(iii) the suspension homomorphisms (9.42) are isomorphisms; and

(iv) if $X = \bigvee\limits_{\alpha\in A} X_\alpha$, then the natural map

$$\widetilde{K}^n(X) \longrightarrow \prod_{\alpha\in A} \widetilde{K}^n(X_\alpha) \tag{9.45}$$

is an isomorphism.

(9.46) *Nonpositive degree K-theory.* From the definition

$$K^0(X) = [X, \mathrm{Fred}(H)] \tag{9.47}$$

developed earlier in this lecture we extract the reduced cohomology

$$\widetilde{K}^0(X) = \big[(X, x_0), (\mathrm{Fred}(H), \mathrm{id}_H)\big] \tag{9.48}$$

of a pointed space, where we use the identity operator as the basepoint in the space of Fredholms. Then for *nonnegative integers* $n \in \mathbb{Z}^{\geqslant 0}$ define

$$\widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X) = [\Sigma^n X, \mathrm{Fred}(H)] = [X, \Omega^n \,\mathrm{Fred}(H)], \tag{9.49}$$

where X is pointed and maps and homotopies preserve basepoints. Just as the space of Fredholms is the classifying space (9.47) for K^0, its n^{th} *loop space* is the classifying space for K^{-n}. To define K^n for $n > 0$ it is clear that we need *deloopings* of $\mathrm{Fred}(H)$. That is the challenge for any cohomology theory.

Remark 9.50. For example, we can define $H^0(X;\mathbb{Z}) = [X, \mathbb{Z}]$, where $\mathbb{Z}$ is a discrete space with basepoint $0 \in \mathbb{Z}$. Its loop spaces are all trivial—they consist of one point—and so $H^{-n}(X;\mathbb{Z}) = 0$ for $n > 0$. But this gives no clue how to define $H^n(X;\mathbb{Z})$ for $n > 0$.

(9.51) *Bott periodicity.* For K-theory the groups $\widetilde{K}^{-n}(X)$, $n>0$, are periodic and so it is easy to extend to positive degree. We sketch the proofs in the next few lectures for both the real and complex cases.

Theorem 9.52 (Bott)**.** *There are homotopy equivalences* $\Omega^2\operatorname{Fred}(H)\simeq\operatorname{Fred}(H)$ *and* $\Omega^8\operatorname{Fred}(H_{\mathbb{R}})\simeq\operatorname{Fred}(H_{\mathbb{R}})$.

(9.53) *Suspensions as Thom complexes.* We give an alternative picture of suspension and a twisted version. Let X be an *unpointed* space. Its n^{th} suspension is

$$\Sigma^n X_+ = X\times S^n \;/\; X\times\{*\}\simeq X\times\mathbb{R}^n \;/\; X\times\big(\mathbb{R}^n\backslash B_r(0)\big), \tag{9.54}$$

where $B_r(0)\subset\mathbb{R}^n$ is the open ball of radius $r>0$.

The construction generalizes to twisted suspensions, replacing $X\times\mathbb{R}^n$ by a real vector bundle $V\to X$. Fix an inner product on the bundle and also fix a real number $r>0$. Define $B_r(0)\subset V$ as the open subspace of vectors of norm strictly less than r. The quotient space

$$X^V = V/\big(V\backslash B_r(0)\big) \tag{9.55}$$

is the *Thom complex* of V, and up to homeomorphism it is independent of the inner product and choice of $r>0$. Note that X^V has a natural basepoint.

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