Problem Set # 2

M392C: Bordism Old and New

Due: November 6, 2012

1. (a) Recall from (14.4) in the lecture notes that a diffeomorphism $f\colon Y \to Y$ of a closed manifold Y determines a bordism X_f. Let f_0, f_1 be diffeomorphisms. Prove that X_{f_0} is diffeomorphic to X_{f_1} (as bordisms) if and only of f_0 is pseudoisotopic to f_1.

   (b) Find a manifold Y and diffeomorphisms $f_0, f_1\colon Y \to Y$ which are pseudoisotopic but not isotopic.

2. (a) Let S be a set with composition laws $\circ_1, \circ_2\colon S \times S \to S$ and distinguished element $1 \in S$. Assume (i) 1 is an identity for both $\circ_1$ and $\circ_2$; and (ii) for all $s_1, s_2, s_3, s_4 \in S$ we have

$$(s_1 \circ_1 s_2) \circ_2 (s_3 \circ_1 s_4) = (s_1 \circ_2 s_3) \circ_1 (s_2 \circ_2 s_4).$$

   Prove that $\circ_1 = \circ_2$ and that this common operation is commutative and associative.

   (b) Let C be a symmetric monoidal category. Apply (a) to $C(1,1)$, where $1 \in C$ is the tensor unit.

3. Let $y \in C$ be a dualizable object in a symmetric monoidal category, and suppose $(y^\vee, c, e)$ and $(\widetilde{y^\vee}, \tilde{c}, \tilde{e})$ are two sets of duality data. Prove there is a unique map $(y^\vee, c, e) \to (\widetilde{y^\vee}, \tilde{c}, \tilde{e})$.

4. For each of the following symmetric monoidal categories determine all of the dualizable objects.

   (a) $(\mathrm{Top}, \amalg)$, the category of topological spaces and continuous maps under disjoint union.

   (b) $(\mathrm{Ab}, \oplus)$, the category of abelian groups and homomorphisms under direct sum.

   (c) $(\mathrm{Mod}_R, \otimes)$, the category of R-modules and homomorphisms under tensor product, where R is a commutative ring.

   (d) $(\mathrm{Set}, \times)$, the category of sets and functions under Cartesian product.

5. Recall that every category C has an associated groupoid $|C|$ obtained from C by inverting all of the arrows. What is $|\mathrm{Bord}_{\langle 1,2\rangle}|$? $|\mathrm{Bord}^{\mathrm{Spin}}_{\langle 1,2\rangle}|$? What are all $\mathrm{Vect}_{\mathbb{C}}$-valued invertible topological quantum field theories with domain $\mathrm{Bord}_{\langle 1,2\rangle}$? $\mathrm{Bord}^{\mathrm{Spin}}_{\langle 1,2\rangle}$?

6. Fix a finite group G. Let C denote the groupoid $G/\!/G$ of G acting on itself by conjugation. Let D denote the groupoid of principal G-bundles over S^1. (A principal G-bundle is a regular, or Galois, cover with group G.) Prove that C and D are equivalent groupoids. You should spell out precisely what these groupoids are.

7. Explain why each of the following fails to be a natural map $\eta\colon F \to G$ of symmetric monoidal functors $F, G\colon C \to D$.

   (a) F, G are the identity functor on Vect_k for some field k, and $\eta(V)\colon V \to V$ is multiplication by 2 for each vector space V.

   (b) C, D are the category of algebras over a field k, the functor F maps $A \mapsto A \otimes A$, the functor G is the identity, and $\eta(A)\colon A \otimes A \to A$ is multiplication.

8. In this problem you construct a simple TQFT $F\colon \mathrm{Bord}_{\langle 0,1\rangle} \to \mathrm{Vect}_{\mathbb{Q}}$. For any manifold M let $\mathcal{C}(M)$ denote the groupoid of principal G-bundles over M, as in Problem 6.

   (a) For a compact 0-manifold Y, define $F(Y)$ as the vector space of functions $\mathcal{C}(Y) \to \mathbb{Q}$. Say what you mean by such functors on a groupoid.

   (b) For a *closed* 1-manifold X define

   $$F(X) = \sum_{[P] \in \pi_0 \mathcal{C}(X)} \frac{1}{\#\,\mathrm{Aut}(P)},$$

   where the sum is over equivalence classes of principal G-bundles. Extend this to all bordisms $X\colon Y_0 \to Y_1$.

   (c) Check that F is a symmetric monoidal functor.

   (d) Calculate F on a set of duality data for the point $\mathrm{pt} \in \mathrm{Bord}_{\langle 0,1\rangle}$. Use it to compute $F(S^1)$.

9. Fix a nonzero number $\lambda \in \mathbb{C}$. Construct an invertible TQFT $F\colon \mathrm{Bord}^{SO}_{\langle 1,2\rangle} \to \mathrm{Vect}_{\mathbb{C}}$ such that for a closed 2-manifold X we have $F(X) = \lambda^{\chi(X)}$, where $\chi(X)$ is the Euler characteristic. Can you extend to the bordism category $\mathrm{Bord}_{\langle 1,2\rangle}$?