Problem Set # 1

Math392C: K-theory

Please take a look at the readings on the website. There are leisurely expositions of basics on bundles in both Atiyah and Hatcher, for example. Look there for definitions that you may need to work these problems. Many of these problems are quite challenging.

Do work in groups and seek help and discussion wherever you can find it—including office hours!.

1. Recall the clutching construction for line bundles on S^2. This problem concerns the complex line bundle $L \to S^2$ of degree 1.

   (a) Construct a hermitian metric on L.

   (b) Let $P \subset L$ denote the subset of unit norm vectors. Prove that P is a manifold diffeomorphic to S^3.

   (c) The fibers of the projection $P \to S^2$ are circles. Show that any two distinct fibers are nontrivially linked in P. What is the linking number?

   (d) What about the line bundle of degree $n \in \mathbb{Z}$?

2. (a) Prove that any (finite rank) complex vector bundle over S^2 is isomorphic to a direct sum of line bundles.

   (b) Show that the corresponding statement for real vector bundles is false.

3. Write $S^n = D^n \cup D^n$ as the union of two closed balls which intersect along the equator S^{n-1}. Let $\mathrm{Vect}^{\cong}(X)$ denote the commutative monoid of isomorphism classes of complex vector bundles on a space X, and $\mathrm{Vect}_{\mathbb{R}}^{\cong}(X)$ the analogous commutative monoid in the real case.

   (a) Show that the set of isomorphism classes of real vector bundles of rank k is the set of components of double coset space

   $$\mathrm{Map}(D^n, O_k) \,\backslash\, \mathrm{Map}(S^{n-1}, O_k) \,/\, \mathrm{Map}(D^n, O_k).$$

   (b) Give generators and relations for the commutative monoids $\mathrm{Vect}^{\cong}(S^n)$, $\mathrm{Vect}_{\mathbb{R}}^{\cong}(S^n)$ for $n = 1, 2, 3$.

   (c) Compute the abelian group completions of the commutative monoids in (b).

   (d) Use your computation of $\mathrm{Vect}_{\mathbb{R}}^{\cong}(S^1)$ and homotopy invariance to prove that there is no retraction $r \colon D^2 \to S^1$ and then also the Brouwer fixed point theorem.

4. In this exercise you will construct the Grassmann manifold and tautological vector bundles over it. Let V be a real vector space of dimension $n \in \mathbb{Z}^{\geq 0}$. Define the Grassmannian $Gr_k(V)$, $0 \leq k \leq n$, to be the set of all k-dimensional subspaces of V.

   (a) Introduce a locally Euclidean topology on $Gr_k(V)$. Here is one way to do so: Suppose $W \in Gr_k(V)$ is a k-dimensional subspace and C an $(n-k)$-dimensional subspace such that $W \oplus C = V$. (We say that C is a complement to W in V.) Then define a subset $\mathcal{O}_{W,C} \subset Gr_k(V)$ by

   $$\mathcal{O}_{W,C} = \{W' \subset V : W' \text{ is the graph of a linear map } W \to C\}.$$

   Show that $\mathcal{O}_{W,C}$ is a vector space, so has a natural topology. Prove that it is consistent to define a subset $U \subset Gr_k(V)$ to be open if and only if $U \cap \mathcal{O}_{W,C}$ is open for all W, C. Note that $\{\mathcal{O}_{W,C}\}$ is a cover of $Gr_k(V)$. (For example, show that $W \in \mathcal{O}_{W,C}$.)

   (b) Use the open sets $\mathcal{O}_{W,C}$ to construct an atlas on $Gr_k(V)$. That is, check that the transition functions are smooth. (Hint: You may first want to check it for two charts with the same W but different complements. Then it suffices to check for two different W which are transverse, using the same complement for both.)

   (c) Now construct the complex Grassmannian: take V complex and use only complex subspaces.

   (d) Prove that $GL(V)$ acts smoothly and transitively on $Gr_k(V)$ in both the real and complex cases. What is the subgroup which fixes $W \in Gr_k(V)$?

   (e) Construct a short exact sequence of vector bundles

   $$0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0$$

   over $Gr_k(V)$ in which rank $S = k$, rank $Q = n - k$, and $\underline{V} = Gr_k(V) \times V$ is the bundle with constant fiber V.

5. If X is compact, prove that $K(X)$ is the free abelian group generated by vector bundles $E \to X$ (denoted '$[E]$') subject to the relation $[E] = [E''] + [E']$ if there exists a short exact sequence

   $$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

6. Let $E \to M$ be a smooth vector bundle over a smooth manifold. Construct a fiber bundle of affine spaces over M whose sections are precisely the covariant derivative operators $\nabla \colon \Omega_M^0(E) \to \Omega_M^1(E)$.