Lecture 15: Groupoids and vector bundles

So far we have two notions of 'space': a topological space and a smooth manifold. But in many situations this is not adequate: the objects represented by the points of a "space" have internal structure, or symmetries. For example, consider the moduli "space" of triangles in the plane, where two triangles represent the same point if there is an isometry of the plane which carries one to the other. Then some triangles, for example an isosceles triangle, admit self-symmetries. We need a mathematical structure which tracks these internal symmetries. In physics too we meet the same phenomenon. For example, some fields in field theory, such as gauge fields (connections), admit internal symmetries. In fact, in both geometry and physics there are geometric objects with more than one layer of internal structure, but in this course we restrict ourselves to a single layer. The intrinsic geometric object we need is called a *stack*. Stacks are presented by *groupoids*, which are more concrete, and we focus on them. Again we consider a topological version (*topological groupoids*) and a smooth version (*Lie groupoids*). In both cases we need to restrict to *local quotient groupoids* to develop K-theory. A key idea in this lecture is *local equivalence* of groupoids: groupoids which are locally equivalent represent the same underlying stack. We prove that a local equivalence of groupoids induces an equivalence of the categories of vector bundles. We also prove homotopy invariance for vector bundles over local quotient groupoids. This enables us to define K-theory for local quotient groupoids, which we will pursue in subsequent lectures. A reference for this material is [FHT1, Appendix].

The important special case of a global quotient groupoid leads to the K-theory of equivariant bundles [Se2].

We begin these notes with background material about categories and simplicial sets, topics which will not be covered in lecture. That makes these notes very definition-heavy, a burden surmountable by careful consideration of examples on the part of the reader.

Categories, functors, and natural transformations

Definition 15.1. A *category* C consists of a collection of objects, for each pair of objects y_0, y_1 a set of morphisms $C(y_0, y_1)$, for each object y a distinguished morphism $\mathrm{id}_y \in C(y, y)$, and for each triple of objects y_0, y_1, y_2 a composition law

(15.2) $$\circ\colon C(y_1, y_2) \times C(y_0, y_1) \longrightarrow C(y_0, y_2)$$

such that $\circ$ is associative and id_y is an identity for $\circ$.

The last phrase indicates two conditions: for all $f \in C(y_0, y_1)$ we have

(15.3) $$\mathrm{id}_{y_1} \circ f = f \circ \mathrm{id}_{y_0} = f$$

and for all $f_1 \in C(y_0, y_1)$, $f_2 \in C(y_1, y_2)$, and $f_3 \in C(y_2, y_3)$ we have

(15.4) $$(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1).$$

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We use the notation $y \in C$ for an object of C and $f\colon y_0 \to y_1$ for a morphism $f \in C(y_0, y_1)$.

Remark 15.5 (set theory). The words 'collection' and 'set' are used deliberately. Russell pointed out that the collection of all sets is not a set, yet we still want to consider a category whose objects are sets. For many categories the objects do form a set. In that case the moniker 'small category' is often used. In these lecture we will be sloppy about the underlying set theory and simply talk about a set of objects.

Definition 15.6. Let C be a category.

(i) A morphism $f \in C(y_0, y_1)$ is *invertible* (or an *isomorphism*) if there exists $g \in C(y_1, y_0)$ such that $g \circ f = \mathrm{id}_{y_0}$ and $f \circ g = \mathrm{id}_{y_1}$.
(ii) If every morphism in C is invertible, then we call C a *groupoid*.

(15.7) *Reformulation.* To emphasize that a category is an algebraic structure like any other, we indicate how to formulate the definition in terms of sets[1] and functions. Then a category $C = (C_0, C_1)$ consists of a set C_0 of objects, a set C_1 of morphisms, and structure maps

$$\begin{aligned} i\colon C_0 &\longrightarrow C_1 \\ s,t\colon C_1 &\longrightarrow C_0 \\ c\colon C_1 \times_{C_0} C_1 &\longrightarrow C_1 \end{aligned} \tag{15.8}$$

which satisfy certain conditions. The map i attaches to each object y the identity morphism id_y, the maps s,t assign to a morphism $(f\colon y_0 \to y_1) \in C_1$ the source $s(f) = y_0$ and target $t(f) = y_1$, and c is the composition law. The fiber product $C_1 \times_{C_0} C_1$ is the set of pairs $(f_2, f_1) \in C_1 \times C_1$ such that $t(f_1) = s(f_2)$. The conditions (15.3) and (15.4) can be expressed as equations for these maps. If C is a groupoid, then there is another structure map

$$\iota\colon C_1 \longrightarrow C_1 \tag{15.9}$$

which attaches to every arrow its inverse.

Definition 15.10. Let C, D be categories.

(i) A *functor* or *homomorphism* $F\colon C \to D$ is a pair of maps $F_0\colon C_0 \to D_0$, $F_1\colon C_1 \to D_1$ which commute with the structure maps (15.8).
(ii) Suppose $F, G\colon C \to D$ are functors. A *natural transformation* η from F to G is a map of sets $\eta\colon C_0 \to D_1$ such that for all morphisms $(f\colon y_0 \to y_1) \in C_1$ the diagram

$$\begin{tikzcd} Fy_0 \arrow[r, "Ff"] \arrow[d, "\eta(y_0)"'] & Fy_1 \arrow[d, "\eta(y_1)"] \\ Gy_0 \arrow[r, "Gf"] & Gy_1 \end{tikzcd} \tag{15.11}$$

commutes. We write $\eta\colon F \to G$.

[1] ignoring set-theoretic complications, as in Remark 15.5

(iii) A natural transformation $\eta\colon F \to G$ is an *isomorphism* if $\eta(y)\colon Fy \to Gy$ is an isomorphism for all $y \in C$.
(iv) A functor $F\colon C \to D$ is an *equivalence of categories* if there exist a functor $F'\colon D \to C$, a natural equivalence $F' \circ F \to \mathrm{id}_C$, and a natural equivalence $F \circ F' \to \mathrm{id}_{C'}$.

In (i) the commutation with the structure maps means that F is a homomorphism in the usual sense of algebra: it preserves compositions and takes identities to identities. A natural transformation is often depicted in a diagram

$$\begin{tikzcd} C \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow \eta"] & D \end{tikzcd} \tag{15.12}$$

with a double arrow.

Definition 15.13. Let $F\colon C \to D$ be a functor. F is *essentially surjective* if for each $z \in D_0$ there exists $y \in C_0$ such that Fy is isomorphic to z. It is *faithful* if for every $y_0, y_1 \in C_0$ the map

$$F\colon C(y_0, y_1) \to D(Fy_0, Fy_1) \tag{15.14}$$

is injective, and it is *full* if (15.14) is surjective.

The following lemma characterizes equivalences of categories; the proof, which we leave to the reader, invokes the axiom of choice to construct an inverse equivalence.

Lemma 15.15. *A functor $F\colon C \to D$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Example 15.16. Let Vect denote the category of vector spaces over a fixed field with linear maps as morphisms. There is a functor $**\colon \mathrm{Vect} \to \mathrm{Vect}$ which maps a vector space V to its double dual V^{**}. But this is not enough to define it—we must also specify the map on morphisms. Thus if $f\colon V_0 \to V_1$ is a linear map, there is an induced linear map $f^{**}\colon V_0^{**} \to V_1^{**}$. (Recall that $f^*\colon V_1^* \to V_0^*$ is defined by $\langle f^*(v_1^*), v_0\rangle = \langle v_1^*, f(v_0)\rangle$ for all $v_0 \in V_0$, $V_1^* \in V_1^*$. Then define $f^{**} = (f^*)^*$.) Now there is a natural transformation $\eta\colon \mathrm{id}_{\mathrm{Vect}} \to **$ defined on a vector space V as

$$\begin{aligned} \eta(V)\colon V &\longrightarrow V^{**} \\ v &\longmapsto \bigl(v^* \mapsto \langle v^*, v\rangle\bigr) \end{aligned} \tag{15.17}$$

for all $v^* \in V^*$. I encourage you to check (15.11) carefully.

Simplices, simplicial sets, and the nerve

Let S be a nonempty finite ordered set. For example, we have the set

$$[n] = \{0, 1, 2, \dots, n\} \tag{15.18}$$

with the given total order. Any S is uniquely isomorphic to $[n]$, where the cardinality of S is $n+1$. Let $A(S)$ be the affine space generated by S and $\Sigma(S) \subset A(S)$ the simplex with vertex set S. So if $S = \{s_1, s_1, \dots, s_n\}$, then $A(S)$ consists of formal sums

$$p = t^0 s_0 + t^1 s_1 + \dots + t^n s_n, \qquad t^i \in \mathbb{R}, \quad t^0 + t^1 + \dots + t^n = 1, \tag{15.19}$$

and $\Sigma(S)$ consists of those sums with $t^i \geqslant 0$. We write $\mathbb{A}^n = A([n])$ and $\Delta^n = \Sigma([n])$. For these standard spaces the point $i \in [n]$ is $(\dots, 0, 1, 0, \dots)$ with 1 in the i^{th} position.

Let Δ be the category whose objects are nonempty finite ordered sets and whose morphisms are order-preserving maps (which may be neither injective nor surjective). The category Δ is generated by the morphisms

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l, dashed] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2]} \arrow[l, dashed, shift left=2] \arrow[l, dashed, shift right=2] & \cdots \end{tikzcd} \tag{15.20}$$

where the right-pointing maps are injective and the left-pointing maps are surjective. For example, the map $d_i\colon [1] \to [2]$, $i = 0, 1, 2$ is the unique injective order-preserving map which does not contain $i \in [2]$ in its image. The map $s_i\colon [2] \to [1]$, $i = 0, 1$, is the unique surjective order-preserving map for which $s_i^{-1}(i)$ has two elements. Any morphism in Δ is a composition of the maps d_i, s_i and identity maps.

Each object $S \in \Delta$ determines a simplex $\Sigma(S)$, as defined above. This assignment extends to a functor

$$\Sigma\colon S \longrightarrow \mathrm{Top} \tag{15.21}$$

to the category of topological spaces and continuous maps. A morphism $\theta\colon S_0 \to S_1$ maps to the affine extension $\theta_*\colon \Sigma(S_0) \to \Sigma(S_1)$ of the map θ on vertices.

Recall the definition **(15.7)** of a category.

Definition 15.22. Let C be a category. The *opposite category* C^{op} is defined by

$$C_0^{\text{op}} = C_0, \quad C_1^{\text{op}} = C_1, \quad s^{\text{op}} = t, \quad t^{\text{op}} = s, \quad i^{\text{op}} = i, \tag{15.23}$$

and the composition law is reversed: $g^{\text{op}} \circ f^{\text{op}} = (f \circ g)^{\text{op}}$.

Here recall that C_0 is the set of objects, C_1 the set of morphisms, and $s, t\colon C_1 \to C_0$ the source and target maps. The opposite category has the same objects and morphisms but with the direction of the morphisms reversed.

The following definition is slick, and at first encounter needs unpacking (see [Fr], for example).

Definition 15.24. A *simplicial set* is a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Set} \tag{15.25}$$

It suffices to specify the sets $X_n = X([n])$ and the basic maps (15.20) between them. Thus we obtain a diagram

$$X_0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_2 \;\cdots \tag{15.26}$$

We label the maps d_i and s_i as before. The d_i are called *face maps* and the s_i *degeneracy maps*. The set X_n is a set of abstract simplices. An element of X_n is degenerate if it lies in the image of some s_i.

The morphisms in an abstract simplicial set are gluing instructions for concrete simplices.

Definition 15.27. Let $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. The *geometric realization* is the topological space $|X|$ obtained as the quotient of the disjoint union

$$\coprod_S X(S) \times \Sigma(S) \tag{15.28}$$

by the equivalence relation

$$(\sigma_1, \theta_* p_0) \sim (\theta^* \sigma_1, p_0), \qquad \theta\colon S_0 \to S_1, \quad \sigma_1 \in X(S_1), \quad p_0 \in \Sigma(S_0). \tag{15.29}$$

The map $\theta_* = \Sigma(\theta)$ is defined after (15.21) and $\theta^* = X(\theta)$ is part of the data of the simplicial set X. Alternatively, the geometric realization map be computed from (15.26) as

$$\coprod_n X_n \times \Delta^n \;/\!\sim, \tag{15.30}$$

where the equivalence relation is generated by the face and degeneracy maps.

Remark 15.31. The geometric realization can be given the structure of a CW complex.

Example 15.32. Let X be a simplicial set whose nondegenerate simplices are

$$X_0 = \{A, B, C, D\}, \qquad X_1 = \{a, b, c, d\}. \tag{15.33}$$

The face maps are as indicated in Figure 4. For example $d_0(a) = B$, $d_1(a) = A$, etc. (This requires a choice not depicted in Figure 4.) The level 0 and 1 subset of the disjoint union (15.30) is pictured in Figure 5. The 1-simplices a, b, c, d glue to the 0-simplices A, B, C, D to give the space depicted in Figure 4. The red 1-simplices labeled A, B, C, D are degenerate, and they collapse under the equivalence relation (15.29) applied to the degeneracy map s_0.

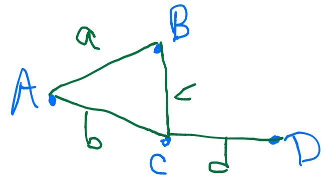

FIGURE 4. The geometric realization of a simplicial set

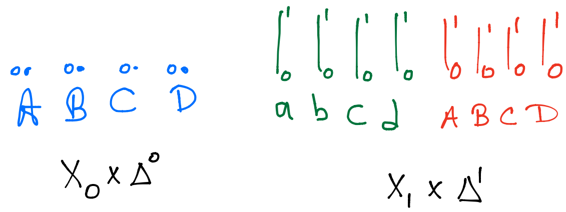

FIGURE 5. Gluing the simplicial set

(15.34) *The nerve of a category.* Let $C = (C_0, C_1)$ be a category, which in part is encoded in the diagram

$$\begin{tikzcd} C_0 \arrow[r, dashed] & C_1 \arrow[l, shift right=1.5ex] \arrow[l, shift left=1.5ex] \end{tikzcd} \tag{15.35}$$

The solid left-pointing arrows are the source s and target t of a morphism; the dashed right-pointing arrow i assigns the identity map to each object. This looks like the start of a simplicial set, and indeed there is a simplicial set NC, the *nerve* of the category C, which begins precisely this way: $NC_0 = C_0$, $NC_1 = C_1$, $d_0 = t$, $d_1 = s$, and $s_0 = i$. A slick definition runs like this: a finite nonempty ordered set S determines a category with objects S and a unique arrow $s \to s'$ if $s \leqslant s'$ in the order. Then

$$NC(S) = \mathrm{Fun}(S, C) \tag{15.36}$$

where $\mathrm{Fun}(-,-)$ denotes the set of functors. As is clear from Figure 6, $NC([n])$ consists of sets of n composable arrows in C. The degeneracy maps in NC insert an identity morphism. The face map d_i omits the i^{th} vertex and composes the morphisms at that spot; if i is an endpoint $i = 0$ or $i = n$, then d_i omits one of the morphisms.

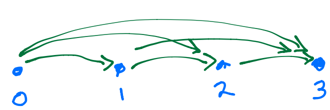

FIGURE 6. A totally ordered set as a category

Example 15.37. Let M be a monoid, regarded as a category with a single object. Then

$$NM_n = M^{\times n}. \tag{15.38}$$

It is a good exercise to write out the face maps.

Definition 15.39. Let C be a category. The *classifying space* BC of C is the geometric realization $|NC|$ of the nerve of C.

Example 15.40. Suppose $G = \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order two, viewed as a category with one object. Then NG_n has a single nondegenerate simplex $(g, \dots, g)$ for each n, where $g \in \mathbb{Z}/2\mathbb{Z}$ is the non-identity element. So BG is glued together with a single simplex in each dimension. We leave the reader to verify that in fact $BG \simeq \mathbb{RP}^\infty$.

Topological and Lie groupoids

From now on we use the formulation **(15.7)** of a category, recall (Definition 15.6) that a groupoid is a category in which all morphisms are invertible, and we identify a groupoid $X = (X_0, X_1)$ with its nerve $NX_\bullet = X_\bullet$, which is a simplicial set **(15.34)**.

Definition 15.41. Let $X = (X_0, X_1)$ be a groupoid.

(i) X is a *topological groupoid* if X_0, X_1 have the structure of topological spaces and if the structure maps i, s, t, c, ι in (15.8), (15.9) are continuous.[2]

(ii) X is a *Lie groupoid* if X_0, X_1 have the structure of smooth manifolds, the structure maps i, s, t, c, ι are smooth, and the source and target maps $s, t\colon X_1 \to X_0$ are submersions.

The submersion condition guarantees that the fiber product $X_1 \times_{X_0} X_1$ is a smooth manifold, which is necessary if the composition c in (15.8) is to be a smooth map.

We now give many examples to illustrate the pervasiveness and utility of topological and Lie groupoids.

Example 15.42 (groups)**.** A groupoid with a single object $X_0 = \{*\}$ is a group; that is, X_1 is a group. A topological groupoid with a single object is a topological group. A Lie groupoid with a single object is a Lie group. The groupoid attached to a (topological, Lie) group G is often denoted BG, but we reserve that notation for classifying spaces. Instead we use the notation '$\mathrm{pt}/\!/G$', explained below in Example 15.44.

Example 15.43 (spaces)**.** A groupoid with only identity arrows ($i\colon X_0 \to X_1$ is a bijection) is a set X_0. A topological groupoid with only identity arrows is a space. A Lie groupoid with only identity arrows is a smooth manifold.

Example 15.44 (group actions)**.** Let X be a set, G a group, and suppose G acts on X on the right.[3] Then we construct a groupoid[4] $Y = X/\!/G$ variously called the *quotient groupoid* or *action groupoid*. We have $Y_0 = X$ and $Y_1 = X \times G$. The source map is projection $X \times G \to X$ and the target is the action $X \times G \to X$. Composition is defined using the group action. If X is a space, G a topological group, and the action is continuous, then Y is a topological groupoid in a natural way. Similarly, if X is a smooth manifold, G a Lie group, and the action is smooth, then Y is a Lie groupoid in a natural way.

[2] It is sometimes convenient to also ask that s, t be *open* maps.

[3] There is a similar construction for left actions.

[4] This notation is not universally admired as it conflicts with the notation for symplectic or Kähler or GIT quotients. Other possibilities include '$X\!:\!G$', '$G \ltimes X$', and '$X \rtimes G$'.

Example 15.45 (principal bundles)**.** As a special case of the previous example, suppose P is a space (or smooth manifold) with a continuous (smooth) right G-action, assume the action is free, and suppose furthermore that continuous (or smooth) local slices exist. That is, for every $p \in P$ there exists a set $U \subset P$ containing p such that the restriction of the projection $\pi\colon P \to P/G$ to U is a homeomorphism (diffeomorphism) onto an open subset of P/G. Then $\pi\colon P \to P/G$ is called a *principal bundle* with base P/G and structure group G. The action groupoid $P/\!/G$ is equivalent to the space P/G (see Definition 15.10(iv)), and we will see below that π defines a *local equivalence*.

Example 15.46 ($G/\!/G$)**.** Let G be a topological or Lie group. Let G act on itself by conjugation, and denote the resulting quotient groupoid as $G/\!/G$. Even for finite groups this is an important groupoid, for example in proofs of the Sylow theorems. It will play a large role in our later study of loop groups and the Verlinde ring.

Example 15.47 (open covers)**.** Let X be a topological space and $\{U_i\}_{i\in I}$ an open cover. Define

$$Y_0 = \coprod_{i\in I} U_i \tag{15.48}$$

as the disjoint union of the sets in the cover, with the obvious topology. There is a projection $\pi\colon Y_0 \to X$ which is a continuous surjection. Given that we can construct a topological groupoid (Y_0, Y_1) by setting

$$Y_1 = Y_0 \times_X Y_0 \tag{15.49}$$

as the fiber product of $\pi\colon Y_0 \to X$ with itself. So a point of Y_0 is an ordered pair of points $x_0 \in U_{i_0}$, $x_1 \in U_{i_1}$ such that $x_0 = x_1$ as points of X. We can take higher fiber products to construct the simplicial set $Y_\bullet$ which is the nerve of the groupoid Y.

The next examples can be considered to be *moduli spaces*, except that they are groupoids rather than spaces.[5] They are parameter spaces for geometric objects with internal symmetries.

Example 15.50 (Galois coverings)**.** Fix a space X and a discrete group G. Then there is a groupoid $Y = \mathrm{Bun}_G(X)$ whose objects are Galois covers $P \to X$ with group (of deck transformations) G and whose morphisms $X(P, P')$ are homeomorphisms $\varphi\colon P \to P'$ which cover the identity map id_X and commute with the G-actions. For $G = \mathbb{Z}/2\mathbb{Z}$ we obtain the groupoid of double covers. Suppose $X = S^1 = \mathbb{R}/\mathbb{Z}$. Let Y be the groupoid whose objects are Galois covers $P \to S^1$ equipped with a basepoint in the fiber P_0; the morphisms need not fix the basepoint. Then we obtain a diagram

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X & & G/\!/G \end{tikzcd} \tag{15.51}$$

in which the left arrow forgets the basepoint and the right arrow maps a cover to its *holonomy*: the path $[0,1] \hookrightarrow \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ has a lift to a Galois cover $P \to \mathbb{R}/\mathbb{Z}$ with initial point the basepoint $* \in P_0$, and the terminal point is $* \cdot h$, where $h \in G$ is the holonomy. The maps in (15.51) are equivalences of groupoids, in fact, local equivalences.

[5] Thus the term 'moduli stack' is more accurately used.

Example 15.52 (connections on principal bundles)**.** We will discuss connections systematically in a later lecture. Here we want to observe that connections give an extension of Example 15.50 when G is not discrete. Namely, fix a smooth manifold M and a Lie group G. Let $\mathrm{Conn}_G(M)$ be the groupoid whose objects are pairs (P, Θ) consisting of a smooth principal G-bundle $P \to M$ and a connection $\Theta \in \Omega^1_P(\mathfrak{g})$. (Here $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of G.) A morphism $(P, \Theta) \to (P', \Theta')$ is a smooth map $\varphi \colon P \to P'$ such that $\varphi^*\Theta' = \Theta$. If G is discrete, then every principal bundle carries a unique connection and $\mathrm{Conn}_G(M) = \mathrm{Bun}_G(M)$. For $M = S^1$ the holonomy map gives an equivalence with the Lie groupoid $G/\!/G$.

Remark 15.53. $\mathrm{Conn}_G(M)$ is not a Lie groupoid as presented. It is (locally) equivalent to a groupoid which is the global quotient of an infinite dimensional manifold by an infinite dimensional Lie group; the details of what type of manifold and Lie group depend on whether we use smooth connections or complete to Banach spaces of connections.

Example 15.54 (Riemannian metrics; complex structures)**.** Connections are *extrinsic* to the geometry of the smooth manifold M. There are also natural groupoids of *intrinsic* geometric structures, which are often quotient groupoids of spaces by the action of the diffeomorphism group $\mathrm{Diff}(M)$. Examples include the space of Riemannian metrics and the space of complex structures. (The latter may be empty, for example if M has odd dimension.) If M is an oriented compact connected 2-manifold of genus g, then the groupoid of compatible complex structures is a model for the moduli stack of curves of genus g.

Example 15.55 (spin structures)**.** Again, we will discuss spin structures in detail later in the course. Here we remark that if M is a fixed smooth manifold, then there is a groupoid whose objects are spin structures and whose morphisms are maps of spin structures. This groupoid is empty if M is not spinable.

One particular type of groupoid is important in differential geometry as a mild generalization of a smooth manifold.

Definition 15.56. A Lie groupoid X is *étale* if the target and source maps $p_0, p_1 \colon X_1 \to X_0$ are local diffeomorphisms.

In this case the underlying topological stack (defined below) is called an *orbifold* or *smooth Deligne-Mumford stack* and the representing groupoid an *orbifold groupoid*. We remark that smooth Deligne-Mumford stacks may be presented by Lie groupoids which are not étale—for example, if $P \to M$ is a principal G-bundle over a smooth manifold, then $P/\!/G$ is locally equivalent to M. Orbifolds have a more concrete differential-geometric description as "V-manifolds" in the work of Satake, Kawasaki, Thurston and others; see [ALR] and the references therein for a discussion of the various approaches.

Local equivalence of groupoids

An equivalence of groupoids (Definition 15.10(iv), Lemma 15.15) has an inverse equivalence, but an equivalence of *topological* groupoids does not necessarily have a *continuous* inverse.

Example 15.57. A principal G-bundle $\pi\colon P \to X$ induces an continuous equivalence $P/\!/X \to X$ which has a continuous inverse if and only if π admits a continuous global section (which only happens if the principal bundle is globally trivializable). As another example, an open cover $\{U_i\}_{i\in I}$ of a topological space X gives rise to an equivalence of groupoids $\pi\colon Y \to X$, where Y is the groupoid of Example 15.47. It admits a global continuous inverse only if each component of X appears as a set in the cover.

The following definition encodes the notion of continuous *local* inverses.

Definition 15.58. Let $f\colon X \to Y$ be a continuous equivalence of topological groupoids. Then f is a *local equivalence* if for each $y_0 \in Y_0$ there exists a neighborhood $i\colon U \hookrightarrow Y_0$ of y_0 and a lift $\tilde{i}$

$$\begin{tikzcd} & \widetilde{X}_0 \arrow[r, dashed] \arrow[d, dashed] & X_0 \arrow[d, "f"] \\ & Y_1 \arrow[r, "t"] \arrow[d, "s"] & Y_0 \\ U \arrow[r, "i"] \arrow[uur, dotted, "\tilde{i}"] & Y_0 & \end{tikzcd} \tag{15.59}$$

which makes the diagram commute.

In the diagram the upper square is a fiber product. Concretely, for $y \in U$ the lift $\tilde{i}(y)$ gives, in a continuous way, $x \in X_0$ and an arrow $\bigl(a\colon y \to f(x)\bigr)$.

Example 15.60. We list examples of local equivalences whose verification we leave to the reader.

(1) A principal G-bundle $\pi\colon P \to X$ induces a local equivalence $P/\!/G \to X$.
(2) An open cover of a space X induces a local equivalence $Y \to X$, where Y is defined in Example 15.47.
(3) Let G be a Lie group. The holonomy map determines a local equivalence $\mathrm{Conn}_G(S^1) \to G/\!/G$.
(4) The composition of local equivalences is a local equivalence. The pullback of a local equivalence is a local equivalence. The fiber product $P \times_X Q \to X$ of local equivalences $P \to X$, $Q \to X$ is a local equivalence.

Remark 15.61*.* Definition 15.58 fits into the theory of *presheaves of groupoids* on the category of topological spaces: it says that f induces a map of stalks. See [FHT1, Remark A.5] for more explanation.

Coarse moduli space

A topological groupoid X has an associated topological space $[X]$. For a groupoid quotient $X/\!/G$ as in Example 15.44, $[X/\!/G] = X/G$ is the quotient space.

Definition 15.62. Let $X = (X_0, X_1)$ be a topological groupoid. Define an equivalence relation $\sim$ on X_0 by $x \sim x'$ if there exists $f \in X_1$ such that $s(f) = x$ and $t(f) = x'$. Let $X_0 \to [X]$ be the quotient map of the equivalence relation, and topologize $[X]$ as a quotient. The space $[X]$ is the *coarse moduli space* of the groupoid X.

Recall (15.8) that s, t are the source and target maps, respectively. We write $f : x \to x'$. The coarse moduli space, or *orbit space*, can be bad, very bad. In particular, it may not be Hausdorff or paracompact.

Example 15.63. Consider an irrational rotation of S^1, which generates a $\mathbb{Z}$-action. The quotient space $S^1/\mathbb{Z}$, which is the coarse moduli space of the quotient groupoid $S^1/\!/\mathbb{Z}$, is not Hausdorff.

We will soon restrict to a class of groupoids (*local quotient groupoids*) whose coarse moduli space is paracompact Hausdorff.

If X is a groupoid, S a space, and $\phi : S \to [X]$ a continuous map, then there is a pullback groupoid $Y = \phi^* X$ with coarse moduli space $[\phi^* X] = S$. Namely, define Y_0 as the pullback

$$\begin{tikzcd} Y_0 \arrow[r, dashed] \arrow[d, dashed] & X_0 \arrow[d, "\pi"] \\ S \arrow[r, "\phi"] & {[X]} \end{tikzcd} \tag{15.64}$$

and Y_1 as the pullback

$$\begin{tikzcd} Y_1 \arrow[r, dashed] \arrow[d, dashed] & X_1 \arrow[d, "\pi\circ s=\pi\circ t"] \\ S \arrow[r, "\phi"] & {[X]} \end{tikzcd} \tag{15.65}$$

The structure maps pullback from the structure maps of X. In particular, we can take f to be an inclusion. For example, an open cover of $[X]$ induces an open cover of X by groupoids.

The course moduli space of a topological stack is invariant under local equivalences if we add the hypothesis that the source (hence target) maps be open.[6]

Lemma 15.66. *If the source map $s : Y_1 \to Y_0$ of a topological groupoid is open, then so too is the quotient map $q : Y_0 \to [Y]$.*

Proof. For $U \subset Y_0$ open we must show $q(U) \subset [Y]$ is open, which is equivalent to $q^{-1}q(U) \subset Y_0$ open. But $q^{-1}q(U) = ts^{-1}(U)$, and t is an open map if s is, since inversion is continuous in a topological groupoid. □

Proposition 15.67. *Let $F : X \to Y$ be a local equivalence of topological groupoids. Assume the source map $Y_1 \to Y_0$ in $Y = (Y_0, Y_1)$ is continuous. Then the induced map $[F] : [X] \to [Y]$ is a homeomorphism.*

Proof. In (iii) since F is an equivalence of discrete groupoids it induces a bijection $[F]$ on equivalence classes. For any continuous map F of groupoids the induced map $[F]$ is continuous. Given $[y_0] \in [Y]$ we choose an open neighborhood $U \subset Y_0$ of y_0 and a local inverse, as in (15.59). The composition

$$U \xrightarrow{\tilde{i}} \widetilde{X}_0 \longrightarrow X_0 \longrightarrow [X] \tag{15.68}$$

[6]The version of Lecture 1 posted online omitted a hypothesis in the definition of a topological groupoid $Y = (Y_0, Y_1)$: the inversion map $\iota : Y_1 \to Y_1$ should also be assumed continuous.

is continuous and factors through the quotient map $U \to [U] \subset [X]$. The resulting map $[U] \to [X]$ is the inverse to $[F]$ restricted to $[U]$. Since the quotient map is open, by Lemma 15.66, $[U]$ is an open neighborhood of $[y_0]$. It follows that $[F]^{-1}$ is continuous at $[y_0]$. □

The homotopy category of groupoids: stacks

Let Top be the category of topological spaces. A *weak equivalence* is a continuous map $\phi\colon X \to Y$ of spaces which induces an isomorphism on π_0 and an isomorphism of all homotopy groups $\pi_q(X;x) \to \pi_q(Y;\phi(x))$ for all $x \in X$. The *homotopy category of spaces* is obtained from Top by formally inverting all weak equivalences. The resulting category can be considered to have the same objects as Top—topological spaces—but only the underlying homotopy type has invariant meaning. Similarly, let TGpd denote the category whose objects are topological groupoids and whose morphisms are continuous maps of groupoids. Now invert the weak equivalences to obtain the homotopy category of *stacks*. See [SiTe, MV] for a detailed development and [FH] for a gentle introduction to sheaves (on the category of smooth manifolds rather than Top, but the basic ideas are the same).

Local quotient groupoids

Even if we require the coarse moduli space $[X]$ of a topological groupoid X to be paracompact Hausdorff, we will not have homotopy invariance. For example, consider the groupoid $X = \mathrm{pt}/\!/\mathbb{R}$, the real line as a group under addition. As we shall see shortly, a vector bundle over X is simply a representation of $\mathbb{R}$. There is a continuous family of nonisomorphic 1-dimensional unitary representations

$$\begin{aligned} \rho_\xi\colon \mathbb{R} &\longrightarrow \mathbb{T} \\ x &\longmapsto e^{i\xi x} \end{aligned} \tag{15.69}$$

parametrized by $\xi \in \mathbb{R}$, where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is the multiplicative group of unit norm complex numbers. Thus there is a complex line bundle over $[0,1] \times X$ not isomorphic to its restriction to the endpoints. The lesson is that the representations of the automorphism groups of the groupoid must form a discrete set if homotopy invariance is to have a chance of working: representations must be discrete. This happens for compact Lie groups.

Definition 15.70. A topological groupoid X is a *local quotient groupoid* if $[X]$ admits a countable open cover $\{U_i\}_{i\in I}$ such that each groupoid X_{U_i} is locally equivalent to a groupoid of the form $S/\!/G$, where S is a paracompact Hausdorff locally contractible space and G is a compact Lie group.

Proposition 15.71.

(*i*) *Let X be a local quotient groupoid. Then $[X]$ is paracompact Hausdorff.*
(*ii*) *Let $F\colon X \to Y$ be a local equivalence of topological groupoids with open source maps. Then X is a local quotient groupoid if and only if Y is.*

Proof. For (i) it suffices, based on Proposition 15.67, to show that the quotient space S/G is paracompact Hausdorff. Paracompactness follows from [En, (5.1.33)] and Hausdorffness from [tD, (I.3.1)]. The second assertion (ii) is immediate from Proposition 15.67. □

Remark 15.72. The source map of a local quotient groupoid is open [tD, (I.3.1)].

Vector bundles over groupoids

A topological groupoid X can be viewed as a space X_0 of points together with gluing data X_1, and a composition law on gluing data. A vector bundle over a topological groupoid, then, is: a vector space for each point $x \in X$, gluing data for each arrow $(x_0 \xrightarrow{f} x_1) \in X_1$, and a consistency condition for each composable pair of arrows $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$. Continuity is ensured by specifying this data all at once. We use the nerve **(15.34)** of the groupoid and write in terms of the face maps d_i.

Remark 15.73. The discussion in this section applies to fiber bundles, not just vector bundles; see [FHT1, §A.3].

Definition 15.74. Let X be a topological groupoid.

(i) A *vector bundle* $E \to X$ is a pair $E = (E_0, \psi)$ consisting of a vector bundle $E_0 \to X_0$ and an isomorphism $\psi\colon d_1^*E_0 \to d_0^*E_0$ on X_1 which satisfies the cocycle constraint

$$\psi_{f_2 \circ f_1} = \psi_{f_2} \circ \psi_{f_1}. \tag{15.75}$$

for $(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2) \in X_2$.

(ii) A map $\phi\colon E \to E'$ of vector bundles over X is a map $\phi\colon E_0 \to E_0'$ of vector bundles over X_0 such that for every $(x_0 \xrightarrow{f} x_1) \in X_1$ the diagram

$$\begin{tikzcd} E_{x_0} \arrow[r, "\psi_f"] \arrow[d, "\phi_{x_0}"'] & E_{x_1} \arrow[d, "\phi_{x_1}"] \\ E'_{x_0} \arrow[r, "\psi'_f"] & E'_{x_1} \end{tikzcd} \tag{15.76}$$

commutes.

The notation is that the isomorphism ψ at $(x_0 \xrightarrow{f} x_1) \in X_1$ is $\psi_f\colon (E_0)_{x_0} \to (E_0)_{x_1}$. This data determines a groupoid $E = \Bigl(E_0 \overset{\tilde d_1}{\underset{\tilde d_0}{\leftleftarrows}} E_1\Bigr)$ where E_1 is the pullback $d_1^*E_0$ and $\tilde d_0\colon E_1 \to E_0$ is the composition $d_1^*E_0 \xrightarrow{\psi} d_0^*E_0 \to E_0$.

There is a category $\mathrm{Vect}(X)$ of vector bundles over a topological groupoid X. If $F\colon X \to Y$ is a continuous map of groupoids, there is a pullback functor

$$F^*\colon \mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X) \tag{15.77}$$

Example 15.78. For a topological group G a vector bundle over $\mathrm{pt}/\!\!/G$ is a continuous representation of G; see Figure 2 in Lecture 1.

Example 15.79. Let a topological group G act on a topological space X. Then a vector bundle over $X/\!\!/G$ is a G-equivariant bundle over X.

Example 15.80. Let G be a finite group. A vector bundle over $G/\!\!/G$ has support over a union of conjugacy classes in $X_0 = G$. If the support is a single conjugacy class, the bundle is determined up to isomorphism by its restriction to any $g \in G$ in that conjugacy class, and that restriction is a representation of the centralizer subgroup $Z(g) \subset G$. So the simple objects in the category of vector bundles over X are parametrized by pairs $(\mathcal{O}, \rho)$ consisting of a conjugacy class in G and an irreducible representation of the centralizer of an element in that conjugacy class.

Example 15.81 (orbifolds)**.** If X is an orbifold groupoid its tangent bundle $TX \to X$ is the vector bundle $TX_0 \to X_0$ with the natural isomorphism $d_1^*TX_0 \to d_0^*TX_0$ from the fact that d_0, d_1 are local diffeomorphisms. A tensor field on an orbifold groupoid X is a tensor field t on X_0 which satisfies $d_0^*t = d_1^*t$. This includes functions, Riemannian metrics, etc.

Example 15.82 (open covers)**.** A vector bundle over the groupoid Y associated to an open cover $\{U_i\}_{i\in I}$ of a space (Example 15.47) is a vector bundle over each U_i together with gluing data on the overlaps $U_{i_0} \cap U_{i_1}$ which satisfies a cocycle condition on triple overlaps $U_{i_0} \cap U_{i_1} \cap U_{i_2}$. Thus Definition 15.74 includes the clutching construction of vector bundles; see **(1.16)**, **(1.18)**.

Proposition 15.83. *Let $F\colon X \to Y$ be a local equivalence of topological groupoids. Then the pullback functor* (15.77) *is an equivalence of categories.*

Proof. We sketch the construction of an inverse equivalence

$$F_*\colon \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(Y) \tag{15.84}$$

called *descent*; we leave the verification of details to the reader. Suppose $E \to X$ is a vector bundle. For each $y_0 \in Y_0$ consider the set of pairs $(x, g) \in X_0 \times Y_1$ where $g\colon y_0 \to Fx$. They form the objects of a groupoid $\mathcal{G}_{y_0}$ which is contractible in the sense that there is a unique arrow between any two objects. The restriction of the vector bundle $E_0 \to X_0$ to this groupoid has a limit which is a vector space isomorphic to the fiber over any object. (You may think of it as the vector space of invariant sections over the contractible groupoid $\mathcal{G}_{y_0}$.) Define the fiber of $(F_*E)_0$ at y_0 to be this vector space. To topologize and see we get a vector bundle $(F_*E)_0 \to Y_0$ use the local lifts $\tilde{i}$ in (15.59). You will need to check that the topology is independent of the local lift. The clutching data ψ also descends: if $(y_0 \xrightarrow{h} y_1) \in Y_1$ and we choose $x_0, x_1 \in X_0$ together with arrows $(y_0 \xrightarrow{g_0} Fx_0)$, $(y_1 \xrightarrow{g_1} Fx_1)$ in Y_1, then the composite $g_1 h g_0^{-1}$ has a unique lift to $(x_0 \xrightarrow{f} x_1) \in X_1$ and we use ψ_f to define an isomorphism between the fibers of $(F_*E)_0$ at y_0 and y_1. $\square$

Homotopy invariance

The definition of local quotient groupoid is designed in part so that the following extension of Theorem 2.1 holds.

Theorem 15.85. *Let X be a local quotient groupoid. Suppose $E \to [0,1]\times X$ is a vector bundle, and denote by $j_t\colon X \to [0,1]\times X$ the inclusion $j_t(x) = (t,x)$. Then there exists an isomorphism*

$$j_0^*E \xrightarrow{\;\cong\;} j_1^*E. \tag{15.86}$$

Proof. First we prove the homotopy invariance for a global quotient. Thus suppose S is a paracompact Hausdorff space with the continuous action of a compact Lie group G, and let $E \to [0,1]\times S$ be a vector bundle. By Theorem 2.1 there exists a nonequivariant trivialization

$$[0,1]\times S\times \mathbb{E} \longrightarrow E \tag{15.87}$$

of vector bundles over $[0,1]\times S$. The G-action on $E \to [0,1]\times S$ transports to a continuous family

$$\psi_g(t,s) \in \operatorname{End}\mathbb{E}, \qquad g\in G, \quad t\in[0,1], \quad s\in S, \tag{15.88}$$

of endomorphisms, thought of as the lift of the arrow $(t,s) \xrightarrow{g} (t,gs)$. For $t \leqslant t'$ we average against Haar measure dg on the compact Lie group G to define

$$\varphi(t,t';s) = \int_G dg\, \psi_g(t',g^{-1}s)\psi_{g^{-1}}(t,s) \quad \in \operatorname{End}\mathbb{E}, \tag{15.89}$$

thought of as a homomorphism $\underline{\mathbb{E}}_{(t,s)} \to \underline{\mathbb{E}}_{(t',s)}$. A direct check shows it is G-invariant. Since isomorphisms are open in $\operatorname{End}\mathbb{E}$, and $\varphi(t,t;s) = \mathrm{id}_{\mathbb{E}}$, we see that $\varphi(t,t';s)$ is an isomorphism for t' sufficiently close to t. Now argue as in the topological proof of Theorem 2.1 in Lecture 1. First, for each $s\in S$ we see by compactness of $[0,1]$ that there exists $0 < t_1 < t_2 < \cdots < t_N < 1$ such that the composition

$$\varphi(t_N,1;s)\circ\cdots\circ\varphi(t_1,t_2;s)\circ\varphi(0,t_1;s) \tag{15.90}$$

is an isomorphism. Then by paracompactness cover S by open sets U for which we string together G-inveriant isomorphisms (15.90) for all $s\in U$. Use a partition of unity for a locally finite refinement to patch.

Returning to the general local quotient groupoid X, cover $[X]$ by open sets on which the restriction of X is locally equivalent to a global quotient S/G. Then apply the argument of Theorem 2.1 again to paste the trivializations of the previous paragraph. ∎

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