

§16. Chern Numbers and Pontrjagin Numbers

In analogy with the Stiefel-Whitney numbers of a compact manifold, introduced on pp. 50-53, this section will introduce the Chern numbers of a compact complex manifold, and the Pontrjagin numbers of a compact oriented manifold. All manifolds are to be smooth.

Partitions

Recall from §6.6 that a *partition* of a non-negative integer k is an unordered sequence $I = i_1, \dots, i_r$ of positive integers with sum k . If $I = i_1, \dots, i_r$ is a partition of k and $J = j_1, \dots, j_s$ is a partition of ℓ , then the juxtaposition

$$IJ = i_1, \dots, i_r, j_1, \dots, j_s$$

is a partition of $k + \ell$. This composition operation is associative, commutative, and has as identity element the vacuous partition of zero which we denote by the empty symbol \emptyset . (In more technical language, the set of all partitions of all non-negative integers can be regarded as a free commutative monoid on the generators $1, 2, 3, \dots$.)

A partial ordering among partitions is defined as follows. A *refinement* of a partition i_1, \dots, i_r will mean any partition which can be written as a juxtaposition $I_1 \dots I_r$ where each I_j is a partition of i_j . If j_1, \dots, j_s is a refinement of i_1, \dots, i_r then it follows of course that $s \geq r$.

Chern Numbers

Let K^n be a compact complex manifold of complex dimension n . Then for each partition $I = i_1, \dots, i_r$ of n , the I -th *Chern number*

$$c_I[K^n] = c_{i_1} \dots c_{i_r}[K^n]$$

is defined to be the integer

$$\langle c_{i_1}(r^n) \dots c_{i_r}(r^n), \mu_{2n} \rangle .$$

Here r^n denotes the tangent bundle of K^n , and μ_{2n} denotes the fundamental homology class determined by the preferred orientation. We adopt the convention that $c_I[K^n]$ is zero if I is a partition of some integer other than n .

As an example, for the complex projective space $P^n(\mathbb{C})$, since $c_i(r^n) = \binom{n+1}{i} a^i$ and $\langle a^n, \mu_{2n} \rangle = +1$ by §14.10, we have the formula

$$c_{i_1} \dots c_{i_r} [P^n(\mathbb{C})] = \binom{n+1}{i_1} \dots \binom{n+1}{i_r}$$

for any partition i_1, \dots, i_r of n .

A complex 1-dimensional manifold K^1 has just one Chern number, namely the Euler characteristic $c_1[K^1]$. For a complex 2-manifold there are two Chern numbers, namely $c_1 c_1[K^2]$ and the Euler characteristic $c_2[K^2]$. In general, a complex n -manifold has $p(n)$ Chern numbers, where $p(n)$ is the number of distinct partitions of n . (Compare p. 80.) We will see in 16.7 that these $p(n)$ Chern numbers are linearly independent; that is there is no linear relation between them which is satisfied for *all* complex n -manifolds.

There is another way of thinking about Chern classes which is important for many purposes. Note that the cohomology group $H^{2n}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ is free abelian of rank $p(n)$. The products $c_{i_1}(r^n) \dots c_{i_r}(r^n)$, where i_1, \dots, i_r ranges over all partitions of n , form a basis for this group. For any complex manifold K^n the tangent bundle r^n is "classified" by a map

$$f : K^n \rightarrow G_n(\mathbb{C}^\infty)$$

with $f^*(r^n) \cong r^n$. Using this classifying map f , the fundamental homology class μ_{2n} of K^n gives rise to a homology class $f_*(\mu_{2n})$ in the free abelian group $H_{2n}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ of rank $p(n)$. To identify this homology class $f_*(\mu_{2n})$, we need only compute the $p(n)$ Kronecker indices

$$\langle c_{i_1}(y^n) \dots c_{i_r}(y^n), f_*(\mu_{2n}) \rangle ,$$

since the products $c_{i_1}(y^n) \dots c_{i_r}(y^n)$ range over a basis for the corresponding cohomology group. But each such Kronecker index is equal to the Chern number

$$\langle f^*(c_{i_1}(y^n) \dots c_{i_r}(y^n)), \mu_{2n} \rangle = c_{i_1} \dots c_{i_r} [K^n] .$$

We see from this approach that it is not necessary to use the basis $\{c_{i_1}(y^n) \dots c_{i_r}(y^n)\}$ for $H^{2n}(G_n(\mathbb{C}^\infty); \mathbb{Z})$. Any other basis would serve equally well. Later we will make use of a quite different basis for this group.

Pontrjagin Numbers

Now consider a smooth, compact, oriented manifold M^{4n} . For each partition $I = i_1, \dots, i_r$ of n , the I -th *Pontrjagin number* $p_I[M^{4n}] = p_{i_1} \dots p_{i_r} [M^{4n}]$ is defined to be the integer

$$\langle p_{i_1}(\tau^{4n}) \dots p_{i_r}(\tau^{4n}), \mu_{4n} \rangle .$$

Here τ^{4n} denotes the tangent bundle and μ_{4n} the fundamental homology class.

As an example, the complex projective space $P^{2n}(\mathbb{C})$, with its complex structure forgotten, is a compact oriented manifold of real dimension $4n$. The Pontrjagin numbers of this manifold are given by the formula

$$p_{i_1} \dots p_{i_r} [P^{2n}(\mathbb{C})] = \binom{2n+1}{i_1} \dots \binom{2n+1}{i_r} ,$$

as one easily verifies using 15.6.

If we reverse the orientation of a manifold M^{4n} , note that its Pontrjagin classes remain unchanged, but its fundamental homology class μ_{4n} changes sign. Hence each Pontrjagin number

$$p_{i_1} \dots p_{i_r} [M^{4n}] = \langle p_{i_1} \dots p_{i_r}, \mu_{4n} \rangle$$

also changes sign. Thus if some Pontrjagin number $p_{i_1} \dots p_{i_r} [M^{4n}]$ is non-zero, then it follows that M^{4n} cannot possess any orientation reversing diffeomorphism.

As an example, the complex projective space $P^{2n}(\mathbb{C})$ does not possess any orientation reversing diffeomorphism. (On the other hand, $P^{2n+1}(\mathbb{C})$ does have an orientation reversing diffeomorphism, arising from complex conjugation.)

This behavior of Pontrjagin numbers is in contrast to the behavior of the Euler number $e[M^{2n}]$ which is invariant under change of orientation. In fact the manifold S^{2n} , with $e[S^{2n}] \neq 0$, certainly does admit an orientation reversing diffeomorphism.

Furthermore, if some Pontrjagin number $p_{i_1} \dots p_{i_r} [M^{4n}]$ is non-zero then, proceeding as in §4.9, we see that M^{4n} cannot be the boundary of any smooth, compact, oriented $(4n+1)$ -dimensional manifold with boundary. (Compare §17.) For example, the projective space $P^{2n}(\mathbb{C})$ cannot be an oriented boundary. In fact the disjoint union $P^{2n}(\mathbb{C}) + \dots + P^{2n}(\mathbb{C})$ of any number of copies of $P^{2n}(\mathbb{C})$ cannot be an oriented boundary, since the l -th Pontrjagin number of such a k -fold union is clearly just k times the l -th Pontrjagin number of $P^{2n}(\mathbb{C})$ itself. Again this argument does not work for $P^{2n+1}(\mathbb{C})$. (In fact $P^{2n+1}(\mathbb{C})$ is the total space of a circle-bundle over a quaternion projective space, and hence is the boundary of an associated disk-bundle.)

Again the corresponding statement for Euler numbers is also false. Thus $e[S^{2n}] \neq 0$ even though S^{2n} clearly bounds an oriented manifold. All of these remarks are due to Pontrjagin.

Symmetric Functions

The following classical algebraic techniques will enable us to define and manipulate certain useful linear combinations of Chern numbers or Pontrjagin numbers.

Let t_1, \dots, t_n be indeterminates. A polynomial function $f(t_1, \dots, t_n)$, say with integer coefficients, is called a *symmetric function* if it is invariant under all permutations of t_1, \dots, t_n . Thus the symmetric functions form a sub-ring

$$\mathfrak{S} \subset \mathbb{Z}[t_1, \dots, t_n] .$$

A familiar and fundamental theorem asserts that \mathfrak{S} itself is also a polynomial ring on n algebraically independent generators,

$$\mathfrak{S} = \mathbb{Z}[\sigma_1, \dots, \sigma_n] ,$$

where $\sigma_k = \sigma_k(t_1, \dots, t_n)$ denotes the k -th *elementary symmetric function*, uniquely characterized by the fact that σ_k is a homogeneous polynomial of degree k in t_1, \dots, t_n with

$$1 + \sigma_1 + \sigma_2 + \dots + \sigma_n = (1+t_1)(1+t_2) \dots (1+t_n) .$$

(Compare p. 84.)

If we make $\mathbb{Z}[t_1, \dots, t_n]$ into a graded ring by assigning each t_i the degree 1, then of course the symmetric functions form a graded subring $\mathfrak{S}^* = \mathbb{Z}[\sigma_1, \dots, \sigma_n]$, where each σ_k has degree k . Thus a basis for the additive group \mathfrak{S}^k , consisting of homogeneous symmetric polynomials of degree k in t_1, \dots, t_n , is given by the set of monomials

$$\sigma_{i_1} \dots \sigma_{i_r}$$

where i_1, \dots, i_r ranges over all partitions of k into integers $\leq n$.

A different and quite useful basis can be constructed as follows. Define two monomials in t_1, \dots, t_n to be *equivalent* if some permutation of t_1, \dots, t_n transforms one into the other. Define $\sum t_1^{a_1} \dots t_r^{a_r}$ to be the summation of all monomials in t_1, \dots, t_n which are equivalent to $t_1^{a_1} \dots t_r^{a_r}$. As an example, using this notation we can write $\sigma_k = \sum t_1 t_2 \dots t_k$.

LEMMA 16.1. *An additive basis for \mathfrak{S}^k , the group of homogeneous symmetric polynomials of degree k in t_1, \dots, t_n , is given by the polynomials $\sum t_1^{a_1} \dots t_r^{a_r}$. Here a_1, \dots, a_r ranges over all partitions of k with length $r \leq n$.*

The proof is not difficult. ■

Now for any partition $I = i_1, \dots, i_r$ of k , define a polynomial s_I in k variables as follows. Choose $n \geq k$ so that the elementary symmetric functions $\sigma_1, \dots, \sigma_k$ of t_1, \dots, t_n are algebraically independent, and let $s_I = s_{i_1, \dots, i_r}$ be the unique polynomial satisfying

$$s_I(\sigma_1, \dots, \sigma_k) = \sum t_1^{i_1} \dots t_r^{i_r}.$$

This polynomial does not depend on n , as one easily verifies by introducing additional variables $t_{n+1} = \dots = t_n' = 0$. In fact, even if $n < k$ the corresponding identity

$$s_I(\sigma_1, \dots, \sigma_n, 0, \dots, 0) = \sum t_1^{i_1} \dots t_r^{i_r}$$

remains valid, as one verifies by a similar argument.

If $n \geq k$, then evidently the $p(k)$ polynomials $s_I(\sigma_1, \dots, \sigma_k)$ are linearly independent, and form a basis for \mathfrak{S}^k . The first twelve such polynomials are given by

$$\begin{aligned} s(\) &= 1, \\ s_1(\sigma_1) &= \sigma_1, \\ \begin{cases} s_2(\sigma_1, \sigma_2) &= \sigma_1^2 - 2\sigma_2 \\ s_{1,1}(\sigma_1, \sigma_2) &= \sigma_2, \end{cases} \\ \begin{cases} s_3(\sigma_1, \sigma_2, \sigma_3) &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ s_{1,2}(\sigma_1, \sigma_2, \sigma_3) &= \sigma_1\sigma_2 - 3\sigma_3 \end{cases} \\ \begin{cases} s_{1,1,1}(\sigma_1, \sigma_2, \sigma_3) &= \sigma_3, \end{cases} \end{aligned}$$

$$\begin{cases}
 s_4 & = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4 \\
 s_{1,3} & = \sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 + 4\sigma_4 \\
 s_{2,2} & = \sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4 \\
 s_{1,1,2} & = \sigma_1\sigma_3 - 4\sigma_4 \\
 s_{1,1,1,1} & = \sigma_4 .
 \end{cases}$$

For further information see Problem 16-A, as well as [van der Waerden, Chapter 26] particularly the exercises, and [Macmahon].

The application of these ideas to Chern classes or Pontrjagin classes is very similar to the application to Stiefel-Whitney classes in §7. Thus if a complex n -plane bundle ω splits as a Whitney sum $\eta_1 \oplus \dots \oplus \eta_n$ of line bundles, then the formula

$$1 + c_1(\omega) + \dots + c_n(\omega) = (1+c_1(\eta_1)) \dots (1+c_1(\eta_n))$$

shows that the Chern class $c_k(\omega)$ can be identified with the k -th elementary symmetric function $\sigma_k(c_1(\eta_1), \dots, c_1(\eta_n))$. The “universal” example of a Whitney sum of line bundles is provided by the n -fold cartesian product $\gamma^1 \times \dots \times \gamma^1$ over the product $P^\infty(\mathbb{C}) \times \dots \times P^\infty(\mathbb{C})$ of complex projective spaces. Note that the cohomology ring of this product is a polynomial ring $\mathbb{Z}[a_1, \dots, a_n]$ where each a_i has degree 2, and where

$$c(\gamma^1 \times \dots \times \gamma^1) = (1+a_1) \dots (1+a_n) .$$

Since the elementary symmetric functions are algebraically independent, it follows that the cohomology $H^*(G_n(\mathbb{C}^\infty); \mathbb{Z})$ of the classifying space maps isomorphically to the ring

$$\mathcal{S}^* \subset \mathbb{Z}[a_1, \dots, a_n]$$

of symmetric polynomials. (This is a theorem of [Borel, 1953]. Compare p. 84.) Thus our new basis for \mathcal{S}^* gives rise to a new basis

$$\{s_I(c_1, \dots, c_k)\}$$

for the cohomology $H^{2k}(G_n(\mathbb{C}^\infty); \mathbb{Z})$.

A Product Formula

Let ω be a complex n -plane bundle with base space B and with total Chern class $c = 1 + c_1 + \dots + c_n$. For any $k \geq 0$ and any partition I of k the cohomology class

$$s_I(c_1, \dots, c_k) \in H^{2k}(B; \mathbb{Z})$$

will be denoted briefly by the symbol $s_I(c)$ or $s_I(c(\omega))$.

LEMMA 16.2 (Thom). *The characteristic class $s_I(c(\omega \oplus \omega'))$ of a Whitney sum is equal to*

$$\sum_{JK=I} s_J(c(\omega)) s_K(c(\omega')) ,$$

to be summed over all partitions J and K with juxtaposition JK equal to I .

As an example, since the single element partition of k can be expressed as a juxtaposition only in two trivial ways, we obtain the following.

COROLLARY 16.3. *The characteristic class $s_k(c(\omega \oplus \omega'))$ of a Whitney sum is equal to $s_k(c(\omega)) + s_k(c(\omega'))$.*

Proof of 16.2. Consider a polynomial ring $\mathbb{Z}[t_1, \dots, t_{2n}]$ in $2n$ indeterminates, and let σ_k [respectively σ'_k] be the k -th elementary symmetric function of the indeterminates t_1, \dots, t_n [respectively t_{n+1}, \dots, t_{2n}]. Then defining

$$\sigma''_k = \sum_{i=0}^k \sigma_i \sigma'_{k-i} ,$$

it is clear that σ''_k is equal to the k -th elementary symmetric function of t_1, \dots, t_{2n} . We will verify the identity

$$s_I(\sigma''_1, \dots, \sigma''_k) = \sum_{JK=I} s_J(\sigma_1, \sigma_2, \dots) s_K(\sigma'_1, \sigma'_2, \dots)$$

for any partition $I = i_1, \dots, i_r$ of k . Since the classes $\sigma_1, \dots, \sigma_k, \sigma'_1, \dots, \sigma'_k$ are algebraically independent (assuming as we may that $k \leq n$), this identity together with the product theorem for Chern classes will clearly complete the proof.

By definition, the element

$$s_I(\sigma''_1, \dots, \sigma''_k) \in \mathbb{Z}[t_1, \dots, t_{2n}]$$

is equal to the sum of all monomials which can be written in the form $t_{a_1}^{i_1} \dots t_{a_r}^{i_r}$, with a_1, \dots, a_r distinct numbers between 1 and $2n$. For each such monomial let J [respectively K] be the partition formed by those exponents i_q such that $1 \leq a_q \leq n$ [respectively $n + 1 \leq a_q \leq 2n$]. The sum of all terms corresponding to a given decomposition $JK = I$ is clearly equal to

$$s_J(\sigma_1, \sigma_2, \dots) s_K(\sigma'_1, \sigma'_2, \dots) .$$

Since every such decomposition occurs, this completes the proof. ■

Now consider a compact complex manifold K^n of complex dimension n . For each partition I of n the notation $s_I(c)[K^n]$, or briefly $s_I[K^n]$, will stand for the characteristic number

$$\langle s_I(c(\tau^n)), \mu_{2n} \rangle \in \mathbb{Z} .$$

This characteristic number is of course equal to a suitable linear combination of Chern numbers.

COROLLARY 16.4. *The characteristic number $s_I[K^m \times L^n]$ of a product of complex manifolds is equal to*

$$\sum_{I_1 I_2 = I} s_{I_1} [K^m] s_{I_2} [L^n] ,$$

to be summed over all partitions I_1 of m and I_2 of n with juxtaposition $I_1 I_2$ equal to I .

For the tangent bundle of $K^m \times L^n$ splits as a Whitney sum

$$\tau \times \tau' \cong (\pi_1^* \tau) \oplus (\pi_2^* \tau')$$

where π_1 and π_2 are the projection maps to the two factors. Hence the characteristic number

$$\langle s_I(\tau \times \tau'), \mu_{2m} \times \mu'_{2n} \rangle$$

is equal to

$$\sum_{I_1 I_2 = I} \langle s_{I_1}(\tau), \mu_{2m} \rangle \langle s_{I_2}(\tau'), \mu'_{2n} \rangle .$$

There are no signs in this formula, since these classes are all even dimensional. ■

As a special case, we clearly have the following.

COROLLARY 16.5. *For any product $K^m \times L^n$ of complex manifolds of dimensions $m, n \neq 0$, the characteristic number $s_{m+n}[K^m \times L^n]$ is zero.*

This corollary suggests the importance of the characteristic number $s_m[K^m]$. Here is an example to show that this characteristic number is not always zero.

Example 16.6. For the complex projective space $P^n(\mathbb{C})$, since $c(\tau) = (1+a)^{n+1}$ it follows that $c_k(\tau)$ is equal to the k -th elementary symmetric function of $n+1$ copies of a . Therefore $s_k(c_1, \dots, c_k)$ is equal to the sum of $n+1$ copies of a^k , that is

$$s_k = (n+1)a^k .$$

Taking $k = n$, it follows that

$$s_n[P^n(\mathbb{C})] = n + 1 \neq 0 .$$

Thus $P^n(\mathbb{C})$ cannot be expressed non-trivially as a product of complex manifolds.

Completely analogous formulas are true for Pontrjagin classes and Pontrjagin numbers. If ξ is a real vector bundle over B , then for any partition I of n the characteristic class

$$s_I(p_1(\xi), \dots, p_n(\xi)) \in H^{4n}(B; \mathbb{Z})$$

is denoted briefly by $s_I(p(\xi))$. The congruence

$$s_I(p(\xi \oplus \xi')) \equiv \sum_{JK=I} s_J(p(\xi)) s_K(p(\xi'))$$

modulo elements of order 2 clearly follows from the proof of 16.2. Hence there is a corresponding equality

$$s_I(p)[M \times N] = \sum_{JK=I} s_J(p)[M] s_K(p)[N]$$

for characteristic numbers. In particular, these characteristic numbers of $M \times N$ are zero unless the dimensions of M and N are divisible by 4.

Linear Independence of Chern Numbers and of Pontrjagin Numbers

The following basic result shows that there are no linear relations between Chern numbers.

THEOREM 16.7 (Thom). *Let K^1, \dots, K^n be complex manifolds with $s_k(c)[K^k] \neq 0$. Then the $p(n) \times p(n)$ matrix*

$$\left[c_{i_1} \dots c_{i_r} [K^{j_1} \times \dots \times K^{j_s}] \right] ,$$

of Chern numbers, where i_1, \dots, i_r and j_1, \dots, j_s range over all partitions of n , is non-singular.

For example, by 16.6, we can take $K^r = P^r(\mathbb{C})$. Similarly:

THEOREM 16.8 (Thom). *If M^4, \dots, M^{4n} are oriented manifolds with $s_k(p)[M^{4k}] \neq 0$, then the $p(n) \times p(n)$ matrix*

$$\left[p_{i_1} \dots p_{i_r} [M^{4j_1} \times \dots \times M^{4j_s}] \right]$$

of Pontrjagin numbers is non-singular.

Again we can take the complex projective space $P^{2k}(\mathbb{C})$, with $p(r) = (1+a^2)^{2k+1}$ and hence

$$s_k(p)[P^{2k}(\mathbb{C})] = 2k + 1 ,$$

as a suitable manifold M^{4k} .

Here is an example. For complex dimension 2 taking $K^n = P^n(\mathbb{C})$ we obtain the matrix

$$\begin{bmatrix} c_1 c_1 [K^1 \times K^1] = 8 & c_1 c_1 [K^2] = 9 \\ c_2 [K^1 \times K^1] = 4 & c_2 [K^2] = 3 \end{bmatrix}$$

of Chern numbers, with determinant -12 . Evidently the direct approach of simply computing the matrix will not help much in the general case.

Proof of 16.7. In place of the Chern numbers themselves, we may use the linear combinations $s_I(c)$. As an immediate generalization of 16.4 we have

$$s_I[K^{j_1} \times \dots \times K^{j_q}] = \sum_{I_1 \dots I_q = I} s_{I_1}[K^{j_1}] \dots s_{I_q}[K^{j_q}] ,$$

to be summed over all partitions I_1 of j_1, I_2 of j_2, \dots , and I_q of j_q with juxtaposition $I_1 \dots I_q$ equal to I . Thus the characteristic number

$s_I[K^{j_1} \times \dots \times K^{j_q}]$ is zero unless the partition $I = i_1, \dots, i_r$ is a refinement of j_1, \dots, j_q . In particular it is zero unless $r \geq q$. Thus if the partitions i_1, \dots, i_r and j_1, \dots, j_q are arranged in a suitably chosen order, then the matrix

$$\left[s_{i_1, \dots, i_r} [K^{j_1} \times \dots \times K^{j_q}] \right]$$

will be triangular, with zeros everywhere above the diagonal. Each diagonal entry $s_{i_1, \dots, i_r} [K^{i_1} \times \dots \times K^{i_r}]$ is clearly equal to the product

$$s_{i_1} [K^{i_1}] \dots s_{i_r} [K^{i_r}] \neq 0 .$$

Hence the matrix is non-singular. The proof of 16.8 is completely analogous. ■

Here are some problems for the reader.

Problem 16-A. Substituting $-t_i$ for x in the identity $(x+t_1)\dots(x+t_n) = x^n + \sigma_1 x^{n-1} + \dots + \sigma_n$ and then summing over i , prove Newton's formula

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} - \dots + \sigma_{n-1} s_1 \pm n\sigma_n = 0 .$$

This formula can be used inductively to compute the polynomial $s_n(\sigma_1, \dots, \sigma_n)$. Alternatively, taking the logarithm of both sides of the identity $(1+t_1)\dots(1+t_n) = 1 + (\sigma_1 + \dots + \sigma_n)$, prove Girard's formula

$$(-1)^k s_k / k = \sum_{i_1+2i_2+\dots+ki_k=k} (-1)^{i_1+\dots+i_k} \frac{(i_1+\dots+i_k-1)!}{i_1! \dots i_k!} \sigma_1^{i_1} \dots \sigma_k^{i_k} .$$

Problem 16-B. The Chern character $ch(\omega)$ of a complex n -plane bundle ω is defined to be the formal sum

$$n + \sum_{k=1}^{\infty} s_k(c(\omega)) / k! \in H^{\Pi}(B; \mathbb{Q}) .$$

Show that this Chern character is characterized by additivity

$$\text{ch}(\omega \oplus \omega') = \text{ch}(\omega) + \text{ch}(\omega') ,$$

together with the property that $\text{ch}(\eta^1)$ is equal to the formal power series $\exp(c_1(\eta^1))$ for any line bundle η^1 . Show that the Chern character is also multiplicative:

$$\text{ch}(\omega \otimes \omega') = \text{ch}(\omega) \text{ch}(\omega') .$$

(As in Problem 7-C, it suffices to consider first the case of two line bundles.)

Problem 16-C. If $2i_1, \dots, 2i_r$ is a partition of $2k$ into even integers, show that the $4k$ -dimensional characteristic class $s_{2i_1, \dots, 2i_r}(c(\omega))$ of a complex vector bundle is equal to the characteristic class $s_{i_1, \dots, i_r}(p(\omega_R))$ of its underlying real vector bundle. As examples, show that the $4k$ -dimensional class $s_{2, \dots, 2}(c(\omega))$ is equal to $p_k(\omega_R)$, and show that the characteristic number $s_{2n}(c)[K^{2n}]$ of a complex $2n$ -manifold is equal to $s_n(p)[K^{2n}]$.

Problem 16-D. If the complex manifold K^n is complex analytically embedded in K^{n+1} with dual cohomology class $u \in H^2(K^{n+1}, \mathbb{Z})$, show that the total tangential Chern class $c(K^n)$ is equal to the restriction to K^n of $c(K^{n+1})/(1+u)$. For any cohomology class $x \in H^{2n}(K^{n+1}; \mathbb{Z})$ show that the Kronecker index $\langle x | K^n, \mu_{2n} \rangle$ is equal to $\langle xu, \mu_{2n+2} \rangle$. (Compare p. 120 as well as Problem 11-C.) Using these constructions, compute $c(K^n)$ for a non-singular algebraic hypersurface K^n of degree d in $P^{n+1}(\mathbb{C})$, and prove that the characteristic number $s_n[K^n]$ is equal to $d(n+2-d^n)$. (An algebraic hypersurface of degree d is the set of zeroes of a homogeneous polynomial of degree d .)

Problem 16-E. Similarly, if $H_{m,n}$ is a non-singular hypersurface of degree $(1, 1)$ in the product $P^m(\mathbb{C}) \times P^n(\mathbb{C})$ of complex projective spaces, with $m, n \geq 2$, prove that the characteristic number $s_{m+n-1}[H_{m,n}]$ is equal to $-(m+n)!/m!n!$. Using disjoint unions of hypersurfaces, prove

that for each dimension n there exists a complex manifold K^n with $s_n[K^n] = p$ if $n + 1$ is a power of the prime p , or with $s_n[K^n] = 1$ if $n + 1$ is not a prime power. (A theorem of Milnor and Novikov asserts that these manifolds K^1, K^2, K^3, \dots freely generate the ring consisting of all "cobordism classes" of manifolds with a complex structure on the stable tangent bundle $\tau \oplus \varepsilon^k$. Compare [Stong].)

Problem 16-F. Develop a corresponding calculus of mod 2 characteristic numbers $s_I(w_1, \dots, w_n)[M^n]$, where I ranges over partitions of n . Using real algebraic hypersurfaces of degree $(1, 1)$ in a product of real projective spaces, prove that there exists a manifold Y^n with $s_n(w)[Y^n] \neq 0$ whenever $n + 1$ is not a power of 2. For n odd show that Y^n is orientable. As in Problem 4-E, let \mathcal{N}_n be the $\mathbb{Z}/2$ vector space consisting of cobordism classes of unoriented n -manifolds. Show that the products $Y^{i_1} \times \dots \times Y^{i_r}$, where i_1, \dots, i_r ranges over all partitions of n into integers not of the form $2^k - 1$, are linearly independent in \mathcal{N}_n . (A theorem of Thom asserts that these products actually form a basis for \mathcal{N}_n , so that the cobordism algebra \mathcal{N}_* is a polynomial algebra freely generated by the manifolds $Y^2, Y^4, Y^5, Y^6, Y^8, \dots$.

§17. The Oriented Cobordism Ring Ω_*

In the next two sections we will define and study the Thom cobordism ring Ω_* . This section contains the basic definition and some preliminary results. For a fuller treatment of cobordism theory, the reader is referred to [Stong].

Smooth Manifolds-with-Boundary

Let us first give a precise definition of this concept, which has already been used briefly in §4 and §16. As a universal model for manifolds-with-boundary, we take the closed half-space H^n , consisting of all points (x_1, \dots, x_n) in the Euclidean space R^n with $x_1 \geq 0$. A subset $X \subset R^A$ is called a *smooth n-dimensional manifold-with-boundary* if, for each point $x \in X$, there exists a smooth mapping

$$h : U \rightarrow R^A$$

which maps some relatively open set $U \subset H^n$ homeomorphically onto a neighborhood of x in X , and for which the matrix of first derivatives $[\partial h_\alpha / \partial u_j]$ has rank n everywhere. (Compare p. 4.)

A point x of X is called an *interior point* if there exists such a local parametrization $h : U \rightarrow R^A$ of X about x such that U is an open subset of R^n (rather than H^n). Evidently the set of interior points forms a smooth n -dimensional manifold which is open as a subset of X . The non-interior points form a smooth $(n-1)$ -dimensional manifold, called the *boundary* ∂X , which is closed as a subset of X .

The tangent bundle τ^n of a smooth manifold-with-boundary X is a smooth n -plane bundle over X . The definition is completely analogous to that on pp. 6, 14. This n -plane bundle has some additional structure

which can be described as follows. If x is a boundary point of X , then the fiber DX_x contains an $(n-1)$ -dimensional subspace $D(\partial X)_x$ consisting of vectors which are tangent to the boundary. This hyperplane $D(\partial X)_x$ separates the tangent space DX_x into two open subsets, consisting respectively of vectors which point "into" or "out of" X . By definition a vector $v \in DX_x$, with $v \notin D(\partial X)_x$, points *into* X if v is the velocity vector $(dp/dt)_{t=0}$ of a smooth path

$$p : [0, \varepsilon) \rightarrow X$$

with $p(0) = x$. Similarly v points *out of* X if v is the velocity vector at $t = 0$ of a path $p : (-\varepsilon, 0] \rightarrow X$ with $p(0) = x$.

Now suppose that the tangent bundle τ^n of X is an oriented n -plane bundle. Then the tangent bundle τ^{n-1} of ∂X has an induced orientation as follows. Choose an oriented basis v_1, \dots, v_n for DX_x at any boundary point x so that v_1 points out of X and v_2, \dots, v_n are tangent to ∂X . Then the ordered basis v_2, \dots, v_n determines the required orientation for $D(\partial X)_x$.

[In the special case of a 1-dimensional manifold-with-boundary, this construction must be modified slightly as follows. An "orientation" of a point x of the 0-dimensional manifold ∂X is just a choice of sign $+1$ or -1 . In fact we assign x the orientation $+1$ or -1 according as the positive direction in DX_x points out of or into X .]

We will need the following statement.

COLLAR NEIGHBORHOOD THEOREM 17.1. *If X is a smooth paracompact manifold-with-boundary, then there exists an open neighborhood of ∂X in X which is diffeomorphic to the product $\partial X \times [0, 1)$.*

The proof is similar to that of Theorem 11.1. (Just as for 11.1, we will actually need this assertion only in the special case where ∂X is compact.) Details will be left to the reader. ■

Oriented Cobordism

If M is a smooth oriented manifold, then the notation $-M$ will be used for the same manifold with opposite orientation. The symbol $+$ will be used for the disjoint union (also called topological sum) of smooth manifolds.

DEFINITION. Two smooth compact oriented n -dimensional manifolds M and M' are said to be *oriented cobordant*, or to belong to the same *oriented cobordism class*, if there exists a smooth, compact, oriented manifold-with-boundary X so that ∂X with its induced orientation is diffeomorphic to $M + (-M')$ under an orientation preserving diffeomorphism.

LEMMA 17.2. *This relation of oriented cobordism is reflexive, symmetric, and transitive.*

Indeed, the disjoint union $M + (-M)$ is certainly diffeomorphic to the boundary of $[0, 1] \times M$ under an orientation preserving diffeomorphism. Furthermore, if $M + (-M') \cong \partial X$, then clearly $M' + (-M) \cong \partial(-X)$. Finally, if $M + (-M') \cong \partial X$ and $M' + (-M'') \cong \partial Y$, then using 17.1 the smoothness structures and the orientations of X and Y can be pieced together along the common boundary M' so as to yield a new smooth oriented manifold-with-boundary bounded by $M + (-M'')$. Details will be left to the reader. ■

Now the set Ω_n consisting of all oriented cobordism classes of n -dimensional manifolds clearly forms an abelian group, using the disjoint union $+$ as composition operation. The zero element of the group is the cobordism class of the vacuous manifold.

Furthermore the cartesian product operation $M_1^m, M_2^n \mapsto M_1^m \times M_2^n$ gives rise to an associative, bilinear product operation

$$\Omega_m \times \Omega_n \rightarrow \Omega_{m+n} .$$

Thus the sequence

$$\Omega_* = (\Omega_0, \Omega_1, \Omega_2, \dots)$$

of oriented cobordism groups has the structure of a graded ring. This ring possesses a 2-sided identity element $1 \in \Omega_0$. Furthermore, it is easily verified that $M_1^m \times M_2^n$ is isomorphic as oriented manifold to $(-1)^{mn} M_2^n \times M_1^m$. Thus this oriented cobordism ring is commutative in the graded sense.

Pontrjagin numbers provide a basic tool for studying these cobordism groups. As already pointed out in §16, we have the following statement.

LEMMA 17.3 (Pontrjagin). *If M^{4k} is the boundary of a smooth, compact, oriented $(4k+1)$ -dimensional manifold-with-boundary, then every Pontrjagin number $p_{i_1} \dots p_{i_r} [M^{4k}]$ is zero.*

Since the identity $p_I[M_1 + M_2] = p_I[M_1] + p_I[M_2]$ is clearly satisfied, this proves the following.

COROLLARY 17.4. *For any partition $I = i_1, \dots, i_r$ of k , the correspondence $M^{4k} \mapsto p_I[M^{4k}]$ gives rise to a homomorphism from the cobordism group Ω_{4k} to \mathbb{Z} .*

Now by 16.8 we obtain the following.

COROLLARY 17.5. *The products $P^{2i_1}(\mathbb{C}) \times \dots \times P^{2i_r}(\mathbb{C})$, where i_1, \dots, i_r ranges over all partitions of k , represent linearly independent elements of the cobordism group Ω_{4k} . Hence Ω_{4k} has rank greater than or equal to $p(k)$, the number of partitions of k .*

Following Thom, we will prove in §18 that the rank is precisely $p(k)$.

To conclude this section, we list without proof the actual structures of the first few oriented cobordism groups. (Compare [Wall, 1960, p. 309].)

$\Omega_0 \cong \mathbb{Z}$. In fact a compact oriented 0-manifold is just a finite set of signed points, and the sum of the signs is a complete cobordism invariant.

$\Omega_1 = 0$, since every compact 1-manifold clearly bounds.

$\Omega_2 = 0$, since a compact *oriented* 2-manifold bounds.

$\Omega_3 = 0$. In contrast to the lower dimensional cases, this assertion, first announced by [Rohlin], is non-trivial. To our knowledge it has never been proved directly.

$\Omega_4 \cong \mathbb{Z}$, generated by the complex projective plane $P^2(\mathbb{C})$.

$\Omega_5 \cong \mathbb{Z}/2$, generated by the manifold Y^5 of Problem 16-F.

$\Omega_6 = 0$.

$\Omega_7 = 0$.

$\Omega_8 \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by $P^4(\mathbb{C})$ and $P^2(\mathbb{C}) \times P^2(\mathbb{C})$.

$\Omega_9 \cong (\mathbb{Z}/2) \oplus (\mathbb{Z}/2)$, generated by Y^9 and the product $Y^5 \times P^2(\mathbb{C})$.

$\Omega_{10} \cong \mathbb{Z}/2$, generated by $Y^5 \times Y^5$.

$\Omega_{11} \cong \mathbb{Z}/2$, generated by Y^{11} .

As manifold Y^5 (respectively Y^9, Y^{11}) we may take the non-singular hypersurface of degree (1, 1) in the product $P^2 \times P^4$ (respectively $P^2 \times P^8$ or $P^4 \times P^8$) of real projective spaces. Using products of the generators listed above, it is easy to show that all of the higher cobordism groups are non-zero.