Lecture 13: Topology of skew-adjoint Fredholm operators

We present background and many of the ideas in the proof of Theorem 12.55, the key result in the proof of Bott periodicity given by Atiyah and Singer [AS3]; the last part of the proof is deferred to the next lecture. In this lecture we get as far as explaining the contractible components of skew-adjoint Fredholms and so making complete Definition 12.42 for n odd. We emphasize the two key deformations in the proof: the deformation retraction to unitary operators (12.2) and modding out by the contractible space of compact operators. For the latter we need to know that a fiber bundle $\mathcal{E} \to M$ with contractible fibers is a homotopy equivalence, conditions for which are set out in Proposition 12.26. The real case is parallel to the complex case. We will not present complete proofs, but highlight most of the main ideas as a reader's guide to [AS3]. We work in an ungraded (non-super) situation which contains most of the key ideas. Along the way we review basic facts about compact operators and the relation to Fredholm operators. We also introduce Banach Lie groups which have the homotopy type of BGL_∞; see [F3] for more along these lines. In the first part of this lecture we give some context; see the end of Lecture 9 for additional relevant material. In particular, we describe the geometric model of K-theory that we are developing, something very important for the rest of the course. (Some of the technical background, especially for the equivariant case that we will use later, may be found in the appendix to [FHT1].)

We continue with some of the notation from previous lectures.

The periodic K-theory spectra

We present the definition of a spectrum and its antecedents: prespectra and Ω-prespectra. These definitions and terms vary in the literature. Spectra are the basic objects of *stable homotopy theory*.

Definition 13.1.

(i) A *prespectrum* $T_\bullet$ is a sequence $\{T_n\}_{n\in\mathbb{Z}^{>0}}$ of pointed spaces and maps $s_n\colon \Sigma T_n \to T_{n+1}$.

(ii) An Ω-*prespectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_n\colon T_n \to \Omega T_{n+1}$ of the structure maps are weak homotopy equivalences.

(iii) A *spectrum* is a prespectrum $T_\bullet$ such that the adjoints $t_n\colon T_n \to \Omega T_{n+1}$ of the structure maps are homeomorphisms.

Obviously a spectrum is an Ω-prespectrum is a prespectrum. We can take the sequence of pointed spaces $T_{n_0}, T_{n_0+1}, T_{n_0+2}, \dots$ to begin at any integer $n_0 \in \mathbb{Z}$. If $T_\bullet$ is a *spectrum* which begins at n_0, then we can extend to a sequence of pointed spaces T_n defined for *all* integers n by setting

$$T_n = \Omega^{n_0-n}T_{n_0}, \qquad n < n_0. \tag{13.2}$$

Note that each T_n, in particular T_0, is an *infinite* loop space:

$$T_0 \simeq \Omega T_1 \simeq \Omega^2 T_2 \simeq \cdots \tag{13.3}$$

Example 13.4. Let X be a pointed space. The *suspension prespectrum* of X is defined by setting $T_n = \Sigma^n X$ for $n \geqslant 0$ and letting the structure maps s_n be the identity maps. In particular, for $X = S^0$ we obtain the sphere prespectrum with $T_n = S^n$.

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(13.5) *Spectra from prespectra.* Associated to each prespectrum $T_\bullet$ is a spectrum[1] $LT_\bullet$ called its *spectrification*. It is easiest to construct in case the adjoint structure maps $t_n\colon T_n \to \Omega T_{n+1}$ are inclusions. Then set $(LT)_n$ to be the colimit of

$$T_n \xrightarrow{t_n} \Omega T_{n+1} \xrightarrow{\Omega t_{n+1}} \Omega^2 T_{n+2} \longrightarrow \cdots \tag{13.6}$$

which is computed as an union. For the suspension spectrum of a pointed space X the 0-space is

$$(LT)_0 = \operatorname*{colim}_{\ell\to\infty} \Omega^\ell \Sigma^\ell X, \tag{13.7}$$

which is usually denoted QX.

(13.8) *Homotopy and homology of prespectra.* Let $T_\bullet$ be a prespectrum. Define its homotopy groups by

$$\pi_n(T) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell} T_\ell, \tag{13.9}$$

where the colimit is over the sequence of maps

$$\pi_{n+\ell}T_\ell \xrightarrow{\pi_{n+\ell}t_\ell} \pi_{n+\ell}\Omega T_{\ell+1} \xrightarrow{\text{adjunction}} \pi_{n+\ell+1}T_{\ell+1} \tag{13.10}$$

For an Ω-prespectrum the composition (13.10) is an isomorphism and there is no need for the colimit. Similarly, define the homology groups as the colimit

$$H_n(T) = \operatorname*{colim}_{\ell\to\infty} \widetilde{H}_{n+\ell} T_\ell, \tag{13.11}$$

where $\widetilde{H}$ denotes the reduced homology of a pointed space. We might be tempted to define the cohomology similarly, but that does not work.[2]

(13.12) *Cohomology theory of a spectrum.* A prespectrum $T_\bullet$ determines a cohomology theory h_T on CW complexes and other nice categories of spaces. Assume for simplicity that $T_\bullet$ is an Ω-prespectrum. Then the reduced cohomology of a pointed space X is

$$\tilde{h}^n_T(X) = [X, T_n], \tag{13.13}$$

where we take homotopy classes of pointed maps. All the computational tools (long exact sequences, spectral sequences, etc.) work for generalized cohomology theories. One account is [DaKi].

[1] The notation 'L' indicates 'left adjoint'.

[2] Homotopy and homology commute with colimits, but cohomology does not: there is a derived functor $\lim^1$ which measures the deviation.

(13.14) *Periodic K-theory spectra.* Theorem 12.55 tells that Fredholm operators give an Ω-prespectrum whose n^{th} space is $\text{Fred}_n(H)$ and whose structure maps are the adjoint of α in (12.54). Bott periodicity (Corollary 12.56) tells that this spectrum is 2-periodic. It is the periodic complex topological K-theory spectrum; the corresponding cohomology groups of a space X are denoted $K^n(X)$. The real version of the theorems gives an Ω-prespectrum whose n^{th} space is $\text{Fred}_n(H_{\mathbb{R}})$; the structure maps are the adjoint of α. Now the spectrum is 8-periodic and the corresponding cohomology groups are $KO^n(X)$, the real K-theory groups.

The geometric model of K-theory

Our point of view in this course is to develop a *geometric model* of K-theory and to see it arise in geometry and physics. Were we to discuss real singular cohomology in place of K-theory the geometric model of interest is restricted to smooth manifolds: the de Rham complex. A closed differential form on a smooth manifold determines a real cohomology class, and this brings topological methods into differential geometry. Absent the geometric model we would not be able to recognize and use the topology underlying Chern-Weil forms, a symplectic form, and many other closed forms which occur naturally in geometry.

The paper [FHT1], especially the appendix, contains much more about this model of K-theory, including the equivariant and twisted cases which we need later in the course.

(13.15) *Untwisted classes.* The model so far consists of a *fixed* super Hilbert space $H = H^0 \oplus H^1$ with the left action of a *fixed* superalgebra $C\ell_n^{\mathbb{C}}$, the complex Clifford algebra. (It is important that every irreducible Clifford module appear infinitely often in the Hilbert space, which is why in Definition 12.42 we write the Hilbert space as $C\ell_n^{\mathbb{C}} \otimes H'$ for an infinite dimensional separable Hilbert space with no Clifford action.) Then a K-theory class in $K^n(X)$ on a space X is represented geometrically by a family $X \to \text{Fred}_n(H)$ of odd skew-adjoint Fredholm operators on the fixed Hilbert space H which commute with the fixed algebra $C\ell_n^{\mathbb{C}}$.

(13.16) *Invertibles.* Kuiper's theorem asserts that the invertibles are a contractible subspace of $\text{Fred}_n(H)$; see **(9.32)** and **(9.33)**. Thus families of invertible operators determine the zero K-theory class, and more generally families of Fredholms $X \to \text{Fred}_n(H)$ determine a class relative to the subspace $A \subset X$ consisting of $x \in X$ such that $T(x)$ is invertible.

(13.17) *Twisted classes.* A more flexible model is obtained by allowing the Hilbert space H and the superalgebra $C\ell_n^{\mathbb{C}}$ to also depend on the point $x \in X$. As usual, we want them to vary in a locally trivial family, so form fiber bundles. We need to pay some point-set attention to define a locally trivial family of Hilbert spaces, though in fact Definition 1.12 goes over *verbatim*. We can generalize the standard Clifford algebras to central simple superalgebras and so consider fiber bundles of such equipped with a supermodule which is a Hilbert space bundle. Now the family of odd skew-adjoint Fredholms act on variable Hilbert spaces; they still commute with the superalgebra action. This is a geometric model for *twisted* K-theory which we will come to shortly.

Of course, there is a real version as well. This model extends nicely to groupoids, as we will discuss in a future lecture.

(13.18) *Finite rank vector bundles.* It is convenient to allow finite rank Hilbert bundles, i.e., ordinary finite rank vector bundles, via a simple construction. Let $E \to X$ be a finite rank complex vector bundle. Fix a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $H = H^0 \oplus H^1$ whose homogeneous subspaces are infinite dimensional. Then $E \oplus \underline{H}$ is a Hilbert bundle and the constant family of Fredholms $0 \oplus \mathrm{id}_H$ has kernel the original vector bundle $E \to X$. In the sequel we use finite rank bundles as geometric representatives of K-theory with no further comment.

(13.19) *Warning about finite dimensional representatives of K^1.* Suppose $E = E^0 \oplus E^1 \to X$ is a finite rank complex super vector bundle with a $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-module structure on each fiber. Let e_1 denote the action of the Clifford generator and ϵ the grading operator. Define $e_2 = ie_1\epsilon$. Then a simple computation shows that $e_2^2 = e_1^2 = -1$ and $e_1e_2 = -e_2e_1$. One interpretation is that $E \to X$ automatically extends to a bundle of $\mathrm{C}\ell^{\mathbb{C}}_2$-modules. Another is that e_2 is an *invertible* odd skew-adjoint endomorphism of finite rank $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-modules, and furthermore the homotopy $t \mapsto te_2$ connects the zero operator on E to an invertible operator. Adding the identity on a fixed infinite dimensional $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-module, as in **(13.18)**, we see that we get the zero element of $K^1(X)$ from this finite rank $\mathrm{C}\ell^{\mathbb{C}}_{-1}$-module over X.

A similar argument works for finite rank real Clifford modules except in degrees congruent to $0, 8 \pmod 8$.

Compact operators

Let H^0, H^1 be *ungraded* Hilbert spaces.

Definition 13.20.

(i) An operator $T\colon H^0 \to H^1$ is *finite rank* if the image $T(H^0) \subset H^1$ is a finite dimensional subspace.

(ii) An operator $T\colon H^0 \to H^1$ is *compact* if the closure $\overline{T\big(B(H^0)\big)} \subset H^1$ of the image of the unit ball is compact.

We topologize the set $\mathrm{cpt}(H^0, H^1) \subset \mathrm{Hom}(H^0, H^1)$ of compact operators by the norm topology. Some basic facts whose proof we leave to the reader: The space of compact operators is closed, and in fact is the closure of the set of finite rank operators. The composition of a bounded operator and a compact operator is compact. Hence the compact operators $\mathrm{cpt}(H) \subset \mathrm{End}(H)$ on a Hilbert space H form a closed 2-sided ideal in the space of bounded operators. A Hilbert space H is finite dimensional if and only if $\mathrm{id}_H\colon H \to H$ is compact.

We will prove a basic fact (Proposition 13.23) relating Fredholm and compact operators. It will be convenient to first prove that the closed range condition is superfluous in the definition (Definition 9.6) of a Fredholm operator.

Lemma 13.21. *Let H^0, H^1 be Hilbert spaces and $T\colon H^0 \to H^1$ a continuous linear map with finite dimensional kernel and finite dimensional cokernel. Then T is Fredholm.*

We use the fact that a finite dimensional subspace of a Hilbert space is closed.

Proof. The kernel $\ker T$ is a closed finite dimensional subspace of H^0 and the image of T equals that of T restricted to the closed subspace $(\ker T)^{\perp}$, so to prove that T is closed range we may assume that T is injective. Choose a finite dimensional complement V to $T(H^0) \subset H^1$; it exists since T has finite dimensional cokernel. Then

$$H^1 = T(H^0) \oplus V = V^\perp \oplus V, \tag{13.22}$$

where the latter is the direct sum of closed subspaces. Let π denote orthogonal projection onto $V^\perp$. Then πT is bijective and continuous, so by the open mapping theorem its inverse F is also continuous.

Now suppose $\{\xi_n\} \subset H^0$ is a sequence such that $T\xi_n \to \eta_\infty$ as $n \to \infty$. Then $\pi T\xi_n \to \pi\eta_\infty$. Apply TF to conclude that $T\xi_n \to TF\pi\eta_\infty$. It follows that $\eta_\infty = TF\pi\eta_\infty$, which shows that η_∞ lies in the image of T. This proves that T has closed range. □

Proposition 13.23. *A bounded operator $T\colon H^0 \to H^1$ is Fredholm if and only if there exist bounded operators $S, S'\colon H^1 \to H^0$ such that $\mathrm{id}_{H^0} - ST$ and $\mathrm{id}_{H^1} - TS'$ are compact.*

We can replace 'compact' by 'finite rank', as is clear from the proof, which also makes clear that we can choose $S' = S$. The operators S, S' are called *parametrices* for T.

Proof. If T is Fredholm, write $H^0 = \ker T \oplus (\ker T)^\perp$ and $H^1 = T(H^0)^\perp \oplus T(H^0)$ as orthogonal sums of closed subspaces. Since T restricted to $(\ker T)^\perp$ is an isomorphism onto $T(H^0)$, it has a continuous inverse on those spaces by the open mapping theorem. Define $S = S'$ to be the extension of this inverse by zero on $T(H^0)^\perp$.

Conversely, if the parametrices exist, restrict $\mathrm{id}_{H^0} - ST$ to $\ker T$ to deduce that $\mathrm{id}_{\ker T}$ is compact. Also, the operator $\mathrm{id}_{H^1} - TS'$ is compact and preserves $T(H^0)$, thus $\mathrm{id}_{\mathrm{coker}\, T}$ is compact. □

We turn now to groups—in fact, complex Banach Lie groups—and so switch notation to emphasize that the linear spaces of operators we have been using are Lie algebras:

$$\begin{aligned} \mathrm{Aut} &\longrightarrow GL \\ \mathrm{End} &\longrightarrow \mathfrak{gl} \\ \mathrm{cpt} &\longrightarrow \mathfrak{cpt} \end{aligned} \tag{13.24}$$

We often omit the Hilbert space from the notation for visual clarity.

Definition 13.25. $GL^{\mathrm{cpt}}(H) = \{P\colon H \to H$ such that P is invertible and $P - \mathrm{id}_H$ is compact$\}$.

GL^{cpt} is a Banach Lie group with Lie algebra $\mathfrak{cpt}$.

Fix a filtration $H_1 \subset H_2 \subset H_3 \subset \cdots$ of H by subspaces with $\dim H_n = n$ such that $\overline{\bigcup_{n=1}^{\infty} H_n} = H$. We can achieve this by choosing a countable basis (H is always assumed separable) and letting H_n be the span of the first n basis vectors. There is an induced increasing sequence of groups

$$GL(H_1) \subset GL(H_2) \subset GL(H_3) \subset \cdots \tag{13.26}$$

where the n^{th} group consists of invertible operators which are the identity on $H_n^\perp$.

Theorem 13.27 (Palais [Pa3])**.** *The inclusion* $\bigcup_{n=1}^{\infty} GL(H_n) \hookrightarrow GL^{\text{cpt}}(H)$ *is a homotopy equivalence.*

The union of the groups (13.26), denoted GL_∞, has the colimit topology: a subset is open iff its intersection which each group in (13.26) is open. We encountered this group—rather its deformation retraction to the unitary subgroup (and with a different notation)—in Remark 3.32.

The Calkin algebra and its subgroups

Definition 13.28. The *Calkin algebra* of a Hilbert space H is the quotient algebra $\operatorname{End}(H)/\operatorname{cpt}(H)$.

Since the ideal of compact operators is closed, the Calkin algebra inherits a Banach space structure. (It is not only a Banach algebra but a C^*-algebra.) So we can talk about unitary elements, skew-adjoint elements, the spectrum of an element, etc. We usually use the notation '$\mathfrak{gl}/\mathfrak{cpt}$' to emphasize that the Calkin algebra is the Lie algebra of a Banach Lie group.

That group is the quotient GL/GL^{cpt}. The quotient map $GL \to GL/GL^{\text{cpt}}$ is a principal bundle. (To prove that we need the existence of local sections, which follows from a theorem of Bartle and Graves, for example; see also [Pa2].) Kuiper's Theorem 12.1 asserts that GL is contractible. It follows from Theorem 12.28 that $GL \to GL/GL^{\text{cpt}}$ is a universal bundle in the sense that it classifies principal GL^{cpt}-bundles (over metrizable bases). In particular, we have proved

Proposition 13.29. *The group* GL/GL^{cpt} *has the homotopy type* BGL_∞*.*

We now have two homotopy types on the table: GL_∞ and BGL_∞. There are also two homotopy types in (12.52). The main result implies a match if we replace BGL_∞ with $\mathbb{Z} \times BGL_\infty$.

Now we bring in the deformation retraction to unitaries (12.2). GL retracts to the contractible group U, as in Corollary 12.3, and GL^{cpt} retracts onto U^{cpt}, which has the same homotopy type. Denote

$$G = U/U^{\text{cpt}}, \tag{13.30}$$

which is a deformation retract of GL/GL^{cpt} and thus has the homotopy type BGL_∞. We summarize the groups defined so far in the diagram

$$\begin{array}{ccccc} U & \xrightarrow{\text{d.r.}} & GL & \hookrightarrow & \mathfrak{gl} \\ \big\downarrow{\scriptstyle U^{\text{cpt}}} & & \big\downarrow{\scriptstyle GL^{\text{cpt}}} & & \big\downarrow{\scriptstyle \pi} \\ G = U/U^{\text{cpt}} & \xrightarrow{\text{d.r.}} & GL/GL^{\text{cpt}} & \hookrightarrow & \mathfrak{gl}/\mathfrak{cpt} \end{array} \tag{13.31}$$

The labeled horizontal arrows are deformation retractions and the first two vertical arrows are principal bundles.

Let $(\mathfrak{gl}/\mathfrak{cpt})^\times \subset \mathfrak{gl}/\mathfrak{cpt}$ denote the Banach Lie group of invertible elements.

Lemma 13.32. GL/GL^{cpt} *is the identity component of* $(\mathfrak{gl}/\mathfrak{cpt})^\times$*.*

We leave the proof to the reader; it can be found in [F3]. The intuition is that GL/GL^{cpt} is a Banach Lie group with Lie algebra the Calkin algebra $\mathfrak{gl}/\mathfrak{cpt}$. So too is $(\mathfrak{gl}/\mathfrak{cpt})^{\times}$.

The following is a restatement of Proposition 13.23.

Proposition 13.33. $\pi^{-1}\bigl((\mathfrak{gl}/\mathfrak{cpt})^{\times}\bigr) = \mathrm{Fred} \subset \mathfrak{gl}$. *Also,* $\pi^{-1}\bigl(GL/GL^{\mathrm{cpt}}\bigr) = \mathrm{Fred}^{(0)}$ *is the space of Fredholm operators of numerical index zero.*

For the latter statement we use Corollary 9.26. We summarize in an expanded version of (13.31):

(13.34)
$$\begin{array}{ccccccccc}
U & \xhookrightarrow{\text{d.r.}} & GL & \hookrightarrow & \mathfrak{gl} & \hookleftarrow & \mathrm{Fred} & \hookleftarrow & \mathrm{Fred}^{(0)} \\
\Big\downarrow{\scriptstyle U^{\mathrm{cpt}}} & & \Big\downarrow{\scriptstyle GL^{\mathrm{cpt}}} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
G = U/U^{\mathrm{cpt}} & \xhookrightarrow{\text{d.r.}} & GL/GL^{\mathrm{cpt}} & \hookrightarrow & \mathfrak{gl}/\mathfrak{cpt} & \hookleftarrow & (\mathfrak{gl}/\mathfrak{cpt})^{\times} & \hookleftarrow & GL/GL^{\mathrm{cpt}}
\end{array}$$

The central vertical map is the quotient by $\mathfrak{cpt}$ which defines the Calkin algebra. To the right are restrictions of that quotient to Fredholm operators. To the left are principal bundles with total space a group.

Corollary 13.35. *The space* $\mathrm{Fred}^{(0)}$ *of Fredholm operators of index zero has the homotopy type* BGL_{∞} *and the space* Fred *of all Fredholm operators has the homotopy type* $\mathbb{Z} \times BGL_{\infty}$.

Since $\mathrm{Fred}_0(H)$ in (12.52) is isomorphic to the space of Fredholm operators on an ungraded Hilbert space, this determines the homotopy type of the spaces in the first line of (12.52).

Remark 13.36. It follows that G also has the homotopy type BGL_{∞}. We caution that [AS3] uses the symbol 'G' for the unitary retraction of $(\mathfrak{gl}/\mathfrak{cpt})^{\times}$, a group whose identity component is our 'G'.

Spaces of skew-adjoint Fredholm operators

Recall that the Lie algebra of the group of unitary operators is the space of skew-adjoint operators. (This is true in finite dimensions.) For any space of operators, or operator algebra, we use a "hat" to denote the subspace of skew-adjoint elements. Thus if we use $\mathcal{F} = \mathrm{Fred}$ for all Fredholm operators, then $\widehat{\mathcal{F}}$ is the notation for skew-adjoint Fredholms.

(13.37) *An ungraded version of* Fred_1. This is essentially a reprise of the text following (12.70). Let $H = H^0 \oplus H^1$ be a super Hilbert space and suppose $A \subset \mathrm{Fred}_1(H)$. Let e_1 denote the action of the Clifford generator. Then $e_1 A$ is even and skew adjoint: $(e_1A)^* = A^*e_1^* = (-A)(-e_1) = Ae_1 = -e_1A$. Let T denote its restriction to the even part of $C\ell_1^{\mathbb{C}} \otimes H$. The loop map (13.41) which appears below is essentially the Atiyah-Singer map (12.70); see (12.71).

(13.38) *Main theorem.* We now state the ungraded version of Theorem 12.55 whose proof we sketch in the next lecture.

Theorem 13.39 ([AS3])**.** *The space* $\widehat{\mathcal{F}}$ *has three components*

(13.40)
$$\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_{+} \amalg \widehat{\mathcal{F}}_{-} \amalg \widehat{\mathcal{F}}_{*}.$$

The components $\widehat{\mathcal{F}}_\pm$ are contractible, and the map

$$\begin{aligned}\alpha\colon \widehat{\mathcal{F}}_* &\longrightarrow \Omega\mathcal{F} \\ T &\longmapsto (\cos \pi t + T \sin \pi t,\ 0 \leqslant t \leqslant 1)\end{aligned} \tag{13.41}$$

is a homotopy equivalence.

Notice that the Atiyah-Singer loop map (13.41) has domain a space of skew-adjoint operators and codomain loops in a space related to a Lie group (Theorem 13.33). So we might expect—after retracting to unitaries which *do* have skew-adjoints as the Lie algebra—that (13.41) is closely related to exponentiation from a Lie algebra to a Lie group. It is.

(13.42) *The contractible components of skew-adjoint Fredholms.* In the diagram

$$\begin{array}{ccc} & & \widehat{\mathcal{F}} \\ & & \big\downarrow{\scriptstyle \mathrm{cpt}} \\ \widehat{G} & \xrightarrow[\mathrm{d.r}]{} & \widehat{GL/GL^{\mathrm{cpt}}} \end{array} \tag{13.43}$$

the vertical arrow is a fiber bundle with contractible fibers and the horizontal arrow is a deformation retraction. So $\widehat{G}$ is homotopy equivalent to $\widehat{\mathcal{F}}$. Now an element $x \in \widehat{G}$ is unitary skew-adjoint, so $xx^* = 1$ and $x = -x^*$, which implies $x^2 = -1$. It follows that $\operatorname{spec} x \subset \{+i -i\}$. Since the spectrum is nonempty, there are three possibilities. This decomposes $\widehat{G}$ into three disjoint subspaces which one can prove are components:

$$\widehat{G} = \widehat{G}_+ \amalg \widehat{G}_- \amalg \widehat{G}_* \tag{13.44}$$

Furthermore, the spaces $\widehat{G}_\pm$ each contain a single element $\pm i$. The decomposition (13.40) of $\widehat{\mathcal{F}}$ follows as does the contractibility of the two components consisting of skew-adjoint Fredholms whose *essential spectrum* is $\{+i\}$ or $\{-i\}$.

Remark 13.45. There are contractible components of $\mathrm{Fred}_n(H_\mathbb{R})$ in the real case if $n \equiv 1 \pmod 4$.

(13.46) *The noncontractible component of skew-adjoint Fredholms.* Replacing the spaces in (13.41) by homotopy equivalent spaces, we reduce the remaining part of Theorem 13.39 to the following.

Theorem 13.47. *The exponential map*

$$\begin{aligned}\epsilon\colon \widehat{G}_* &\longrightarrow \Omega G \\ x &\longmapsto (\exp \pi t x,\ 0 \leqslant t \leqslant 1)\end{aligned} \tag{13.48}$$

is a homotopy equivalence.

Now, as promised, the loop map is exponentiation, since $x^2 = -1$ implies

$$\cos \pi t + x \sin \pi t = \exp \pi t x. \tag{13.49}$$

We sketch a proof of Theorem 13.47 in the next lecture.

References

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