Lecture 12: Kuiper's theorem, classifying spaces, Atiyah-Singer loop map, Atiyah-Bott-Shapiro construction

We begin this lecture by giving a proof of Kuiper's theorem on the contractibility of the general linear group of an infinite dimensional Hilbert space. We follow [ASe1, Appendix C] closely; see also the original paper [Ku]. One consequence is the contractibility of infinite dimensional Stiefel manifolds, which we use to construct geometric classifying spaces for compact Lie groups. Then we introduce spaces of Fredholm operators which incorporate Clifford algebras. We state the Atiyah-Singer looping construction **(12.59)**, which is the main theorem in [AS3], and show how together with algebraic Bott periodicity **(11.41)** it implies Bott periodicity. In the next two lectures we sketch the main points in the proof of that theorem, which relies on Kuiper's contractibility result. We conclude with some important material not covered in class: the Atiyah-Bott-Shapiro construction.

Kuiper's Theorem

Theorem 12.1 ([Ku])**.** *Let H be an infinite dimensional real or complex separable Hilbert space. Then the general linear group $\mathrm{Aut}(H)$ is contractible in the norm topology.*

The general linear group deformation retracts onto the unitary group of automorphisms which preserve the inner product. Namely, to any operator P is associated a nonnegative self-adjoint operator $|P|$ such that $|P|^2 = P^*P$. The operator $|P|$ is positive if P is invertible. Then the retraction is

$$P_t = P\big((1-t)\,\mathrm{id}_H + t|P|^{-1}\big). \tag{12.2}$$

Corollary 12.3. *The unitary group $U(H)$ is contractible in the norm topology*

In the real case 'unitary' is usually called 'orthogonal' and the group is denoted $O(H)$.

Definition 12.4. A continuous map $f\colon X \to Y$ of topological spaces is a *weak homotopy equivalence* if $f_*\colon \pi_0 X \to \pi_0 Y$ is a bijection and for every $x \in X$ the map $f_*\colon \pi_q(X,x) \to \pi_q(Y, f(x))$ is an isomorphism for all $q > 0$.

Whitehead proved that a weak homotopy equivalence of CW complexes is a homotopy equivalence, and the same is true for spaces with the homotopy type of a CW complex [Mi]. This applies in particular to open subsets of a Banach space,[1] so to the general linear group in the norm topology. Therefore, to prove Theorem 12.1 it suffices to show all homotopy groups of $\mathrm{Aut}(H)$ vanish.

Let X be a compact simplicial complex and $f\colon X \to \mathrm{Aut}(H)$ a continuous map. We prove by a series of deformations that $f = f_0$ is homotopic to the constant map with value id_H.

Lemma 12.5. *There exists a homotopy $f_0 \simeq f_1$ and a finite dimensional subspace $V \subset \mathrm{End}(H)$ such that $f_1(x) \in V$ for all $x \in X$.*

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[1] Milnor [Mi] proves that an absolute neighborhood retract (ANR) has the homotopy type of a CW complex. A (paracompact) Banach manifold is an ANR [Pa2].

Of course, we make a choice of the homotopy whose existence is proved! (Similar comment for the lemmas which follow.)

Proof. Since $Aut(H) \subset \operatorname{End}(H)$ is open it has a cover by balls, and since X is compact it is covered by a finite number of their inverse images. Subdivide X so that every simplex lies in such a ball. Define f_1 to agree with f_0 on vertices and be affine on each simplex. Then $f_t = (1-t)f_0 + tf_1$ is the desired homotopy and V is the span of $f(x)$ in $\operatorname{End}(H)$ over the vertices x in X. $\square$

Lemma 12.6. *There exists an orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3$ such that (i) $\alpha(H_1) \perp H_3$ for all $\alpha \in V$, (ii) H_1 is infinite dimensional, and (iii) there is an isomorphism $T\colon H_1 \to H_3$.*

Proof. Let $P_1 \subset H$ be a line and $P_2 \subset H$ a finite dimensional orthogonal subspace so that $\alpha(P_1) \subset P_1 \oplus P_2$ for all $\alpha \in V$. Let P_3 be a line orthogonal to $P_1 \oplus P_2$. We begin an iterative process. Choose a line Q_1 orthogonal to $P_1 \oplus P_2 \oplus P_3$ and a finite dimensional subspace Q_2 such that the sum $P_1 \oplus Q_1 \oplus P_2 \oplus Q_2$ is orthogonal and contains $\alpha(Q_1)$ for all $\alpha \in V$. Let Q_3 be a line orthogonal to $P_1 \oplus Q_1 \oplus P_2 \oplus Q_2 \oplus P_3$. Set $P_i^{(1)} = P_i \oplus Q_i$, $i = 1, 2, 3$. Then $\alpha(P_1^{(1)}) \subset P_1^{(1)} \oplus P_2^{(1)}$, $\dim P_1^{(1)} > \dim P_1$, and $P_1^{(1)} \cong P_3^{(1)}$. Iterate to find $P_i^{(1)} \subset P_i^{(2)} \subset \cdots$ with these properties. Set

$$H_i = \overline{\bigcup_{k=1}^{\infty} P_i^{(k)}}, \qquad i = 1, 3, \tag{12.7}$$

and choose $H_2 = (H_1 \oplus H_3)^{\perp}$. $\square$

Lemma 12.8. *There exists a homotopy $f_1 \simeq f_3$ so that $f_3\big|_{H_1} = \mathrm{id}_{H_1}$.*

Proof. For $x \in X$ let $H_x = \big(f_1(x)H_1 \oplus H_3\big)^{\perp}$. The identity transformation is connected to the rotation

$$\begin{aligned} H_1 \oplus H_x \oplus H_1 &\longrightarrow H_1 \oplus H_x \oplus H_1 \\ \xi \oplus \eta \oplus \zeta &\longmapsto (-\zeta) \oplus \eta \oplus \xi \end{aligned} \tag{12.9}$$

by a path (of unitaries). Conjugate this path by

$$f_1(x) \oplus \mathrm{id}_{H_x} \oplus T\colon H_1 \oplus H_x \oplus H_1 \longrightarrow f_1(x)H_1 \oplus H_x \oplus H_3 \tag{12.10}$$

to obtain a path from id_H to

$$\begin{aligned} \varphi_x\colon f_1(x)H_1 \oplus H_x \oplus H_3 &\longrightarrow f_1(x)H_1 \oplus H_x \oplus H_3 \\ f_1(x)\xi \oplus \eta \oplus T\zeta &\longmapsto -f_1(x)\zeta \oplus \eta \oplus T\xi \end{aligned} \tag{12.11}$$

Set $f_2(x) = \varphi_x^{-1} f_1(x)$. Then f_2 is continuous, $f_1 \simeq f_2$, and $f_2(x)\big|_{H_1} = -T\colon H_1 \to H_3$ for all $x \in X$. Now compose with the rotation

$$\begin{aligned} H_1 \oplus H_2 \oplus H_3 &\longrightarrow H_1 \oplus H_2 \oplus H_3 \\ \xi \oplus \lambda \oplus T\zeta &\longmapsto -\zeta + \lambda \oplus T\xi \end{aligned} \tag{12.12}$$

to obtain f_3 homotopic to f_2 with $f_3\big|_{H_1} = \mathrm{id}_{H_1}$. $\square$

Proof of Theorem 12.1. We execute the *Eilenberg swindle*. First, relative to the orthogonal decomposition $H = H_1^\perp \oplus H_1$ we have

$$f_3(x) = \begin{pmatrix} u(x) & 0 \\ * & 1 \end{pmatrix}, \tag{12.13}$$

where '1' denotes the identity operator. By a simple homotopy multiplying the operator '$*$' by t we move to a homotopic family

$$f_4(x) = \begin{pmatrix} u(x) & 0 \\ 0 & 1 \end{pmatrix}. \tag{12.14}$$

Since H_1 is infinite dimensional and separable we can write it as a countable orthogonal direct sum

$$H_1 = K_1 \oplus K_2 \oplus K_3 \oplus \cdots \tag{12.15}$$

of infinite dimensional subspaces, each equipped with an isomorphism to $H_1^\perp$. Now the path of operators

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix} \tag{12.16}$$

on $H_1^\perp \oplus H_1^\perp$ begins at $t = 0$ with the identity operator and concludes at $t = 1$ with $\left(\begin{smallmatrix} u^{-1} & \\ & u \end{smallmatrix}\right)$. Exchanging the roles of u, u^{-1} we obtain a path of operators from $\left(\begin{smallmatrix} u & \\ & u^{-1} \end{smallmatrix}\right)$ to the identity. Therefore, we obtain a homotopy

$$f_4 = \begin{pmatrix} u &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& \ddots \end{pmatrix} \simeq \begin{pmatrix} u &&&& \\ & u^{-1} &&& \\ && u && \\ &&& u^{-1} & \\ &&&& \ddots \end{pmatrix} \simeq \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& \ddots \end{pmatrix} \tag{12.17}$$

of operators on $H = H_1^\perp \oplus K_1 \oplus K_2 \oplus K_3 \oplus \cdots$. $\square$

Stiefel manifolds and classifying spaces for principal bundles

Recall first the definition.

Definition 12.18. Let G be a Lie group. A *principal G bundle* is a fiber bundle $\pi\colon P \to M$ over a smooth manifold M equipped with a right G-action $P \times G \to P$ which is simply transitive on each fiber.

The hypothesis that π is a fiber bundle means it admits local trivializations. For a *principal* bundle a local trivialization is equivalent to a local section. In one direction, if $U \subset M$ and $s\colon U \to P$ is a section of $\pi\big|_U\colon P\big|_U \to U$, then there is an induced local trivialization

$$\begin{aligned} \varphi\colon U \times G &\longrightarrow P \\ x, g &\longmapsto s(x)\cdot g \end{aligned} \tag{12.19}$$

where '$\cdot$' denotes the G-action on P.

(12.20) *From vector bundles to principal bundles and back.* Let $\pi\colon E \to M$ be a vector bundle of rank k. Assume for definiteness that π is a *real* vector bundle. There is an associated principal $GL_k(\mathbb{R})$-bundle $\mathcal{B}(E) \to M$ whose fiber at $x \in M$ is the spaces of bases $b\colon \mathbb{R}^k \xrightarrow{\cong} E_x$. These fit together into a principal bundle which admits local sections: a local section of the principal bundle $\mathcal{B}(E) \to M$ is a local trivialization of the vector bundle $E \to M$. Conversely, if $P \to M$ is a principal $G = GL_k(\mathbb{R})$-bundle, then there is an *associated* rank k vector bundle $E \to M$ defined as

$$E = P \times \mathbb{R}^k/G, \tag{12.21}$$

where the right G-action on $P \times \mathbb{R}^k$ is

$$(p, \xi)\cdot g = (p\cdot g, g^{-1}\xi), \qquad p \in P, \quad \xi \in \mathbb{R}^k, \quad g \in G, \tag{12.22}$$

and we use the standard action of $GL_k(\mathbb{R})$ on $\mathbb{R}^k$ to define $g^{-1}\xi$.

A section of the associated bundle E is a G-equivariant map $s\colon P \to \mathbb{R}^k$. Note that G acts on the *right* on both P and $\mathbb{R}^k$. A special case of this construction is the frame bundle of a smooth k-dimensional manifold M, a case the reader should thing through carefully if this is new.

(12.23) *More general associated bundles.* Let $P \to M$ be a principal G-bundle and F a space equipped with a *left* G-action. Then there is an associated fiber bundle with fiber F; the total space is the quotient

$$F_P = (P \times F)/G = P \times_G F \tag{12.24}$$

where $g \in G$ acts on the right of $(p, f) \in P \times F$ to give $(p\cdot g, g^{-1}\cdot f)$. A section of the associated bundle is a G-equivariant function $P \to F$. The fibers of $F_P \to M$ are identified with F only up to the action of G. The principal bundle controls this uncertainty. More precisely, each point $p \in P_x$ gives an identification of the fiber $(F_P)_x$ with F. In that sense points of a principal bundle are generalized bases for all associated fiber bundles, and it is the principal bundle which controls the geometry and topology.

This geometric viewpoint on fiber bundles was advocated by Steenrod [St].

(12.25) *Fiber bundles with contractible fiber.* We quote the following general proposition in the theory of fiber bundles.

Proposition 12.26. *Let $\pi\colon \mathcal{E} \to M$ be a fiber bundle whose fiber F is contractible and a metrizable topological manifold, possibly infinite dimensional. Assume that the base M is metrizable. Then π admits a section. Furthermore, if $\mathcal{E}, M, F$ all have the homotopy type of a CW complex, then π is a homotopy equivalence.*

See [Pa2] for a proof of the first assertion. The last assertion follows from the long exact sequence of homotopy groups and Whitehead's theorem, stated after Definition 12.4. The takeaway, after stripping off the technical hypotheses, is that a fiber bundle with contractible fibers is a homotopy equivalence. But don't forget that there are technical hypotheses!

(12.27) *Classifying maps for principal bundles.* Now we characterize universal principal bundles.

Theorem 12.28. *Let G be a Lie group. Suppose $\pi^{\mathrm{univ}}\colon P^{\mathrm{univ}} \to B$ is a principal G-bundle and P^{univ} is a contractible metrizable topological manifold.*[2] *Then for any continuous principal G-bundle $P \to M$ with M metrizable, there is a classifying diagram*

(12.29)
$$\begin{tikzcd} P \arrow[r, "\tilde\varphi"] \arrow[d] & P^{\mathrm{univ}} \arrow[d] \\ M \arrow[r, "\varphi"] & B \end{tikzcd}$$

In the commutative diagram (12.29) the map $\tilde\varphi$ commutes with the G-actions on P, P^{univ}, i.e., it is a map of principal G-bundles.

Proof. A G-map $\tilde\varphi$ is equivalently a section of the associated fiber bundle

$$(P \times P^{\mathrm{univ}})/G \to M \tag{12.30}$$

formed by taking the quotient by the diagonal right G-action. The fiber of the bundle (12.30) is P^{univ}. Sections exist by Proposition 12.26, since P^{univ} is contractible. □

(12.31) *Stiefel manifolds.* Let H be a separable (complex) Hilbert space. Introduce the infinite dimensional *Stiefel manifold*

$$St_k(H) = \{b\colon \mathbb{C}^k \to H : b \text{ is an isometry}\}. \tag{12.32}$$

It is an open subset of the linear space $\mathrm{Hom}(\mathbb{C}^k, H) \cong H \oplus \cdots \oplus H$, which we give the topology of a Hilbert space. Then the open subset $St_k(H)$ is a Hilbert manifold. There is an obvious projection

$$\pi\colon St_k(H) \longrightarrow Gr_k(H) \tag{12.33}$$

[2] We allow an infinite dimensional manifold modeled on a Hilbert space, say.

to the Grassmannian

$$Gr_k(H) = \{W \subset H : \dim W = k\}. \tag{12.34}$$

which maps b to its image $b(\mathbb{C}^k) \subset H$. We leave the reader to check that π is smooth. In fact, π is a principal bundle with structure group the unitary group U_k.

Theorem 12.35. *$St_k(H)$ is contractible.*

Proof. The unitary group $U(H)$ acts transitively on $St_k(H)$ by left composition. The stabilizer of a k-frame $b_0\colon \mathbb{C}^k \to H$ is the unitary group of the orthogonal complement $b(\mathbb{C}^k)^\perp$. Both unitary groups are contractible, by Kuiper (Corollary 12.3). So too is the quotient homogeneous space, which is diffeomorphic to $St_k(H)$. □

Corollary 12.36. *The bundle (12.33) is a universal U_k-bundle.*

Remark 12.37. There is a simpler proof that $St_k(H)$ is contractible based on the contractibility of the unit sphere in H.

(12.38) *Other Lie groups.* Let G be a *compact* Lie group. (Note G need not be connected.) The *Peter-Weyl theorem* asserts that there is an embedding $G \subset U_k$ for some $k > 0$. Let $EG = St_k(H)$ be the Stiefel manifold for a *complex* separable Hilbert space H. Then the restriction of the free $GL_k(\mathbb{C})$-action to G is also free; let BG be the quotient. It is a Hilbert manifold, and

$$EG \longrightarrow BG \tag{12.39}$$

is a universal principal G-bundle, by Theorem 12.28.

This gives Hilbert manifold models for the classifying space of any compact Lie group.

Fredholms and Clifford algebras

Fix a $\mathbb{Z}/2\mathbb{Z}$-graded complex Hilbert space $H = H^0 \oplus H^1$ for which both H^0 and H^1 are infinite dimensional. With few modifications what we do applies to real Hilbert spaces, but we defer that to the next lecture.

Remark 12.40. Following **(11.25)** strictly, *which I strongly recommend* [DF], leads to some awkward conventions [DM, §4.4]. First, H should have an inner product $\langle -, - \rangle$, which we assume is even so that $H^0 \perp H^1$. But then the sign rule implies that $\langle \xi, \xi \rangle \in i\mathbb{R}$ for $\xi \in H^1$. Worse, an odd skew-adjoint operator on H has eigenvalues which are multiples of a primitive eighth root of unity; they are neither purely real nor purely imaginary. We will take the easy way out to conform with the (old) literature, which does not follow the Koszul sign rule. So each of H^0, H^1 is a Hilbert space with a usual inner product, and a continuous odd skew-adjoint operator A on $H = H^0 \oplus H^1$ has the form

$$A = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} \tag{12.41}$$

where $T\colon H^0 \to H^1$ is continuous and $T^*\colon H^1 \to H^0$ is its usual adjoint relative to the inner products on H^0, H^1.

Definition 12.42. Let $H = H^0 \oplus H^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space. The following spaces of operators are endowed with the norm topology.

(i) $\mathrm{Fred}_0(H)$ is the space of odd skew-adjoint Fredholm operators on H.
(ii) For $n > 0$, $\mathrm{Fred}_{-n}(H) \subset \mathrm{Fred}_0(\mathrm{C}\ell^{\mathbb{C}}_{-n} \otimes H)$ is the subspace[3] of operators which (graded) commute with the left action of $\mathrm{C}\ell^{\mathbb{C}}_{-n}$.

We endow $\mathrm{C}\ell^{\mathbb{C}}_{-n}$ with the Hermitian inner product which renders products $e_{i_1} \cdots e_{i_q}$ of basis elements orthonormal; then the pin group Pin_{-n} acts unitarily and the generators e_i act by odd skew-adjoint unitary isomorphisms. An odd operator A on $\mathrm{C}\ell^{\mathbb{C}}_{-n} \otimes H$ commutes with $\mathrm{C}\ell^{\mathbb{C}}_{-n}$ if and only if $Ae_i = -e_iA$ for $i = 1, \dots, n$.

Remark 12.43. The matrix (12.41) makes clear that $\mathrm{Fred}_0(H)$ is canonically identified with $\mathrm{Fred}(H^0, H^1)$, the space studied in Lecture 1. There is also an ungraded interpretation of $\mathrm{Fred}_{-n}(H)$. First observe that the $n-1$ elements $f_i = e_ie_n,\ i = 1, \dots, n-1$ generate a Clifford algebra isomorphic to $\mathrm{C}\ell^{\mathbb{C}}_{-(n-1)}$. Then if $A \in \mathrm{Fred}_{-n}(H)$ the restriction of e_nA to the even subspace of $\mathrm{C}\ell^{\mathbb{C}}_{-n} \otimes H$ is a skew-adjoint Fredholm operator whcih anticommutes with the f_i. This is the form of $\mathrm{Fred}_{-n}(H)$ which is most studied in [AS3].

(12.44) *Periodicity of* $\mathrm{Fred}_{-n}(H)$. Recall from Theorem 11.42 that there is an isomorphism

$$\mathrm{C}\ell^{\mathbb{C}}_{-2} \cong \mathrm{End}(\mathbb{S}) \cong \mathbb{S} \otimes \mathbb{S}^* \tag{12.45}$$

for a complex super vector space of dimension $1|1$.

Proposition 12.46. *The map*

$$\begin{aligned} \mathrm{Fred}_0(\mathbb{S}^* \otimes H) &\longrightarrow \mathrm{Fred}_{-2}(H) \subset \mathrm{Fred}_0(\mathbb{S} \otimes \mathbb{S}^* \otimes H) \\ A &\longmapsto \mathrm{id}_{\mathbb{S}} \otimes A \end{aligned} \tag{12.47}$$

is a homeomorphism.

This follows since the only central endomorphisms of $\mathbb{S}$ are multiples of $\mathrm{id}_{\mathbb{S}}$.

Kuiper's Theorem 12.1 implies that there is a contractible space of isomorphisms $H \xrightarrow{\cong} \mathbb{S}^* \otimes H$.

Corollary 12.48. *Up to a contractible choice the isomorphism* (12.45) *leads to a homeomorphism*

$$\mathrm{Fred}_0(H) \xrightarrow{\cong} \mathrm{Fred}_{-2}(H). \tag{12.49}$$

The same argument leads to contractible spaces of isomorphisms

$$\begin{aligned} \mathrm{Fred}_0(H) \cong \mathrm{Fred}_{-2}(H) \cong \mathrm{Fred}_{-4}(H) \cong \cdots \\ \mathrm{Fred}_{-1}(H) \cong \mathrm{Fred}_{-3}(H) \cong \mathrm{Fred}_{-5}(H) \cong \cdots \end{aligned} \tag{12.50}$$

[3] We add another condition in the next lecture to get rid of contractible components.

It is natural to extend the definition

$$\mathrm{Fred}_n(H) \subset \mathrm{Fred}_0(\mathrm{C}\ell_n^{\mathbb{C}} \otimes H), \qquad n \in \mathbb{Z}, \tag{12.51}$$

to allow positive integers as well. Note that the generators e_i of the Clifford algebra are odd *self*-adjoint if $n > 0$. Then (12.50) extends in both directions:

$$\begin{gathered} \cdots \cong \mathrm{Fred}_2(H) \cong \mathrm{Fred}_0(H) \cong \mathrm{Fred}_{-2}(H) \cong \cdots \\ \cdots \cong \mathrm{Fred}_1(H) \cong \mathrm{Fred}_{-1}(H) \cong \cdots \end{gathered} \tag{12.52}$$

(12.53) *Atiyah-Singer loop map.* For each $n > 0$ define

$$\begin{aligned} \alpha\colon\ \mathrm{Fred}_{-n}(H) &\longrightarrow \Omega\,\mathrm{Fred}_{-(n-1)}(\mathrm{C}\ell_{-1}^{\mathbb{C}} \otimes H) \\ A &\longmapsto \bigl(t \mapsto e_n \cos \pi t + A \sin \pi t\bigr), \qquad 0 \leqslant t \leqslant 1. \end{aligned} \tag{12.54}$$

The Clifford algebra $\mathrm{C}\ell_{-1}^{\mathbb{C}}$ in the codomain has generator e_n. Note that the operators in the domain and codomain both act on the same Hilbert space $\mathrm{C}\ell_{-n}^{\mathbb{C}} \otimes H$. The codomain consists of paths from e_n to $-e_n$, the endpoints fixed independent of A. The space of such paths is homotopy equivalent to the based loop space with any basepoint. (Recall **(9.32)**.) The reader should check that $e_n \cos \pi t + A \sin \pi t$ is indeed Fredholm, and in fact is invertible if $t \neq 1/2$.

Theorem 12.55 ([AS3])**.** *α is a homotopy equivalence.*

Then algebraic Bott periodicity in the form Corollary 12.48 combines with Theorem 12.55 to prove Bott periodicity (Theorem 9.52).

Corollary 12.56. *There is a homotopy equivalence*

$$\Omega^2\,\mathrm{Fred}_0(H) \simeq \mathrm{Fred}_0(H) \tag{12.57}$$

We sketch a proof of Theorem 12.55 in the next few lectures.

Atiyah-Bott-Shapiro construction (Bonus material)

We did not have time in lecture for this important construction, which we explain in the next section is "adjoint" to the Atiyah-Singer loop map in some sense. Clifford modules implement the suspension in the definition (9.49) of the negative K-groups. The constructions work equally over $\mathbb{R}$ and $\mathbb{C}$, though our notation assumes the latter. The picture of suspension is (9.54), and the construction applies as well to the twisted suspension, or Thom complex, in (9.55).

(12.58) *A family of operators parametrized by a real vector space.* Let (V, Q) be a real quadratic vector space with Q negative definite and $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ a $\mathrm{C}\ell(V, Q)$-module. Then $\xi \in V$ determines an odd endomorphism $c(\xi) \in (\mathrm{End}\,\mathbb{S})^1$. Since Q is negative definite we can choose a compatible inner product on $\mathbb{S}$; then $c(\xi)$ is skew-adjoint. The family of operators $\xi \mapsto c(\xi)$ is supported at $0 \in V$, i.e., the operator $c(\xi)$ is invertible if $\xi \neq 0$. This defines an element in the relative K-theory group $K^0(V, V\backslash 0) \cong K^0(V, V\backslash B_r(0))$, as in **(9.33)**.

(12.59) *A vector bundle over the sphere.* Let $V \oplus \mathbb{R}$ have the direct sum inner product which is negative definite. The sphere $S(V \oplus \mathbb{R})$ is the 1-point compactification of V and is naturally decomposed as

$$S(V \oplus \mathbb{R}) = D^+ \cup_{S(V)} D^-, \qquad D^{\pm} = \{(\xi, t) \in S(V \oplus \mathbb{R}) : \pm t > 0\}. \tag{12.60}$$

We identify D^+ as the closed unit ball in V. Glue the trivial bundles $D^+ \times \mathbb{S}^0$ and $D^- \times \mathbb{S}^1$ using the isomorphisms $c(\xi)\colon \mathbb{S}^0 \to \mathbb{S}^1$ for $\xi \in S(V)$. This gives a vector bundle over $S(V \oplus \mathbb{R})$ whose K-theory class agrees with the class constructed in **(12.58)**. Note that the vector bundle comes trivialized on D^-, which is the 1-point compactification of the complement of the open unit ball in V.

(12.61) *The clutching function.* By Proposition 3.26 the K-theory class is determined by the homotopy class of the clutching function

$$\begin{aligned} S(V) &\longrightarrow \mathrm{Aut}(\mathbb{S}^0) \\ \xi &\longmapsto c(\xi_0)^{-1} c(\xi), \end{aligned} \tag{12.62}$$

where $\xi_0 \in S(V)$ is a basepoint.

Note that **(12.58)**, **(12.59)**, and **(12.61)** describe three geometric objects which represent the same information, all defined from a Clifford module.

Example 12.63. Take $V = \mathbb{R}^2$ and $\mathbb{S} = \mathbb{C}^{1|1}$ the complex Clifford module with

$$c(e_1) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \qquad c(e_2) = \begin{pmatrix} & i \\ i & \end{pmatrix}. \tag{12.64}$$

Then

$$c(\cos\theta\, e_1 + \sin\theta\, e_2) = \begin{pmatrix} & -e^{-i\theta} \\ e^{i\theta} & \end{pmatrix} \tag{12.65}$$

and the product with $c(e_1)^{-1}$ on the left is $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$. Restricted to $\mathbb{S}^0$ we get the clutching function for the hyperplane bundle over S^2.

(12.66) *Modding out by modules which extend.* Suppose $\mathbb{S}$ extends to a $C\ell(V \oplus \mathbb{R})$-module. Then we use the extra generator e_{n+1} to make a homotopy of clutching functions (12.62):

$$c(\xi_0)^{-1} \left[\cos \pi t/2\, c(\xi) + \sin \pi t/2\, c(e_{n+1})\right], \qquad 0 \leqslant t \leqslant 1. \tag{12.67}$$

This shows that the bundle we obtain over $S(V \oplus \mathbb{R})$ is trivializable. ABS [ABS, §11] use this to define a map from the graded ring of Clifford modules modulo those which extend to the reduced K-theory of the sphere—working both over the reals and the complexes—and they prove that this map is an isomorphism of graded rings. The Atiyah-Singer theorem [AS3] we are proving is a generalization.

(12.68) *Parametrized version; twisted K-theory.* Let $V \to X$ be a real vector bundle equipped with a family Q of negative definite quadratic forms. Then we obtain a bundle $C\ell(V,Q) \to X$ of Clifford algebras. Let $S \to X$ be a $C\ell(V,Q)$-module, that is, a super vector bundle with an action of $C\ell(V,Q)$ fiberwise. The previous constructions can be carried out fiber by fiber to construct a class in the Thom space $K^0(X^V)$. This can be identified with a K-theory class of degree $-n$ in the base, but in general it lives in *twisted* K-theory, a topic we will return to shortly.

Comparison of ABS and AS suspension maps (Bonus material)

Consider the ABS construction **(12.58)** in a special case. Suppose $E \to X$ is a complex vector bundle. The ABS construction yields a complex vector bundle $\underline{C\ell^{\mathbb{C}}_{-1}} \otimes pr_1^*E \to X \times \mathbb{R}$ equipped with the family of odd skew-adjoint operators which at $(x,s) \in X \times \mathbb{R}$ is Clifford multiplication multiplication by

$$se_1 \otimes \mathrm{id}_E\,. \tag{12.69}$$

The operator is invertible except at $s = 0$, when it is the zero operator. This family of operators is clutching data for a vector bundle over ΣX_+.

The AS map (12.54) is a sort of adjoint in the world of Fredholm operator representatives for K-theory. The relevant special case of (12.54) is

$$\begin{aligned} \alpha\colon\ \mathrm{Fred}_{-1}(H) &\longrightarrow \Omega\,\mathrm{Fred}_0(C\ell^{\mathbb{C}}_{-1} \otimes H) \\ A &\longmapsto \big(e_1 \cos \pi t + A \sin \pi t,\ 0 \leqslant t \leqslant 1\big). \end{aligned} \tag{12.70}$$

Write

$$e_1 \cos \pi t + A \sin \pi t = e_1(\cos \pi t - e_1 A \sin \pi t) \tag{12.71}$$

and note that $(e_1A)^* = A^*e_1^* = (-A)(-e_1) = Ae_1 = -e_1A$ is skew-adjoint. Since $\cos \pi t$ times the identity operator is invertible self-adjoint except at $t = 1/2$, it follows that the operator in (12.70) is invertible except at $t = 1/2$. At $t = 1/2$ it is the Fredholm operator A. Recalling that invertible operators are a "fat basepoint" in Fredholms, we can homotop the family (12.70) to a family of Fredholm operators

$$s \longmapsto se_1 \otimes \mathrm{id} + \mathrm{id} \otimes A, \qquad s \in \mathbb{R}, \tag{12.72}$$

which is more parallel to the ABS construction. It is in this sense that the AS loop map is adjoint to the ABS suspension.

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