# SURVEY OF COBORDISM THEORY

# A SURVEY OF COBORDISM THEORY[1]

by J. Milnor

This paper will start out with a discussion of known results and then will taper off into a discussion of unsolved problems.

The theory of cobordism was initiated by L. Pontrjagin and V. A. Rohlin [10, 12]. It came of age with the work of R. Thom [17]. The basic question in this theory is the following. Let $\mathscr{M}$ be some class of compact manifolds. Given $V \in \mathscr{M}$ how can one decide whether or not $V$ is the boundary of some other manifold in $\mathscr{M}$ ? Of course a necessary condition is that $V$ itself must be a *closed* manifold: that is the boundary $\partial V$ must be vacuous.

## 1. The classical cobordism groups $N_k$ and $\Omega_k$.

As a first illustration of this problem let $\mathscr{D}$ denote the class of all compact differentiable manifolds. The manifolds $V \in \mathscr{D}$ need not be connected or orientable, and are allowed to have boundaries.

Theorem 1 (Pontrjagin, Thom). — *A closed* k*-dimensional manifold* V $\in \mathscr{D}$ *is the boundary of some* (k + 1) *-dimensional manifold in* $\mathscr{D}$ *if and only if the Stiefel-Whitney numbers* $w_{i_1} \ldots w_{i_n}$ [V] *are all zero.*

(Explanation: The Stiefel-Whitney cohomology classes[2]) $w_i \in H^i(V; J_2)$ are defined for example in Steenrod [15]. If $i_{i_1} + \ldots + i_n = k$ is any partition of $k$ then the cup product $w_{i_1} \ldots w_{i_n}$ is a top dimensional cohomology class. Applying the canonical " integration " homomorphism

$$[V]: H^k(V; J_2) \to J_2$$

we obtain a " Stiefel-Whitney number " $w_{i_1} \ldots w_{i_n}[V] \in J_2$.)

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[1]) Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

[2]) The notation J will be used for the integers and $J_2$ for the integers modulo 2.

The *non-oriented cobordism group* $N_k = H_k(\mathscr{D})$ is constructed as follows. Given two $k$-manifolds $V, V' \in \mathscr{D}$ the *sum* $V + V'$ will mean the (disjoint) topological sum, provided with a differentiable structure in the obvious way.

*Definition.* Two closed manifolds $V, V' \in \mathscr{D}$ are *congruent modulo* $\partial\mathscr{D}$ if $V + V'$ is the boundary of some manifold in $\mathscr{D}$. The set of all congruence classes of closed $k$-manifolds, under the composition operation $+$, forms the required group $N_k$. We will also use the notation $H_k(\mathscr{D})$ for this group since it is something like a homology group. (The Russian term for " cobordism " is " intrinsic homology ".)

It follows from Theorem 1 that each $N_k$ is a finite abelian group of the form $J_2 \oplus \ldots \oplus J_2$.

The cartesian product operation between differentiable manifolds gives rise to a bilinear pairing

$$N_k \oplus N_l \to N_{k+l} .$$

Thus the graded group $N_* = (N_0, N_1, \ldots)$ has the structure of a graded ring.

THEOREM 2 (Thom). — *The non-oriented cobordism ring* $N_*$ *has the structure of a polynomial algebra*

$$J_2 [X_2, X_4, X_5, X_6, X_8, X_9, \ldots]$$

*with one generator* $X_k \in N_k$ *for each dimension which is not of the form* $2^m - 1$.

If $k$ is even then the real projective $k$-space can be taken as generator. For $k$ odd generators have been constructed by Dold [4].

Thom's proof of Theorems 1 and 2 involves a brilliant mixture of algebra and geometry. A key step in the argument is his proof that $N_k$ is isomorphic to a certain homotopy group. I will not try to give details.

Next consider the class $\mathscr{D}_o$ consisting of all *oriented* compact differentiable manifolds.

THEOREM 1'. — *A closed manifold in* $\mathscr{D}_o$ *is the boundary of a manifold in* $\mathscr{D}_o$ *if and only if both its Stiefel-Whitney numbers and its Pontrjagin numbers are zero.*

This result is due to Pontrjagin, Thom, Milnor, Averbuh, and Wall. (See [2, 9, 19].) For the definition of the Pontrjagin numbers $p_{i_1} \dots p_{i_n} [V] \in J$ the reader is refered to Hirzebruch [6]. These numbers are defined only if the dimension $k$ is a multiple of 4.

The *oriented cobordism ring* $\Omega_* = H_* (\mathscr{D}_o)$ is defined as follows. For $V \in \mathscr{D}_o$ let $- V$ denote the same manifold $V$ with the opposite orientation. We will say that

$$V \equiv V' \quad (mod\ \partial \mathscr{D}_o)$$

if $(- V) + V'$ is the boundary of some manifold in $\mathscr{D}_o$. As an example, for any closed manifold $V$ we have $V \equiv V \pmod{\partial \mathscr{D}_o}$ since

$$(- V) + V \approx \partial (V \times I)$$

where $I$ denotes the unit interval. The set of all such congruence classes form the required group $\Omega_k$. Again the cartesian product operation makes $\Omega_* = (\Omega_0, \Omega_1, \dots)$ into a graded ring.

It follows from Theorem 1′ that $\Omega_k$ is a finitely generated group of the form

$$J \oplus \dots \oplus J \oplus J_2 \oplus \dots \oplus J_2$$

where infinite cyclic summands can occur only if $k \equiv 0 \pmod 4$.

THEOREM 2′. — *The ring* $\Omega_*$, *modulo the ideal consisting of* 2-*torsion elements, is a polynomial ring* $J[Y_4, Y_8, Y_{12}, \dots]$ *with one generator in each dimension divisible by* 4.

The complex projective space of real dimension $4m$ can be taken as generator for $m = 1, 2, 3$. However a different generator is needed in dimension 16.

For a description of the 2-torsion in $\Omega_*$ the reader is referred to Wall's paper.

## 2. MANIFOLDS WITH $X$-STRUCTURE.

In this section we will define the concept of an " $X$-structure " on the tangent bundle of a differentiable manifold; and study the corresponding cobordism theory.

First recall Steenrod's definition of a tensor field [15, § 6.4 and § 9.1 with mild alterations]. Every differentiable $k$-manifold $V$ can be made Riemannian and hence has a tangent bundle with structural group $O_k$. Let $X$ be any topological space on which the group $O_k$ acts. Then we can form the weakly associated bundle with base space $V$ and fibre $X$. This may be called the " tensor bundle of type $X$ " and its cross-sections are " tensor fields ". As an example, if $k = 2m$, then $O_{2m}$ acts on the coset space $O_{2m}/U_m$.

A cross-section of the corresponding bundle is called a *quasi-* (or almost) *complex structure* on $V$. (See [15, § 41.10].)

We will modify this definition as follows, so that it makes sense for all dimensions simultaneously. Let $O$ denote the union of the orthogonal groups $O_1 \subset O_2 \subset O_3 \subset \dots$ in the fine topology. Then we require that this infinite orthogonal group $O$ act on the space $X$. It follows that each $O_k$ acts on $X$. Hence there is a tensor bundle of type $X$ over any manifold $V \in \mathcal{D}$.

*Definition:* A homotopy class of cross-sections of the tensor bundle with fibre $X$ over $V$ is called an *$X$-structure* on $V$. A manifold $V \in \mathcal{D}$ together with an $X$-structure on $V$ is called an *$X$-manifold.* We will still use the single symbol $V$ to denote this pair.

Now if $V$ is an $X$-manifold then $\partial V$ is also. Given any closed $X$-manifold $V$ one can define a second $X$-manifold $-$ V so that

$$\partial(V \times I) \approx V + (-V) .$$

Thus one can define a cobordism group for the class of $X$-manifolds. The resulting group will be denoted by $N_k(X)$ and called the X-*cobordism group.* (Following Atiyah [1] this could also be called the $k$-th " bordism group " of the $O$-space $X$.)

*Example 1.* Let $O/U$ denote the union of the spaces

$$O_2/U_1 \subset O_4/U_2 \subset O_6/U_3 \subset \dots$$

in the fine topology with $O$ acting on $O/U$ in the usual way. Then a manifold with an $O/U$-structure will be called a *weakly complex manifold.* (Compare Hirzebruch [7].) For example any complex manifold is quasi-complex and hence weakly complex. Any sphere can be given an $O/U$-structure although only $S^2$ and $S^6$ possess quasi-complex structures.

The following results are due to Milnor and Novikov.

THEOREM 1″. — *A closed weakly complex manifold* V *is the boundary of a weakly complex manifold if and only if its Chern numbers* $c_{i_1} \dots c_{i_n} [V]$ *are all zero.*

(Explanation: an $O/U$-structure on $V$ determines a preferred $U$-bundle over $V$. Hence Chern classes are defined.) It follows that $N_k(O/U)$ is zero for $k$ odd and is free abelian for $k$ even.

THEOREM 2″. — *The graded group* $N_*(O/U)$ *has a natural ring structure, making it into a polynomial ring* $J[Y_2, Y_4, Y_6, \dots]$ *with one generator in each even dimension.*

As generators one can take certain algebraic varieties with their natural complex structures. (Compare [7]. It is not known whether connected varieties will suffice.)

*Example 2.* More generally one could use any subgroup $G$ of the infinite orthogonal group in place of $U$. For example using the infinite symplectic group $Sp$ we would obtain a cobordism ring $N_*(O/Sp)$ which is appropriate for the study of " weakly quaternionic manifolds ". The following six groups seem particularly interesting:

$$1 \subset Sp \subset SU \subset U \subset SO \subset O\,.$$

Starting from the right, the ring $N_*(O/O)$ is just the non-oriented cobordism ring $N_*$ and $N_*(O/SO)$ is the oriented cobordism ring $\Omega_*$. The rings $N_*(O/SU)$ and $N_*(O/Sp)$ are more or less unknown. (Compare the concluding remarks in [9].)

The ring $N_*(O/1) = N_*(O)$ has essentially been studied by Pontrjagin [11]. An $O$-structure on $V$ is a trivialization of the tangent $O$-bundle of $V$ (the " stable " tangent bundle). Manifolds which admit such a structure are called " $\pi$-manifolds ". It turns out that $N_k(O)$ is isomorphic to the stable homotopy groups $\pi_{k+n}(S^n)$ of the $n$-sphere, with $n$ large. This fact is the basis for Pontrjagin's method of studying homotopy groups.

*Example 3.* Let $X$ be a space on which $O$ operates trivially. Then an $X$-structure on $V$ is just a prefered homotopy class of maps $V \to X$. As cases of particular interest $X$ might be an Eilenberg-MacLane space or the classifying space of a group. How does one compute he groups $N_k(X)$ ?

The above definitions can be modified slightly by admitting only oriented manifolds. Thus one obtains groups $\Omega_k(X)$ where $X$ is any space on which the rotation group $SO$ acts. Again I do not know how to compute these groups. (Added in proof: See Conner and Floyd [21].)

*Example 4.* Let $P$ denote the infinite real projective space, with the infinite rotation group $SO$ acting in the natural way. The cobordism groups $\Omega_k(P)$ for oriented manifolds with $P$-structure can be called the *spinor cobordism groups.* This name is appropriate since a $P$-structure is roughly a " lifting " of the structural group of the tangent bundle to the infinite spinor group. A manifold admits a $P$-structure if and only if its Stiefel-Whitney class $w_2$ is zero. The groups $\Omega_k(P)$ have no odd torsion, but otherwise I do not know much about them.

## 3. Miscellaneous cobordism theories.

So far we have concentrated on differentiable manifolds. However one could equally well define a cobordism group based on the class $\mathscr{T}$ of all compact topological manifolds. (Compare Brown [3, Theorem 3].) The natural correspondence $\mathscr{D} \to \mathscr{T}$ induces a homomorphism from the differentiable cobordism group $N_k = H_k(\mathscr{D})$ to the topological cobordism group $H_k(\mathscr{T})$.

Since Thom [16] has shown that Stiefel-Whitney classes can be defined topologically, we have:

Theorem 3 (Thom). — *The homomorphism* $\mathrm{N}_k \to H_k(\mathscr{T})$ *has kernel zero.*

Problem: Is this homomorphism onto ?

Another possibility would be to consider the class $\mathscr{C}_o$ of all compact, oriented, combinatorial manifolds. Whitehead [20] has shown that each differentiable manifold has a preferred class of triangulations. Hence there is a natural homomorphism from

$\Omega_k = H_k(\mathscr{D}_o)$ to $H_k(\mathscr{C}_o)$. Thom, Rohlin and Švarč have shown that Pontrjagin classes can be defined for combinatorial manifolds. Therefore we have:

THEOREM 3′. — *The homomorphism* $\Omega_k \to H_k(\mathscr{C}_o)$ *has kernel zero.*

However examples show that this homomorphism is not onto. The reader is referred to [13, 18].

Another interesting possibility would be to look at the class of compact homology manifolds.

Returning to the differentiable case, interesting cobordism groups can be obtained by restricting the connectivities of the manifolds involved. As an extreme case we can consider only differentiable manifolds which are either homotopy spheres or homotopy cells. The resulting cobordism groups are closely related to the problem of classifying differentiable structures on spheres. The reader is referred to Milnor [8] and Smale [14].

As a final, quite different, example consider differentiable imbeddings of the circle $S^1$ in the 3-sphere $S^3$. Such an object (a knot) is said to *bound* if it can be extended to a differentiable imbedding of the disk $D^2$ in the disk $D^4$. The resulting cobordism group has been studied by Fox and Milnor [5]. This group is not finitely generated.

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Princeton University.