Chapter 8

Bordism, Spectra, and Generalized Homology

This chapter contains a mixture of algebraic and differential topology and serves as an introduction to generalized homology theories. We will give a precise definition of a generalized homology theory later, but in the meantime you should think of a generalized homology theory as a functor from pairs of spaces to graded abelian groups (or graded R-modules) satisfying all Eilenberg–Steenrod axioms but the dimension axiom.

The material in this chapter will draw on the basic notions and theorems of differential topology, and you should re-familiarize yourself with the notion of smooth maps between smooth manifolds, submanifolds, tangent bundles, orientation of a vector bundle, the normal bundle of a submanifold, the Sard theorem, transversality and the tubular neighborhood theorem. One of the projects for this chapter is to prepare a lecture on these topics. A good reference for this material is Hirsch's book [**16**]; more elementary references include [**27**] and [**15**].

In this chapter (in contrast to the rest of this book), the word "manifold" will mean a *compact, smooth* manifold with or without boundary and a submanifold $V \subset M$ will mean a compact submanifold whose boundary is contained in the boundary of M in such a way that V meets the boundary of M transversely. The normal bundle of a submanifold $i : V \hookrightarrow M$ is the quotient bundle $i^*(TM)/TV$, and we will use the notation $\nu(V \hookrightarrow M)$ or $\nu(i)$. If M is a submanifold of $\mathbf{R}^n$, or more generally if M has a Riemannian metric, then the normal bundle $\nu(V \hookrightarrow M)$ can be identified with the subbundle of $TM|_V$ consisting of all tangent vectors in T_pM which are perpendicular to T_pV, where $p \in V$. A *tubular neighborhood* of a submanifold

$i : V \hookrightarrow M$ is a embedding $f : \nu(i) \to M$ which restricts to the identity on (the zero section) V. Informally, we say that the open set $U = f(\nu(i)) \subset M$ is a tubular neighborhood of V.

8.1. Framed bordism and homotopy groups of spheres

Pontrjagin and Thom in the 1950's noted that in many situations there is a one-to-one correspondence between problems in geometric topology (= manifold theory) and problems in algebraic topology. Usually the algebraic problem is more tractable, and its solution leads to geometric consequences. In this section we discuss the quintessential example of this correspondence; a reference is the last section of Milnor's beautiful little book [**27**].

We start with an informal discussion of the passage from geometric topology to algebraic topology.

Definition 8.1. A *framing* of a submanifold V^{k-n} of a closed manifold M^k is a embedding ϕ of $V \times \mathbf{R}^n$ in M so that $\phi(p, 0) = p$ for all $p \in V$. If (W^{k+1-n}, ψ) is a framed submanifold of $M \times I$, then the two framed submanifolds of M given by intersecting W with $M \times \{0\}$ and $M \times \{1\}$ are *framed bordant*. Let $\Omega^{\text{fr}}_{k-n,M}$ be the set of framed bordism classes of $(k - n)$-dimensional framed submanifolds of M.

A framed submanifold defines a *collapse map* $M \to S^n = \mathbf{R}^n \cup \{\infty\}$ by sending $\phi(p, v)$ to v and all points outside the image of ϕ to ∞. Note that $0 \in S^n$ is a regular value and the inverse image of 0 is V. A framed bordism gives a homotopy of the two collapse maps. A framed bordism from a framed submanifold to the empty set is a *null-bordism*. In the special case of a framed submanifold V^{k-n} of S^k, a null-bordism is given by an extension to a framed submanifold W^{k+1-n} of D^{k+1}.

Theorem 8.2. *The collapse map induces a bijection* $\Omega^{\text{fr}}_{k-n,M} \to [M, S^n]$.

This method of translating between bordism and homotopy sets is called the *Pontrjagin–Thom construction*.

Here are some examples (without proof) to help your geometric insight. A (framed) point in a S^k gives a map $S^k \to S^k$ which generates $\pi_k S^k \cong \mathbf{Z}$. Any framed circle in S^2 is null-bordant, for example the equator with the obvious framing is the boundary of the 2-disk in the 3-ball. However, a framed S^1 in S^3 so that the circle $\phi(S^1 \times \{(1, 0)\})$ links the S^1 with linking number 1 represents the generator of $\pi_3(S^2) \cong \mathbf{Z}$. (Can you reinterpret this in terms of the Hopf map? Why can't one see the complexities of knot theory in framed bordism?) Now S^3 is naturally framed in S^4, S^4 in S^5, etc.

so we can suspend the linking number 1 framing of S^1 in S^3 to get a framing of S^1 in S^{k+1} for $k > 2$. This represents the generator of $\pi_{k+1}S^k \cong \mathbf{Z}_2$.

More generally, one can produce examples of framed manifolds by twisting and suspending. If (V^{k-n}, ϕ) is a framed submanifold of M^k and $\alpha : V \to O(n)$, then the *twist* is the framed submanifold $(V, \phi.\alpha)$ where $\phi.\alpha(p, v) = \phi(p, \alpha(p)v)$. The framed bordism class depends only on (V, ϕ) and the homotopy class of α. (See Exercise 132 below for more on this construction.) Next if (V^{k-n}, ϕ) is a framed submanifold of S^k, then the *suspension* of (V^{k-n}, ϕ) is the framed submanifold $(V^{k-n}, S\phi)$ of S^{k+1} is defined using the obvious framing of S^k in S^{k+1}, with $S^k \times \mathbf{R}_{>0}$ mapping to the upper hemisphere of S^{k+1}. Then the generator of $\pi_3(S^2)$ mentioned earlier can be described by first suspending the inclusion of a framed circle in the 2-sphere, and then twisting by the inclusion of the circle in $O(2)$.

To prove Theorem 8.2 we first want to reinterpret $\Omega^{\text{fr}}_{k-n,M}$ in terms of normal framings. The key observation is that a framed submanifold determines n linearly independent normal vector fields on M.

Definition 8.3.

1. A *trivialization* of a vector bundle $p : E \to B$ with fiber $\mathbf{R}^n$ is a collection $\{\sigma_i : B \to E\}_{i=1}^n$ of sections which form a basis pointwise. Thus $\{\sigma_1(b), \ldots, \sigma_n(b)\}$ is linearly independent and spans the fiber E_b for each $b \in B$.

   Equivalently, a trivialization is a specific bundle isomorphism $E \cong B \times \mathbf{R}^n$. A trivialization is also the same as a choice of section of the associated principal frame bundle.
2. A *framing* of a vector bundle is a homotopy class of trivializations, where two trivializations are called *homotopic* if there is a continuous 1-parameter family of trivializations joining them. In terms of the associated frame bundle this says the two sections are homotopic in the space of sections of the frame bundle.

A section of a normal bundle is called a normal vector field.

Definition 8.4. A *normal framing* of a submanifold V of M is a homotopy class of trivializations of the normal bundle $\nu(V \hookrightarrow M)$. If W is a normally framed submanifold of $M \times I$, then the two normally framed submanifolds of M given by intersecting W with $M \times \{0\}$ and $M \times \{1\}$ are *normally framed bordant*. (You should convince yourself that restriction of $\nu(W \hookrightarrow M \times I)$ to $V_0 = (M \times \{0\}) \cap W$ is canonically identified with $\nu(V_0 \hookrightarrow M)$).

Exercise 127. Show that a framed submanifold (V, ϕ) of M determines a normal framing of V in M. Use notation from differential geometry and denote the standard coordinate vector fields on $\mathbf{R}^n$ by $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$.

Exercise 128. Define a map from the set of bordism classes of $(k-n)$-dimensional framed submanifolds of M to the set of bordism classes of $(k-n)$-dimensional normally framed submanifolds of M and show it is a bijection. (The existence part of the tubular neighborhood theorem will show the map is surjective, while the uniqueness part will show the map is injective.)

Henceforth we let $\Omega^{\mathrm{fr}}_{k-n,M}$ denote both the bordism classes of framed submanifolds and the bordism classes of normally framed submanifolds of M.

Proof of Theorem 8.2. To define an inverse

$$d : [M, S^n] \to \Omega^{\mathrm{fr}}_{k-n,M}$$

to the collapse map

$$c : \Omega^{\mathrm{fr}}_{k-n,M} \to [M, S^n]$$

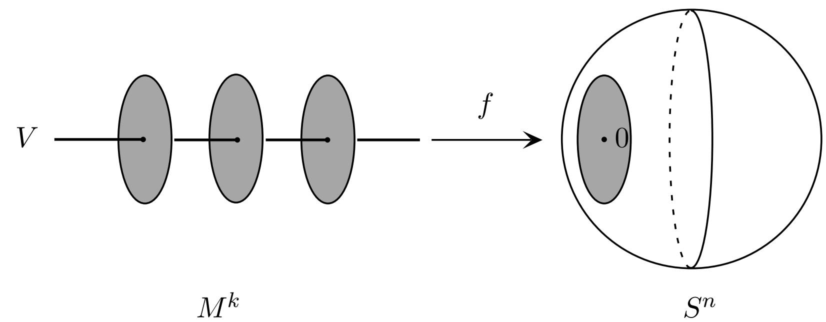
one must use differential topology; in fact, this was the original motivation for the development of transversality.

Any element of $[M, S^n]$ can be represented by a map $f : M \to S^n = \mathbf{R}^n \cup \{\infty\}$, which is smooth in a neighborhood of $f^{-1}(0)$ and transverse to 0 (i.e. 0 is a regular value). Thus:

1. The inverse image $f^{-1}(0) = V$ is a smooth submanifold of M^k of codimension n (i.e. of dimension $k-n$), and
2. The differential of f identifies the normal bundle of V in M^k with the pullback of the normal bundle of $0 \in S^n$ via f. More precisely, the differential of f, $df : TM^k \to TS^n$ restricts to $TM^k|_V$ and factors through the quotient $\nu(V \hookrightarrow M^k)$ to give a map of vector bundles

$$\begin{array}{ccc} \nu(V \hookrightarrow M^k) & \xrightarrow{df} & \nu(0 \hookrightarrow S^n) \\ \downarrow & & \downarrow \\ V & \longrightarrow & 0 \end{array}$$

which is an isomorphism in each fiber.



Since the normal bundle of 0 in $\mathbf{R}^n \cup \{\infty\}$ is naturally framed by the standard basis, the second assertion above implies that the normal bundle of V in M^k is also framed, i.e. there is a bundle isomorphism

$$\begin{array}{ccc} \nu(V \hookrightarrow M^k) & \xrightarrow{\cong} & V \times \mathbf{R}^n \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

The map d is defined by sending $[f]$ to $f^{-1}(0)$ with the above framing. To see that d is well-defined, consider a homotopy

$$F : M \times I \to S^n.$$

where $F|_{M\times\{0,1\}}$ is transverse to $0 \in S^n$. Consider the "trace of F"

$$\begin{aligned} \hat{F} : M \times I &\to S^n \times I \\ (m,t) &\mapsto (F(m,t),t), \end{aligned}$$

which has the advantage that it takes boundary points to boundary points. The (relative) transversality approximation theorem says that $\hat{F}$ is homotopic (rel $M \times \{0,1\}$) to a map transverse to $0 \times I$. The inverse image of $0 \times I$ equipped with an appropriate normal framing gives a normally framed bordism between $F|^{-1}_{M\times\{0\}}(0)$ and $F|^{-1}_{M\times\{1\}}(0)$.

Our final task is to show that c and d are mutual inverses. It is easy to see that $d \circ c$ is the identity, but to show $c \circ d$ is the identity takes some work. First represent an element of $[M, S^n]$ by a a map f transverse to $0 \in \mathbf{R}^n \cup \{\infty\} = S^n$. It seems plausible that the collapse map associated to $V = f^{-1}(0)$ with the normal framing induced by df is homotopic to f, but there are technical details.

Here goes. Let $\nu = \nu(V \hookrightarrow M)$, let $g : \nu \to M$ be a tubular neighborhood of V, assume ν has a metric, and let $D = g(D(\nu))$ correspond to the disk bundle. Define $\Phi : \nu \to \mathbf{R}^n$ by $\Phi(x) = \lim_{t\to 0} \ t^{-1} f(g(tx))$. Then $\Phi(x)$

is the velocity vector of a curve, and by the chain rule Φ is the composite of the identification of ν with $\nu(V \hookrightarrow \nu)$ and $df \circ dg$. In particular Φ gives an isomorphism from each fiber of ν to $\mathbf{R}^n$.

There is a homotopy $f_t : D \to \mathbf{R}^n \cup \{\infty\}$ for $-1 \leq t \leq 1$ given by

$$f_t(g(x)) = \begin{cases} \frac{1}{1+t\|x\|}\Phi(x) & \text{if } -1 \leq t \leq 0, \\ t^{-1}f(g(tx)) & \text{if } 0 < t \leq 1. \end{cases}$$

We now have a map

$$\partial D \times [-1,1] \ \cup \ (M - \text{Int } D) \times \{1\} \ \cup \ (M - \text{Int } D) \times \{-1\} \to S^n - \{0\}$$

defined by f_t on the first piece, by f on the second piece, and by the constant map at infinity on the third piece. This extends to a map $(M - \text{Int } D) \times [-1,1] \to S^n - \{0\}$ by the Tietze extension theorem.

We can then paste back in f_t to get a homotopy

$$F : M \times [-1,1] \to S^n$$

from our original f to a map h so that

$$h^{-1}\mathbf{R}^n = \text{Int } D \cong V \times \mathbf{R}^n$$

where the diffeomorphism $\cong$ is defined by mapping to V by using the original tubular neighborhood and by mapping to $\mathbf{R}^n$ by h. Thus $f \simeq h$ where h is in the image of c. It follows that c is surjective and thus that c and d are mutual inverses. □

In reading the above proof you need either a fair amount of technical skill to fill in the details or you need to be credulous. For an alternate approach see [**27**, Chapter 7].

For a real vector bundle over a point, i.e. a vector space, a framing is the same as a choice of orientation of the vector space, since $GL(n, \mathbf{R})$ has two path components. Thus a normal framing of $V \subset S^k$ induces an orientation on the normal bundle $\nu(V \hookrightarrow S^k)$. (See Section 10.7 for more information about orientation.)

Exercise 129. Let V be a normally framed submanifold of a manifold M. Show that an orientation on M induces an orientation on V. (Hint: consider the isomorphism $TV \oplus \nu = TM|_V$.)

Theorem 8.5 (Hopf degree theorem)**.** *Let M^k be a connected, closed, smooth manifold.*

1. *If M^k is orientable, then two maps $M^k \to S^k$ are homotopic if and only if they have the same degree.*
2. *If M^k is nonorientable, then two maps $M^k \to S^k$ are homotopic if and only if they have the same degree mod 2.*

Exercise 130. Prove the Hopf degree theorem in two ways: obstruction theory and framed bordism.

The function $\pi_k S^n \to [S^k, S^n]$ obtained by forgetting base points is a bijection. For $n > 1$ this follows from the fact that S^n is simply connected and so vacuously the fundamental group acts trivially. For $n = 1$ this is still true because $\pi_k S^1$ is trivial for $k > 1$ and abelian for $k = 1$.

The result that $\pi_n S^n \cong \mathbf{Z}$ is a nontrivial result in algebraic topology; it is cool that this can be proven using differential topology.

Exercise 131. We only showed that the isomorphism of Theorem 8.2 is a bijection of sets. However, since $\pi_k S^n$ is an abelian group, the framed bordism classes inherit an abelian group structure. Prove that this group structure on framed bordism is given by taking the disjoint union:

$$[V_0] + [V_1] := V_0 \amalg V_1 \subset S^k \# S^k \cong S^k$$

with negatives given by changing the orientation of the framing (e.g. replacing te first vector field in the framing by its negative)

$$-[V_0] = [-V_0].$$

We will generalize Theorem 8.2 by considering the effect of the suspension map $S : \pi_k S^n \to \pi_{k+1} S^{n+1}$ and eventually passing to the limit $\lim_{\ell \to \infty} \pi_{k+\ell} S^{n+\ell}$. This has the effect of eliminating the thorny embedding questions of submanifolds in S^k; in the end we will be able to work with abstract framed manifolds V without reference to an embedding of V in some sphere.

Exercise 132. (The J-homomorphism) Let $V^{k-n} \subset M^k$ be a non-empty normally framed manifold. Use twisting to define a function

$$J : [V^{k-n}, O(n)] \to [M^k, S^n].$$

Now let V be the equatorial $S^{k-n} \subset S^k$ with the canonical framing coming from the inclusions $S^{k-n} \subset S^{k-n+1} \subset \cdots \subset S^k$, and show that the function

$$J : \pi_{k-n}(O(n)) \to \pi_k S^n$$

is a homomorphism provided $k > n$. It is called the *J-homomorphism* and can be used to construct interesting elements in $\pi_k S^n$.

Draw an explicit picture of a framed circle in $\mathbf{R}^3 = S^3 - \{\infty\}$ representing $J(\iota)$ where $\iota \in \pi_1 O(2) = \mathbf{Z}$ is the generator.

8.2. Suspension and the Freudenthal theorem

Recall that the (reduced) suspension of a space $X \in \mathcal{K}_*$ with nondegenerate basepoint is the space

$$SX = X \times I/\sim$$

where the subspace $(x_0 \times I) \cup (X \times \{0,1\})$ is collapsed to a point. This construction is functorial with respect to based maps $f : X \to Y$. In particular, the suspension defines a function

$$S : [X,Y]_0 \to [SX, SY]_0.$$

By Proposition 6.35, $SS^k = S^{k+1}$, so that when $X = S^k$, the suspension defines a function, in fact a homomorphism

$$S : \pi_k(Y) \to \pi_{k+1}(SY)$$

for any space Y. Taking Y to be a sphere one obtains

$$S : \pi_k(S^n) \to \pi_{k+1}(S^{n+1}).$$

We next identify $S^k \subset S^{k+1} = SS^k$ as the equator, and similarly $S^n \subset S^{n+1}$, and interpret the above map in terms of framed bordism.

If $f : S^k \to S^n$ is smooth, then the suspension

$$Sf : S^{k+1} \to S^{n+1}$$

is smooth away from the base points. If $x \in S^n$ is a regular value different from the base point, and $V = f^{-1}(x)$ is the normally framed submanifold of S^k associated to f, then clearly

$$V = (Sf)^{-1}(x) \subset S^k \subset S^{k+1}.$$

Let us compare normal bundles and normal framings.

$$\begin{aligned} \nu(V \hookrightarrow S^{k+1}) &= \nu(V \hookrightarrow S^k) \oplus \nu(S^k \hookrightarrow S^{k+1})|_V \\ &= \nu(V \hookrightarrow S^k) \oplus \varepsilon_V \end{aligned}$$

where $\varepsilon_V = V \times \mathbf{R} =$ trivial line bundle.

Similarly, $\nu(x \hookrightarrow S^{n+1}) = \nu(x \hookrightarrow S^n) \oplus \varepsilon_{\{x\}}$, and the differential of Sf preserves the trivial factor, since, locally (near the equator $S^k \subset S^{k+1}$),

$$Sf \cong f \times Id : S^k \times (-\epsilon, \epsilon) \to S^n \times (-\epsilon, \epsilon).$$

We have shown the following.

Theorem 8.6. *Taking the suspension of a map corresponds, via the Pontrjagin-Thom construction, to the same manifold V, but embedded in the equator $S^k \subset S^{k+1}$, and with normal framing the direct sum of the old normal framing and the trivial 1-dimensional framing.* □

Now consider the effect of multiple suspensions.

$$S^\ell : \pi_k S^n \to \pi_{k+\ell} S^{n+\ell}$$

For each suspension, the effect on the normally framed submanifold V is to replace it by the same manifold embedded in the equator, with the new normal framing $\nu_{\text{new}} = \nu_{\text{old}} \oplus \varepsilon_V$. Thus after ℓ suspensions,

$$\nu_{\text{new}} = \nu_{\text{old}} \oplus \varepsilon_V^\ell.$$

The following fundamental result is the starting point for the investigation of "stable" phenomena in homotopy theory. We will not give a proof at this time, since a spectral sequence proof is the easiest way to go. The proof is given in Section 10.3.

Theorem 8.7 (Freudenthal suspension theorem)**.** *Suppose that X is an $(n-1)$–connected space $(n \geq 2)$. Then the suspension homomorphism*

$$S : \pi_k X \to \pi_{k+1} SX$$

is an isomorphism if $k < 2n-1$ and an epimorphism if $k = 2n-1$. □

The most important case is when $X = S^n$, and here the Freudenthal suspension theorem can also be given a differential topology proof using framed bordism and the facts that any j-manifold embeds in S^n for $n \geq 2j+1$, uniquely up to isotopy if $n \geq 2j+2$, and that any embedding of a j-manifold in S^{n+1} is isotopic to an embedding in S^n if $n \geq 2j+1$.

Exercise 133. Show that for any k-dimensional CW-complex X and for any $(n-1)$-connected space Y $(n \geq 2)$ the suspension map

$$[X, Y]_0 \to [SX, SY]_0$$

is bijective if $k < 2n-1$ and surjective if $k = 2n-1$. (Hint: Instead consider the map $[X, Y]_0 \to [X, \Omega SY]_0$. Convert the map $Y \to \Omega SY$ to a fibration and apply cross-section obstruction theory as well as the Freudenthal suspension theorem.)

For a based space X, $\pi^n X = [X, S^n]_0$ is called the *n-th cohomotopy set.* If X is a CW-complex with dim $X < 2n-1$, then Exercise 133 implies that $\pi^n X$ is a group, with group structure given by suspending and using the suspension coordinate in SX. The reader might ponder the geometric meaning (framed bordism) of the cohomotopy group structure when X is a manifold.

Definition 8.8. The k-th *stable homotopy group* of a based space X is the limit

$$\pi_k^S X = \lim_{\ell \to \infty} \pi_{k+\ell} S^\ell X.$$

The *stable k-stem* is

$$\pi_k^S = \pi_k^S S^0.$$

The computation of the stable k-stem for all k is the holy grail of the field of homotopy theory.

The Hurewicz theorem implies that if X is $(n-1)$-connected, then SX is n-connected, since $\widetilde{H}_\ell SX = \widetilde{H}_{\ell-1}X = 0$ if $\ell \leq n$ and $\pi_1 SX = 0$ if X is path connected. The following corollary follows from this fact and the Freudenthal theorem.

Corollary 8.9. *If X is path connected,*

$$\pi_k^S X = \pi_{2k}(S^k X) = \pi_{k+\ell}(S^\ell X) \qquad \text{for } \ell \geq k.$$

For the stable k-stem,

$$\pi_k^S = \pi_{2k+2}(S^{k+2}) = \pi_{k+\ell}(S^\ell) \qquad \text{for } \ell \geq k+2.$$

□

Recall from Equation (6.3) that $\pi_k(O(n-1)) \to \pi_k(O(n))$, induced by the inclusion $O(n-1) \hookrightarrow O(n)$, is an isomorphism for $k < n-2$, and therefore letting $O = \lim_{n\to\infty} O(n)$, $\pi_k O = \pi_k(O(n))$ for $k < n-2$. It follows from the definitions that the following diagram commutes

$$\begin{array}{ccc} \pi_k(O(n-1)) & \xrightarrow{J} & \pi_{k+n-1}(S^{n-1}) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle s} \\ \pi_k(O(n)) & \xrightarrow{J} & \pi_{k+n}(S^n) \end{array}$$

with the horizontal maps the J-homomorphisms, the left vertical map induced by the inclusion, and the right vertical map the suspension homomorphism. If $k < n-2$, then both vertical maps are isomorphisms, and so one obtains the *stable J-homomorphism*

$$J : \pi_k(O) \to \pi_k^S.$$

Corollary 8.10. *The Pontrjagin-Thom construction defines an isomorphism from π_k^S to the normally framed bordism classes of normally framed k-dimensional closed submanifolds of S^n for any $n \geq 2k+2$.* □

8.3. Stable tangential framings

We wish to remove the restriction that our normally framed manifolds be submanifolds of S^n. To this end we need to eliminate the reference to the normal bundle. This turns out to be easy and corresponds to the fact that the normal and tangent bundles of a submanifold of S^n are inverses in a certain stable sense. Since the tangent bundle is an intrinsic invariant of a smooth manifold, and so is defined independently of any embedding in S^k, this will enable us to replace normal framings with tangential framings. On the homotopy level, however, we will need to take suspensions when describing in what way the bundles are inverses. In the end this means that we will obtain an isomorphism between *stably tangentially framed* bordism classes and *stable* homotopy groups.

In what follows, ε^j will denote a *trivialized* j-dimensional real bundle over a space.

Lemma 8.11. *Let $V^k \subset S^n$ be a closed, oriented, normally framed submanifold of S^n. Then*

1. *A normal framing $\gamma : \nu(V \hookrightarrow S^n) \cong \varepsilon^{n-k}$ induces a trivialization*
$$\overline{\gamma} : TV \oplus \varepsilon^{n-k+1} \cong \varepsilon^{n+1}.$$
2. *A trivialization $\overline{\gamma} : TV \oplus \varepsilon \cong \varepsilon^{k+1}$ induces a trivialization*
$$\nu(V \hookrightarrow S^n) \oplus \varepsilon^{k+1} \cong \varepsilon^{n+1}.$$

Proof. The inclusion $S^n \subset \mathbf{R}^{n+1}$ has a trivial 1-dimensional normal bundle which can be framed by choosing the outward unit normal as a basis. This shows that the once stabilized tangent bundle of S^n is *canonically* trivialized
$$TS^n \oplus \varepsilon \cong \varepsilon^{n+1}$$
since the tangent bundle of $\mathbf{R}^{n+1}$ is canonically trivialized.

There is a canonical decomposition
$$(TS^n \oplus \varepsilon)|_V = \nu(V \hookrightarrow S^n) \oplus TV \oplus \varepsilon.$$
Using the trivialization of $TS^n \oplus \varepsilon$, one has a canonical isomorphism
$$\varepsilon^{n+1} \cong \nu(V \hookrightarrow S^n) \oplus TV \oplus \varepsilon.$$
Thus a normal framing $\gamma : \nu(V \hookrightarrow S^k) \cong \varepsilon^{n-k}$ induces an isomorphism
$$\varepsilon^{n+1} \cong \varepsilon^{n-k} \oplus TV \oplus \varepsilon,$$
and, conversely a trivialization $\overline{\gamma} : TV \oplus \varepsilon \cong \varepsilon^{k+1}$ induces an isomorphism
$$\varepsilon^{n+1} \cong \nu(V \hookrightarrow S^n) \oplus \varepsilon^{k+1}.$$
□

Definition 8.12. A *stable (tangential) framing* of an k-dimensional manifold V is an equivalence class of trivializations of
$$TV \oplus \varepsilon^n$$
where ε^n is the trivial bundle $V \times \mathbf{R}^n$. Two trivializations
$$t_1 : TV \oplus \varepsilon^{n_1} \cong \varepsilon^{k+n_1}, \quad t_2 : TV \oplus \varepsilon^{n_2} \cong \varepsilon^{k+n_2}$$

are considered equivalent if there exists some N greater than n_1 and n_2 such that the direct sum trivializations

$$t_1 \oplus \mathrm{Id} : TV \oplus \varepsilon^{n_1} \oplus \varepsilon^{N-n_1} \cong \varepsilon^{k+n_1} \oplus \varepsilon^{N-n_1} = \varepsilon^{k+N}$$

and

$$t_2 \oplus \mathrm{Id} : TV \oplus \varepsilon^{n_2} \oplus \varepsilon^{N-n_2} \cong \varepsilon^{k+n_2} \oplus \varepsilon^{N-n_2} = \varepsilon^{k+N}$$

are homotopic.

Similarly, a *stable normal framing* of a submanifold V of S^ℓ is an equivalence class of trivializations of $\nu(V \hookrightarrow S^\ell) \oplus \varepsilon^n$ and a *stable framing* of a bundle η is an equivalence class of trivializations of $\eta \oplus \varepsilon^n$.

A tangential framing is easier to work with than a normal framing, since one does not need to refer to an embedding $V \subset S^n$ to define a tangential framing. However, stable normal framings and stable tangential framings are equivalent; essentially because the tangent bundle of S^n is canonically stably framed. Lemma 8.11 generalizes to give the following theorem.

Theorem 8.13. *There is a 1-1 correspondence between stable tangential framings and stable normal framings of a manifold V. More precisely:*

1. *Let $i : V \hookrightarrow S^n$ be an embedding. A stable framing of TV determines stable framing of $\nu(i)$ and conversely.*
2. *Let $i_1 : V \hookrightarrow S^{n_1}$ and $i_2 : V \hookrightarrow S^{n_2}$ be embeddings. For n large enough there exists a canonical (up to homotopy) identification*

$$\nu(i_1) \oplus \varepsilon^{n-n_1} \cong \nu(i_2) \oplus \varepsilon^{n-n_2}.$$

A stable framing of $\nu(i_1)$ determines one of $\nu(i_2)$ and vice versa.

Proof. 1. The proof of Lemma 8.11 gives a canonical identification

$$\nu(V \hookrightarrow S^n) \oplus \varepsilon^\ell \oplus TV \cong \varepsilon^{n+\ell}$$

for all $\ell > 0$. Associativity of $\oplus$ shows stable framings of the normal bundle and tangent bundles coincide.

2. Let $i_1 : V \hookrightarrow S^{n_1}$ and $i_2 : V \hookrightarrow S^{n_2}$ be embeddings. There is a formal proof that stable framings of $\nu(i_1)$ and $\nu(i_2)$ coincide. Namely, a stable framing of $\nu(i_1)$ determines a stable framing of TV by part 1, which in turn determines a stable framing of $\nu(i_2)$. However, the full statement of part 2 applies to submanifolds with non-trivial normal bundle, and theorems from differential topology must be used.

Choose n large enough so that any two embeddings of V in S^n are isotopic. (Transversality theorems imply that $n > 2k + 1$ suffices.)

The composite $V \xrightarrow{i_1} S^{n_1} \xrightarrow{j_1} S^n$, with $S^{n_1} \xrightarrow{j_1} S^n$ the equatorial embedding, has normal bundle

$$\nu(j_1 \circ i_1) = \nu(i_1) \oplus \varepsilon^{n-n_1}.$$

Similarly, the composite $V \xrightarrow{i_2} S^{n_2} \xrightarrow{j_2} S^n$ has normal bundle

$$\nu(j_2 \circ i_2) = \nu(i_2) \oplus \varepsilon^{n-n_2}.$$

Then $j_2 \circ i_2$ is isotopic to $j_1 \circ i_1$, and the isotopy induces an isomorphism $\nu(j_2 \circ i_2) \cong \nu(j_1 \circ i_1)$.

If $n > 2(k+1)+1$, then any self-isotopy is isotopic to the constant isotopy, so that the identification $\nu(j_2 \circ i_2) \cong \nu(j_1 \circ i_1)$ is canonical (up to homotopy). □

Definition 8.14. Two real vector bundles E, F over V are called *stably equivalent* if there exists non-negative integers i, j so that $E \oplus \varepsilon^i$ and $F \oplus \varepsilon^j$ are isomorphic.

Since every smooth compact manifold embeds in S^n for some n, the second part of Theorem 8.13 has the consequence that the stable normal bundle (i.e. the stable equivalence class of the normal bundle for some embedding) is a well defined invariant of a smooth manifold, independent of the embedding, just as the tangent bundle is. However, something stronger holds. If $\nu(i_1)$ and $\nu(i_2)$ are normal bundles of two different embeddings of a manifold in a sphere, then not only are $\nu(i_1)$ and $\nu(i_2)$ stably equivalent, but the stable isomorphism is determined up to homotopy.

Returning to bordism, we see that the inclusion $S^n \subset S^{n+1}$ sets up a correspondence between the suspension operation and stabilizing a normal (or equivalently tangential) framing. Consequently Corollary 8.10 can be restated as follows.

Corollary 8.15. *The stable k-stem π_k^S is isomorphic to the stably tangentially framed bordism classes of stably tangentially framed k-dimensional smooth, oriented compact manifolds without boundary.* □

This statement is more appealing since it refers to k-dimensional manifolds intrinsically, without reference to an embedding in some S^n.

Here is a list of some computations of stable homotopy groups of spheres for you to reflect on. (Note: π_k^S has been computed for $k \leq 64$. There is no reasonable conjecture for π_k^S for general k, although there are many results known. For example, in Chapter 10, we will show that the groups are finite for $k > 0$; $\pi_0^S = \mathbf{Z}$ by the Hopf degree theorem.)

k	1	2	3	4	5	6
π_k^S	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/24$	0	0	$\mathbf{Z}/2$
k	7	8	9	10	11	12
π_k^S	$\mathbf{Z}/240$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/6$	$\mathbf{Z}/504$	0
k	13	14	15	16	17	18
π_k^S	$\mathbf{Z}/3$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/480 \oplus \mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$\mathbf{Z}/8 \oplus \mathbf{Z}/2$
k	19	20	21	22	23	24
π_k^S	$\mathbf{Z}/264 \oplus \mathbf{Z}/2$	$\mathbf{Z}/24$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	†	$(\mathbf{Z}/2)^2$

† π_{23}^S is $\mathbf{Z}/16 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/9 \oplus \mathbf{Z}/3 \oplus \mathbf{Z}/5 \oplus \mathbf{Z}/7 \oplus \mathbf{Z}/13$.

The reference [**32**] is a good source for the tools to compute π_k^S.

We will give stably framed manifolds representing generators of π_k^S for $k < 9$; you may challenge your local homotopy theorist to supply the proofs. In this range there are (basically) two sources of framed manifolds: normal framings on spheres coming from the image of the stable J-homomorphism $J : \pi_k(O) \to \pi_k^S$, and tangential framing coming from Lie groups. There is considerable overlap between these sources.

Bott periodicity (Theorem 6.49) computes $\pi_k(O)$.

k	0	1	2	3	4	5	6	7	8
$\pi_k O$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$

Then $J : \pi_k O \to \pi_k^S$ is an isomorphism for $k = 1$, an epimorphism for $k = 3, 7$, and a monomorphism for $k = 8$.

Another source for framed manifolds are Lie groups. If G is a compact k-dimensional Lie group and $T_eG \cong \mathbf{R}^k$ is an identification of its tangent space at the identity, then one can use the group multiplication to identify $TG \cong G\times\mathbf{R}^k$ and thereby frame the tangent bundle. This is the so-called Lie invariant framing. The generators of the cyclic groups $\pi_0^S, \pi_1^S, \pi_2^S, \pi_3^S, \pi_6^S, \pi_7^S$ are given by $e, S^1, S^1 \times S^1, S^3, S^3 \times S^3, S^7$ with invariant framings. (The unit octonions S^7 fail to be a group because of the lack of associativity, but nonetheless, they do have an invariant framing.)

Finally, the generators of π_8^S are given by S^8 with framing given by the J-homomorphism and the unique exotic sphere in dimension 8. (An exotic sphere is a smooth manifold homeomorphic to a sphere and not diffeomorphic to a sphere.)

We have given a bordism description of the groups π_k^S. If X is any space, $\pi_k^S X$ can be given a bordism description also. In this case one adds the structure of a map from the manifold to X. (A map from a manifold to a space X is sometimes called a *singular manifold in* X.)

Definition 8.16. Let $(V_i, \gamma_i : TV_i \oplus \varepsilon^a \cong \varepsilon^{k+a})$, $i = 0, 1$ be two stably framed k-manifolds and $g_i : V_i \to X$, $i = 0, 1$ two maps.

We say (V_0, γ_0, g_0) is *stably framed bordant to* (V_1, γ_1, g_1) *over* X if there exists a stably framed bordism (W, τ) from (V_0, γ_0) to (V_1, γ_1) and a map

$$G : W \to X$$

extending g_0 and g_1.

We introduce the notation:

1. Let X_+ denote $X \amalg \mathrm{pt}$, the union of X with a disjoint base point.
2. Let $\Omega_k^{\mathrm{fr}}(X)$ denote the stably framed bordism classes of stably framed k-manifolds over X.

Since every space maps uniquely to point, and since $S^0 = \mathrm{pt}_+$, we can restate Corollary 8.15 in this notation as

$$\Omega_k^{\mathrm{fr}}(\mathrm{pt}) = \pi_k^S(\mathrm{pt}_+)$$

since $\pi_k^S = \pi_k^S(S^0) = \pi_k^S(\mathrm{pt}_+)$.

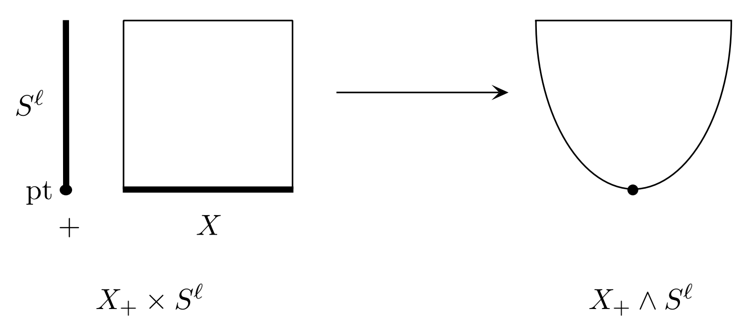
More generally one can easily prove the following theorem.

Theorem 8.17. $\Omega_k^{\mathrm{fr}}(X) = \pi_k^S(X_+)$.

The proof of this theorem is essentially the same as for $X = \mathrm{pt}$; one just has to carry the map $V \to X$ along for the ride. We give an outline of the argument and indicate a map $\pi_k^S(X_+) \to \Omega_k^{\mathrm{fr}}(X)$.

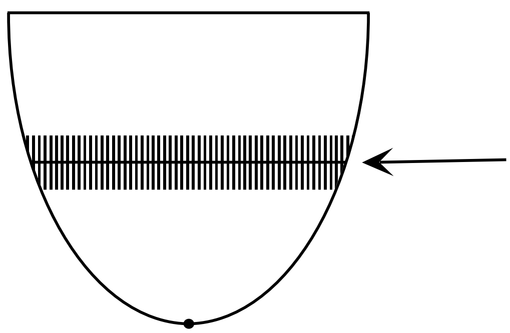
Sketch of proof. Choose ℓ large so that $\pi_k^S(X_+) = \pi_{k+\ell}(X_+ \wedge S^\ell)$.

The smash product $X_+ \wedge S^\ell = X_+ \times S^\ell / X_+ \vee S^\ell = X \times S^\ell / X \times \mathrm{pt}$ is called the *half smash* of X and S^ℓ and is depicted in the following picture.



Given $f : S^{k+\ell} \to X_+ \wedge S^\ell$, make f transverse to $X \times \{x\}$, where $x \in S^\ell$ is a point different from the base point. (You should think carefully about what transversality means since X is just a topological space. The point is that smoothness is only needed in the normal directions, since one can project to the sphere.)

Then $f^{-1}(X \times \{x\}) = V$ is a smooth, compact manifold, and since a neighborhood of $X \times \{x\}$ in $X_+ \wedge S^\ell$ is homeomorphic to $X \times \mathbf{R}^\ell$ as indicated in the following figure,



the submanifold V has a framed normal bundle, and $f_{|_V} : V \to X \times \{x\} = X$. This procedure shows how to associate a stably framed manifold with a map to X to a (stable) map $f : S^{k+\ell} \to X_+ \wedge S^\ell$. One can show as before, using the Pontrjagin-Thom construction, that the induced map $\pi_{k+\ell}(X_+ \wedge S^\ell) \to \Omega_k^{\text{fr}}(X)$ is an isomorphism. □

Exercise 134. Define the reverse map $\Omega_k^{\text{fr}}(X) \to \pi_k^S(X_+)$.

8.4. Spectra

The collection of spheres, $\{S^n\}_{n=0}^\infty$, together with the maps (in fact homeomorphisms)

$$k_n : SS^n \xrightarrow{\cong} S^{n+1}$$

forms a system of spaces and maps from which one can construct the stable homotopy groups $\pi_n^S(X)$. Another such system is the collection of Eilenberg–MacLane spaces $K(\mathbf{Z}, n)$ from which we can recover the cohomology groups by the identification $H^n(X; \mathbf{Z}) = [X, K(\mathbf{Z}, n)]$ according the the results of Chapter 7.

The notion of a spectrum abstracts from these two examples and introduces a category which measures "stable" phenomena, that is, phenomena which are preserved by suspending. Recall that $\widetilde{H}^n(X) = \widetilde{H}^{n+1}(SX)$ and by definition $\pi_n^S(X) = \pi_{n+1}^S(SX)$. Thus cohomology and stable homotopy groups are measuring stable information about a space X.

Definition 8.18. A *spectrum* is a sequence of pairs $\{K_n, k_n\}$ where the K_n are based spaces and $k_n : SK_n \to K_{n+1}$ are basepoint preserving maps, where SK_n denotes the suspension.

In Exercise 95 you saw that the the n-fold reduced suspension of S^nX of X is homeomorphic to $S^n \wedge X$. Thus we can rewrite the definition of stable homotopy groups as

$$\pi_n^S X = \lim_{\ell\to\infty} \pi_{n+\ell}(S^\ell \wedge X)$$

where the limit is taken over the homomorphisms

$$\pi_{n+\ell}(S^\ell \wedge X) \to \pi_{n+\ell+1}(S^{\ell+1} \wedge X).$$

These homomorphisms are composites of the suspension

$$\pi_{n+\ell}(S^\ell \wedge X) \to \pi_{n+\ell+1}(S(S^\ell \wedge X)),$$

the identification $S(S^\ell \wedge X) = S^1 \wedge (S^\ell \wedge X) = S(S^\ell) \wedge X$, and the map $\pi_{n+\ell+1}(S(S^\ell) \wedge X) \to \pi_{n+\ell+1}(S^{\ell+1} \wedge X)$ induced by the map $k_\ell : S(S^\ell) \to S^{\ell+1}$.

Thus we see a natural link between the sphere spectrum

$$\mathbf{S} = \{S^n, k_n : S(S^n) \cong S^{n+1}\}$$

and the stable homotopy groups

$$\pi_n^S(X) = \lim_{\ell\to\infty} \pi_{n+\ell}(S^\ell \wedge X).$$

Another example is provided by ordinary integral homology. The path space fibration and the long exact sequence in homotopy, shows that the loop space of the Eilenberg–MacLane space $K(\mathbf{Z}, n+1)$ is homotopy equivalent to $K(\mathbf{Z}, n)$. Fixing a model for $K(\mathbf{Z}, n)$ for each n, there exists a sequence of homotopy equivalences

$$h_n : K(\mathbf{Z}, n) \to \Omega K(\mathbf{Z}, n+1).$$

Then h_n defines, by taking its adjoint, a map

$$k_n : S(K(\mathbf{Z}, n)) \to K(\mathbf{Z}, n+1).$$

In this way we obtain the *Eilenberg–MacLane* spectrum

$$\mathbf{K}(\mathbf{Z}) = \{K(\mathbf{Z}, n), k_n\}.$$

We have seen in Theorem 7.22 that $H^n(X; \mathbf{Z}) = [X, K(\mathbf{Z}, n)]$.

Ordinary homology and cohomology are derived from the Eilenberg–MacLane spectrum, as the next theorem indicates. This point of view generalizes to motivate the definition of homology and cohomology with respect to any spectrum.

Theorem 8.19. *For any space X,*

1. $$H_n(X;\mathbf{Z}) = \lim_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge K(\mathbf{Z},\ell)).$$
2. $$H^n(X;\mathbf{Z}) = \lim_{\ell\to\infty} [S^\ell(X_+), K(\mathbf{Z}, n+\ell)]_0$$

□

Recall that for $n \geq 0$, $H^n(X) = \widetilde{H}^n(X_+) = \widetilde{H}^{n+1}(SX_+) = H^{n+1}(SX_+)$; in fact the diagram

$$\begin{array}{ccc} [X_+, K(\mathbf{Z},n)]_0 & \xrightarrow{S} & [SX_+, SK(\mathbf{Z},n)]_0 \\ \downarrow{\scriptstyle h_n \cong} & & \downarrow{\scriptstyle k_n} \\ [X_+, \Omega K(\mathbf{Z},n+1)]_0 & \xrightarrow{\cong} & [SX_+, K(\mathbf{Z},n+1)]_0 \end{array}$$

commutes. This shows that we could have *defined* the cohomology of a space by

$$H^n(X;\mathbf{Z}) = \lim_{\ell\to\infty} [S^\ell X_+; K(\mathbf{Z}, n+\ell)]_0,$$

and verifies the second part of this theorem. The first part can be proven by starting with this fact and using Spanier-Whitehead duality. See the project on Spanier-Whitehead duality at the end of this chapter.

These two examples and Theorem 8.19 leads to the following definition. Recall that X_+ denotes the space X with a disjoint base point. In particular, if $A \subset X$, then $(X_+/A_+) = X/A$ if A is non-empty and equals X_+ if A is empty.

Definition 8.20. Let $\mathbf{K} = \{K_n, k_n\}$ be a spectrum. Define the *(unreduced) homology and cohomology with coefficients in the spectrum* $\mathbf{K}$ to be the functor taking a space X to the abelian group

$$H_n(X;\mathbf{K}) = \lim_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge K_\ell)$$

and

$$H^n(X;\mathbf{K}) = \lim_{\ell\to\infty} [S^\ell(X_+); K_{n+\ell}]_0,$$

the *reduced homology and cohomology with coefficients in the spectrum* $\mathbf{K}$ to be the functor taking a based space X to the abelian group

$$\widetilde{H}_n(X;\mathbf{K}) = \lim_{\ell\to\infty} \pi_{n+\ell}(X \wedge K_\ell)$$

and

$$\widetilde{H}^n(X;\mathbf{K}) = \lim_{\ell\to\infty} [S^\ell X; K_{n+\ell}]_0,$$

and the *homology and cohomology of a pair with coefficients in the spectrum* **K** to be the functor taking a pair of space (X, A) to the abelian group

$$H_n(X, A; \mathbf{K}) = \lim_{\ell\to\infty} \pi_{n+\ell}((X_+/A_+) \wedge K_\ell)$$

and

$$H^n(X, A; \mathbf{K}) = \lim_{\ell\to\infty} [S^\ell(X_+/A_+); K_{n+\ell}]_0,$$

It is a theorem that these are *generalized (co)homology theories*; they satisfy all the Eilenberg–Steenrod axioms except the dimension axiom. We will discuss this in more detail later.

For example, stable homotopy theory $\widetilde{H}_n(X; \mathbf{S}) = \pi_n^S X$ is a reduced homology theory; framed bordism $H_n(X; \mathbf{S}) = \pi_n^S X_+ = \Omega_n^{\text{fr}}(X)$ is an unreduced homology theory.

Note that $H_n(\text{pt}; \mathbf{K})$ can be non-zero for $n \neq 0$, for example $H_n(\text{pt}; \mathbf{S}) = \pi_n^S$. Ordinary homology is characterized by the fact that $H_n(\text{pt}) = 0$ for $n \neq 0$, (see Theorem 1.31). The groups $H_n(\text{pt}; \mathbf{K})$ are called the *coefficients* of the spectrum.

There are many relationships between reduced homology, unreduced homology, suspension, and homology of pairs, some of which are obvious and some of which are not. We list some facts for homology.

- For a based space X, $\widetilde{H}_n(X; \mathbf{K}) = \widetilde{H}_{n+1}(SX; \mathbf{K})$.
- For a space X, $H_n(X; \mathbf{K}) = \widetilde{H}_n(X_+; \mathbf{K})$.
- For a pair of spaces, $H_n(X, A; \mathbf{K}) \cong \widetilde{H}_n(X/A; \mathbf{K})$.
- For a CW-pair, $H_n(X, A; \mathbf{K})$ fits into the long exact sequence of a pair.

8.5. More general bordism theories

(Stably) framed bordism is a special case of a general bordism theory, where one considers bordisms respecting some *specific stable structure* on the normal bundle of a smooth manifold. We will give examples of stable structures now, and then ask you to supply a general definition in Exercise 135. Basically a property of vector bundles is stable if whenever a bundle η has that property, then so does $\eta \oplus \varepsilon^k$ for all k.

8.5.1. Framing. A stable framing on a bundle $[\eta]$ is, as we have seen, a choice of homotopy class of bundle isomorphism

$$\gamma : \eta \oplus \varepsilon^k \cong \varepsilon^{n+k}$$

subject to the equivalence relation generated by the requirement that

$$\gamma \sim \gamma \oplus \text{Id} : \eta \oplus \varepsilon^k \oplus \varepsilon \cong \varepsilon^{n+k+1}.$$

8.5.2. The empty structure. This refers to bundles with no extra structure.

8.5.3. Orientation. This is weaker than requiring a framing. The most succinct way to define an orientation of a n-plane bundle η of is to choose a homotopy class of trivialization of the highest exterior power of the bundle,

$$\gamma : \wedge^n(\eta) \cong \varepsilon.$$

Equivalently, an orientation is a reduction of the structure group to $GL_+(n, \mathbf{R})$, the group of n-by-n matrices with positive determinant. A oriented manifold is a manifold with an orientation on its tangent bundle.

Since $\wedge^{a+b}(V \oplus W)$ is canonically isomorphic to $\wedge^a V \otimes \wedge^b W$ if V is a a-dimensional vector space and W is a b-dimensional vector space, it follows that $\wedge^n(\eta)$ is canonically isomorphic to $\wedge^{n+k}(\eta \oplus \varepsilon^k)$ for any $k \geq 0$. Thus an orientation on η induces one on $\eta \oplus \varepsilon$, so an orientation is a well-defined stable property.

8.5.4. Spin structure. Let $\mathrm{Spin}(n) \to SO(n)$ be the double cover where $\mathrm{Spin}(n)$ is connected for $n > 1$. A *spin structure* on an n-plane bundle η over a space M is a reduction of the structure group to $\mathrm{Spin}(n)$. This is equivalent to giving a principal $\mathrm{Spin}(n)$-bundle $P \to M$ and an isomorphism $\eta \cong (P \times_{\mathrm{Spin}(n)} \mathbf{R}^n \to M)$. A *spin manifold* is a manifold whose tangent bundle has a spin structure. Spin structures come up in differential geometry and index theory.

The stabilization map $SO(n) \to SO(n+1)$ induces a map $\mathrm{Spin}(n) \to \mathrm{Spin}(n+1)$. Thus a principal $\mathrm{Spin}(n)$-bundle $P \to M$ induces a principal $\mathrm{Spin}(n+1)$-bundle $P \times_{\mathrm{Spin}(n)} \mathrm{Spin}(n+1) \to M$, and hence a spin structure on η gives a spin structure on $\eta \oplus \varepsilon$. A spin structure is a stable property.

A framing on a bundle gives a spin structure. A spin structure on a bundle gives an orientation. It turns out that a spin structure is equivalent to a framing on the 2-skeleton of M.

8.5.5. Stable complex structure. An *complex structure* on a bundle η is a bundle map $J : \eta \to \eta$ so that $J \circ J = -\mathrm{Id}$. This forces the (real) dimension of η to be even. Equivalently complex structure is a reduction of the structure group to $GL(k, \mathbf{C}) \subset GL(2k, \mathbf{R})$. The tangent bundle of a complex manifold admits a complex structure. One calls a manifold with a complex structure on its tangent bundle an *almost complex manifold* and it may or may not admit the structure of a complex manifold. (It can be shown that S^6 is an almost complex manifold, but whether or not S^6 is a complex manifold is still an open question.)

One way to define a *stable complex structure* on a bundle η is as a section

$$J \in \Gamma(\mathrm{Hom}(\eta \oplus \varepsilon^k,\ \eta \oplus \varepsilon^k))$$

satisfying $J^2 = -\mathrm{Id}$ in each fiber. Given such a J, one can extend it canonically to

$$\widehat{J} = J \oplus i \in \Gamma(\mathrm{Hom}(\eta \oplus \varepsilon^k \oplus \varepsilon^{2\ell},\ \eta \oplus \varepsilon^k \oplus \varepsilon^{2\ell}))$$

by identifying $\varepsilon^{2\ell}$ with $M \times \mathbf{C}^\ell$ and using multiplication by i to define $i \in \Gamma(\mathrm{Hom}(M \times \mathbf{C}^\ell, M \times \mathbf{C}^\ell))$. As usual, two such structures are identified if they are homotopic. Note that odd-dimensional manifolds cannot have almost complex structures but may have stable almost complex structures.

If $\gamma : \eta \oplus \varepsilon^k \cong \varepsilon^\ell$ is a stable framing, up to equivalence we may assume that ℓ is even. Then identifying ε^ℓ with $M \times \mathbf{C}^{\ell/2}$ induces an stable complex structure on $\eta \oplus \varepsilon^k$. Thus stably framed bundles have a stable complex structure.

Similarly, a complex structure determines an orientation, since a complex vector space has a canonical (real) orientation. To see this, notice that if $\{e_1, \ldots, e_r\}$ is a complex basis for a complex vector space, then $\{e_1, ie_1, \cdots, e_r, ie_r\}$ is a real basis whose orientation class is independent of the choice of the basis $\{e_1, \ldots, e_r\}$.

The orthogonal group $O(n)$ is a strong deformation retract of the general linear group $GL(n, \mathbf{R})$; this can be shown using the Gram-Schmidt process. This leads to a one-to-one correspondence between isomorphism classes of vector bundles and isomorphism classes of $\mathbf{R}^n$-bundles with structure group $O(n)$ over a paracompact base space. An $\mathbf{R}^n$-bundle with a metric has structure group $O(n)$. Conversely an $\mathbf{R}^n$-bundle with structure group $O(n)$ over a connected base space admits a metric, uniquely defined up to scaling. Henceforth in this chapter all bundles will have metrics with orthogonal structure group.

The following exercise indicates how to define a structure on a stable bundle in general.

Exercise 135. Let $\mathbf{G} = \{G_n\}$ be a sequence of *topological groups* with continuous homomorphisms $G_n \to G_{n+1}$ and $G_n \to O(n)$ so that the diagram

$$\begin{array}{ccc} G_n & \to & G_{n+1} \\ \downarrow & & \downarrow \\ O(n) & \hookrightarrow & O(n+1) \end{array}$$

commutes for each n, where the injection $O(n) \to O(n+1)$ is defined by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

Use this to define a stable $\mathbf{G}$-structure on a bundle η. (Hint: either use classifying spaces or else consider the overlap functions for the stable bundle.)

Define what a homomorphism $\mathbf{G} \to \mathbf{G}'$ should be in such a way that a bundle with a stable $\mathbf{G}$-structure becomes a bundle with a stable $\mathbf{G}'$-structure.

There are many examples of $\mathbf{G}$-structures. As a perhaps unusual example, one could take G_n to be $O(n)$ or $SO(n)$ with the *discrete* topology. This spectrum arises in the study of flat bundles and algebraic K-theory.

For our previous examples, a framing corresponds to $G_n = 1$, the trivial group for all n. The empty structure corresponds to $G_n = O(n)$. An orientation corresponds to $G_n = SO(n) \subset O(n)$. A spin structure corresponds to $G_n = \mathrm{Spin}(n) \to SO(n)$. An stable complex structure corresponds to $G_n = U([n/2]) \subset O(n)$.

Concepts such as orientation and almost complex structure are more natural on the tangent bundle, while the Pontrjagin-Thom construction and hence bordism naturally deals with the stable normal bundle. The following exercise generalizes Theorem 8.13 and shows that in some cases one can translate back and forth.

Exercise 136.

1. Show that an orientation on the stable tangent bundle of a manifold determines one on the stable normal bundle and conversely.
2. Show that a complex structure on the stable tangent bundle of a manifold determines one on the stable normal bundle and conversely.

(Hint/discussion: The real point is that the tangent bundle and normal bundle are (stably) Whitney sum inverses, so one may as well consider bundles α and β over a finite-dimensional base space with a framing of $\alpha \oplus \beta$. A complex structure on α is classified a map to $G_n(\mathbf{C}^k)$ and β is equivalent to the pullback of the orthogonal complement of canonical bundle over the complex grassmannian, and hence β is equipped with a complex structure. Part 1 could be done using exterior powers or using the grassmannian of oriented n-planes in $\mathbf{R}^k$.)

Definition 8.21. Given a $\mathbf{G}$-structure, define the *n-th $\mathbf{G}$-bordism group of a space X* to be the $\mathbf{G}$-bordism classes of n-dimensional closed manifolds mapping to X with stable $\mathbf{G}$-structures on the normal bundle of an embedding of the manifold in a sphere. Denote this abelian group (with disjoint union as the group operation) by

$$\Omega_n^G(X).$$

Thus an element of $\Omega_n^G(X)$ is represented by an embedded closed submanifold $M^n \subset S^k$, a continuous map $f : M \to X$, and a stable $\mathbf{G}$-structure γ on the normal bundle $\nu(M \hookrightarrow S^k)$. Bordism is the equivalence relation generated by replacing k by $k+1$, and by

$$(M_0 \subset S^k, f_0, \gamma_0) \sim (M_1 \subset S^k, f_1, \gamma_1)$$

provided that there exists a compact manifold $W \subset S^k \times I$ with boundary $M_0 \times \{0\} \cup M_1 \times \{1\}$ (which we identify with $M_0 \amalg M_1$), a map $F : W \to X$ and a stable $\mathbf{G}$-structure Γ on $\nu(W \hookrightarrow S^k \times I)$ which restricts to $(M_0 \amalg M_1, f_0 \amalg f_1, \gamma_0 \amalg \gamma_1)$.

We previously used the notation Ω^{fr} for framed bordism, i.e. $\Omega^{\text{fr}} = \Omega^{\mathbf{1}}$ where $\mathbf{1} = \mathbf{G} = \{G_n\}$, the trivial group for all n.

We next want to associate spectra to bordism theories based on a stable structure. We have already seen how this works for framed bordism:

$$\Omega_n^{\text{fr}}(X) = \pi_n^S(X_+) = \lim_{\ell \to \infty} \pi_{n+\ell}(X_+ \wedge S^\ell)$$

i.e. framed bordism corresponds to the sphere spectrum $\mathbf{S} = \{S^n, k_n\}$.

What do the other bordism theories correspond to? Does there exist a spectrum $\mathbf{K}$ for each structure $\mathbf{G}$ so that

$$\Omega_n^G(X) = H_n(X; \mathbf{K}) = \lim_{\ell \to \infty} \pi_{n+\ell}(X_+ \wedge K_\ell)?$$

The answer is *yes*; the spectra for bordism theories are called *Thom spectra* $\mathbf{MG}$. In particular, one can define $\mathbf{G}$-*cobordism* by taking

$$H^n(X; \mathbf{MG}) = \lim_{\ell \to \infty} [S^\ell X_+; MG_{n+\ell}]_0.$$

We are using the algebraic topology terminology where *cobordism* is the theory dual (in the Spanier-Whitehead sense) to *bordism.* It is traditional for geometric topologists to call bordant manifolds "cobordant," but we will avoid this terminology in this book.

Thus we know that $\mathbf{M1}$ is the sphere spectrum. We will give a construction for $\mathbf{MG}$ for any structure $\mathbf{G}$.

8.6. Classifying spaces

The construction of Thom spectral is accomplished most easily via the theory of classifying spaces. The basic result about classifying spaces is the following. The construction and the proof of this theorem is one of the student projects for Chapter 4.

Theorem 8.22. *Given any topological group G, there exists a principal G-bundle $EG \to BG$ where EG is a contractible space. The construction is* *functorial, so that any continuous group homomorphism $\alpha : G \to H$ induces a bundle map*

$$\begin{array}{ccc} EG & \xrightarrow{E\alpha} & EH \\ \downarrow & & \downarrow \\ BG & \xrightarrow{B\alpha} & BH \end{array}$$

compatible with the actions, so that if $x \in EG, g \in G$,

$$E\alpha(x \cdot g) = (E\alpha(x)) \cdot \alpha(g).$$

The space BG is called a classifying space for G.

The function

$$\Phi : \mathit{Maps}(B, BG) \to \{\mathit{Principal}\ G\text{-}\mathit{bundles\ over}\ B\}$$

defined by pulling back (so $\Phi(f) = f^*(EG)$) induces a bijection *from the homotopy set $[B, BG]$ to the set of isomorphism classes of principal G-bundles over B, when B is a CW-complex (or more generally a paracompact space).* □

The long exact sequence for the fibration $G \to EG \to BG$ shows that $\pi_n BG = \pi_{n-1} G$. In fact, ΩBG is (weakly) homotopy equivalent to G, as one can see by taking the extended fiber sequence $\cdots \to \Omega EG \to \Omega BG \to G \to EG \to BG$, computing with homotopy groups, and observing that EG and ΩEG are contractible. Thus the space BG is a *delooping* of G.

The following lemma is extremely useful.

Lemma 8.23. *Let $p : E \to B$ be a principal G-bundle, and let $f : B \to BG$ be the classifying map. Then the homotopy fiber of f is weakly homotopy equivalent to E.*

Proof. Turn $f : B \to BG$ into a fibration $q : B' \to BG$ using Theorem 6.18 and let F' denote the homotopy fiber of $q : B' \to BG$. Thus there is a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle q} & \\ & BG & \end{array}$$

with h a homotopy equivalence. The fact that f is the classifying map for $p : E \to B$ implies that there is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & EG \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow{f} & BG \end{array}$$

and since EG is contractible, $f \circ p = q \circ h \circ p : E \to BG$ is nullhomotopic. By the homotopy lifting property for the fibration $q : B' \to BG$ it follows that $h \circ p : E \to B'$ is homotopic into the fiber F' of $q : B' \to BG$ and so one obtains a homotopy commutative diagram of spaces

$$\begin{array}{ccc} E & \longrightarrow & F' \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow{h} & B' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ BG & \xrightarrow{=} & BG. \end{array}$$

The left edge is a fibration, h is a homotopy equivalence, and by the five lemma the map $\pi_n(E) \to \pi_n(F')$ is an isomorphism for all n. □

In Lemma 8.23 one can usually conclude that the homotopy fiber of $f : B \to BG$ is in fact a homotopy equivalence. This would follow if we know that B' is homotopy equivalent to a CW-complex. This follows for most G by a theorem of Milnor [**24**].

Exercise 137. Show that given a principal G-bundle $E \to B$ there is fibration

$$\begin{array}{ccc} E & \longrightarrow & EG \times_G E \\ & & \downarrow \\ & & BG \end{array}$$

where $EG \times_G E$ denotes the Borel construction . How is this fibration related to the fibration of Lemma 8.23?

8.7. Construction of the Thom spectra

We proceed with the construction of the Thom spectra. We begin with a few preliminary notions.

Definition 8.24. If $E \to B$ is any vector bundle over a CW-complex B with metric then *the Thom space of* $E \to B$ is the quotient $D(E)/S(E)$, where $D(E)$ denotes the unit disk bundle of E and $S(E) \subset D(E)$ denotes the unit sphere bundle of E.

Notice that the zero section $B \to E$ defines an embedding of B into the Thom space.

The first part of the following exercise is virtually a tautology, but it is key to understanding why the spectra for bordism are given by Thom spaces.

Exercise 138.

1. If $E \to B$ is a smooth vector bundle over a smooth compact manifold B, then the Thom space of E is a smooth manifold away from one point and the 0-section embedding of B into the Thom space is a smooth embedding with normal bundle isomorphic to the bundle $E \to B$.
2. The Thom space of a vector bundle over a compact base is homeomorphic to the one-point compactification of the total space.

Now let a **G**-structure be given. Recall that this means we have a sequence of continuous groups G_n and homomorphisms $G_n \to O(n)$ and $G_n \to G_{n+1}$ such that the diagram

$$\begin{array}{ccc} G_n & \to & G_{n+1} \\ \downarrow & & \downarrow \\ O(n) & \hookrightarrow & O(n+1) \end{array}$$

commutes.

We will construct the Thom spectrum for this structure from the Thom spaces of vector bundles associated to the principal bundles $G_n \to EG_n \to BG_n$.

Composing the homomorphism $G_n \to O(n)$ with the standard action of $O(n)$ on $\mathbf{R}^n$ defines an action of G_n on $\mathbf{R}^n$. Use this action to form the universal $\mathbf{R}^n$-vector bundle over BG_n

$$\begin{array}{c} EG_n \times_{G_n} \mathbf{R}^n \\ \Big\downarrow \\ BG_n \end{array}$$

Let us denote this vector bundle by $V_n \to BG_n$. Notice that by our assumption that G_n maps to $O(n)$, this vector bundle has a metric, and so the unit sphere and disk bundles are defined.

Functoriality gives vector bundle maps (which are linear injections on fibers).

$$\begin{array}{ccc} V_n & \longrightarrow & V_{n+1} \\ \downarrow & & \downarrow \\ BG_n & \longrightarrow & BG_{n+1} \end{array}$$

Let MG_n denote the Thom space of $V_n \to BG_n$. Thus MG_n is obtained by collapsing the unit sphere bundle of V_n in the unit disk bundle to a point.

Lemma 8.25.

1. *If $E \to B$ is a vector bundle, then the Thom space of $E \oplus \varepsilon$ is the reduced suspension of the Thom space of E.*
2. *A vector bundle map*

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

which is an isomorphism preserving the metric on each fiber induces a map of Thom spaces.

Proof. To see why the first statement is true, note that an $O(n)$-equivariant homeomorphism $D^{n+1} \to D^n \times I$ determines an homeomorphism of $D(E\oplus\varepsilon)$ with $D(E) \times I$ which induces a homeomorphism $D(E \oplus \varepsilon)/S(E \oplus \varepsilon)$ with

$$(D(E) \times I)/(S(E) \times I \;\cup\; D(E) \times \{0, 1\}).$$

But it is easy to see that this identification space is the same as the (reduced) suspension of $D(E)/S(E)$.

The second statement is clear. □

The following theorem states that the collection $\mathbf{MG} = \{MG_n\}$ forms a spectrum, and that the corresponding homology theory is the bordism theory defined by the corresponding structure.

Theorem 8.26. *The fiberwise injection $V_n \to V_{n+1}$ extends to a (metric preserving) bundle map $V_n \oplus \varepsilon \to V_{n+1}$ which is an isomorphism on each fiber, and hence defines a map*

$$k_n : SMG_n \to MG_{n+1}.$$

Thus $\{MG_n, k_n\} = \mathbf{MG}$ is a spectrum, called the Thom spectrum.

Moreover, the bordism groups $\Omega_n^G(X)$ are isomorphic to $H_n(X;\mathbf{MG})$.

Proof. Since the diagram

$$\begin{array}{ccc} G_n & \to & G_{n+1} \\ \downarrow & & \downarrow \\ O(n) & \hookrightarrow & O(n+1) \end{array}$$

commutes, where $O(n) \hookrightarrow O(n+1)$ is the homomorphism

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix},$$

it follows by the construction of V_n that the pullback of V_{n+1} by the map $\gamma_n : BG_n \to BG_{n+1}$ splits canonically into a direct sum $\gamma_n^*(V_{n+1}) = V_n \oplus \varepsilon$. Thus the diagram

$$\begin{array}{ccc} V_n & \longrightarrow & V_{n+1} \\ \downarrow & & \downarrow \\ BG_n & \longrightarrow & BG_{n+1} \end{array}$$

extends to a diagram

$$\begin{array}{ccc} V_n \oplus \varepsilon & \longrightarrow & V_{n+1} \\ \downarrow & & \downarrow \\ BG_n & \longrightarrow & BG_{n+1} \end{array}$$

which is an isomorphism on each fiber; this isomorphism preserves the metrics since the actions are orthogonal.

By Lemma 8.25, the above bundle map defines a map

$$k_n : SMG_n \to MG_{n+1}$$

establishing the first part of the theorem.

We now outline how to establish the isomorphism

$$\Omega_n^G(X) = \lim_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge MG_\ell).$$

This is a slightly more complicated version of the Pontrjagin-Thom construction we described before, using the basic property of classifying spaces.

We will first define the collapse map

$$c : \Omega_n^G(X) \to \lim_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge MG_\ell).$$

Suppose $[W, f, \gamma] \in \Omega_n^G(X)$. So W is an n-manifold with a $\mathbf{G}$-structure on its stable normal bundle, and $f : W \to X$ is a continuous map. Embed W in $S^{n+\ell}$ for some large ℓ so that the normal bundle $\nu(W)$ has a G_ℓ-structure.

Let $F \to W$ be the principal $O(\ell)$-bundle of orthonormal frames in $\nu(W)$. The statement that $\nu(W)$ has a G_ℓ-structure is equivalent to saying that there is a principal G_ℓ-bundle $P \to W$ and a bundle map

$$\begin{array}{ccc} P & \longrightarrow & F \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

which is equivariant with respect to the homomorphism

$$G_\ell \to O(\ell).$$

Let $c_1 : W \to BG_\ell$ classify the principal bundle $P \to W$. Then by definition $\nu(W)$ is isomorphic to the pullback $c_1^*(V_\ell)$.

Let U be a tubular neighborhood of W in $S^{n+\ell}$ and $D \subset U \subset S^{n+\ell}$ correspond to the disk bundle. Define a map

$$h : S^{n+\ell} \to MG_\ell$$

by taking everything outside of D to the base point, and on D, take the composite

$$D \cong D(\nu(W)) \to D(V_\ell) \to MG_\ell.$$

The product

$$f \times h : S^{n+\ell} \to X \times MG_\ell$$

composes with the collapse

$$X \times MG_\ell \to X_+ \wedge MG_\ell$$

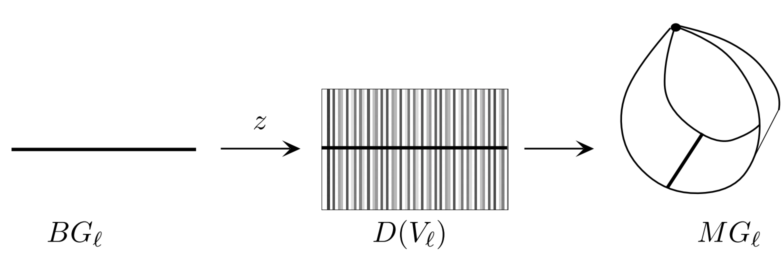
to give a map

$$\alpha = f \wedge h : S^{n+\ell} \to X_+ \wedge MG_\ell.$$

We have thus defined the collapse map

$$c : \Omega_n^G(X) \to \lim_{\ell \to \infty} \pi_{n+\ell}(X_+ \wedge MG_\ell) = H_n(X; \mathbf{MG}).$$

To motivate the definition of the inverse of c, we will make a few comments on the above construction. The figure below illustrates that the composite of the zero section $z : BG_\ell \to D(V_\ell)$ and the quotient map $D(V_\ell) \to MG_\ell$ is a embedding.



We thus will consider BG_ℓ to be a subset of MG_ℓ. Then in the above construction of the collapse map c, $W = \alpha^{-1}(X \times BG_\ell)$.

Next we use transversality to define the inverse of this the collapse map c. Represent $\hat{\alpha} \in H_n(X; \mathbf{MG})$ by

$$\alpha : S^{n+\ell} \to X_+ \wedge MG_\ell.$$

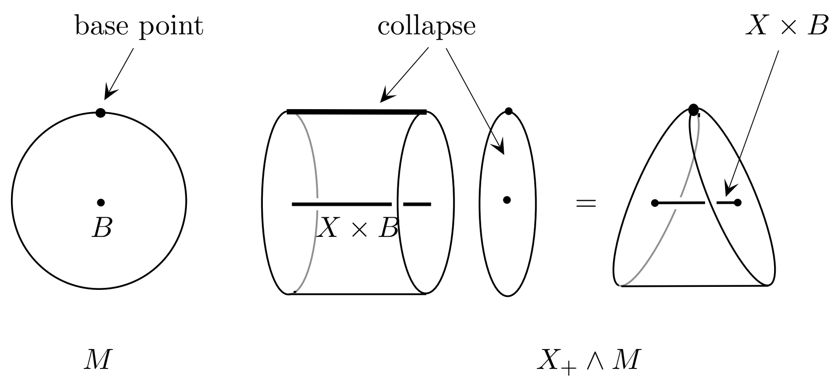
Observe that the composite

$$X \times BG_\ell \hookrightarrow X_+ \times MG_\ell \to X_+ \wedge MG_\ell$$

is an embedding, since:

1. BG_ℓ misses the base point of MG_ℓ, and
2. the base point of X_+ misses X.

(The following figure gives an analogue by illustrating the embedding of $X \times B$ in $X_+ \wedge M$ if B is a point, M is a D^2-bundle over B, and X is a interval.)



Furthermore

$$X \times BG_\ell \subset X_+ \wedge MG_\ell$$

has a neighborhood which is isomorphic to the pullback $\pi_2^* V_\ell$ where $\pi_2 : X \times BG_\ell \to BG_\ell$ is the projection on the second factor. Transversality, adapted to this setting, says that $\alpha : S^{n+\ell} \to X_+ \wedge MG_\ell$ can be homotoped slightly to a map β so that $W = \beta^{-1}(X \times BG_\ell)$ is a smooth manifold, and whose tubular neighborhood, i.e. the normal bundle of W, has a **G**-structure. The composite of $\beta : W \to X \times BG_\ell$ and $\mathrm{pr}_1 : X \times BG_\ell \to X$ give the desired element $(W \to X) \in \Omega_n^G(X)$.

We sort of rushed through the construction of the inverse map to c, so we will backtrack and discuss some details. For every point in BG_ℓ, there is a neighborhood $U \subset BG_\ell$ over which the bundle $V_\ell \to BG_\ell$ is trivial and so there is a map

$$\alpha^{-1}(X \times U) \to D^\ell / S^\ell$$

defined by composing α with projection on the fiber. Transversality then applies to this map between manifolds and one can patch together to get β using partitions of unity. Furthermore, transversality gives a diagram of bundle maps, isomorphisms in each fiber,

$$\begin{array}{ccccc} \nu(W \hookrightarrow S^{n+\ell}) & \longrightarrow & X \times V_\ell & \longrightarrow & V_\ell \\ \downarrow & & \downarrow & & \downarrow \\ W & \xrightarrow{\beta} & X \times BG_\ell & \xrightarrow{\mathrm{pr}_2} & BG_\ell \end{array}$$

so that the normal bundle of W inherits a $\mathbf{G}$-structure.

Next note that replacing ℓ by $\ell + 1$ leads to the same bordism element. Stabilizing the normal bundle

$$\nu(W \hookrightarrow S^{n+\ell}) \longrightarrow \nu(W \hookrightarrow S^{n+\ell}) \oplus \varepsilon = \nu(W \hookrightarrow S^{n+\ell+1})$$

corresponds to including $W \subset S^{n+\ell} \subset S^{n+\ell+1}$. Since the composite

$$SS^{n+\ell} \xrightarrow{Sf} S(X_+ \wedge MG_\ell) \xrightarrow{k_\ell \wedge Id} X_+ \wedge MG_{\ell+1}$$

replaces the tubular neighborhood of $X \times BG_\ell$, i.e. $X \times V_\ell$ by $X \times (V_\ell \oplus \varepsilon)$, the construction gives a well-defined stable $\mathbf{G}$-structure on the stable normal bundle of W.

The full proof that the indicated map $H_n(X; \mathbf{MG}) \to \Omega_n^G(X)$ is well-defined and is the inverse of c is a careful but routine check of details involving bordisms, homotopies, and stabilization. □

Taking X to be a point, we see that the groups (called the coefficients) $\Omega_n^G = \Omega_n^G(\mathrm{pt})$ are isomorphic to the homotopy groups $\lim\limits_{\ell \to \infty} \pi_{n+\ell}(MG_\ell)$, since $\mathrm{pt}_+ \wedge M = M$.

As an example of how these coefficients can be understood geometrically, consider oriented bordism, corresponding to $G_n = SO(n)$. The coefficients Ω_n^{SO} equal $\pi_{n+\ell}(MSO_\ell)$ for ℓ large enough. Some basic computations are the following.

1. An oriented closed 0-manifold is just a signed finite number of points. This bounds a 1-manifold if and only if the sum of the signs is zero. Hence $\Omega_0^{SO} \cong \mathbf{Z}$. Also, $\pi_\ell MSO_\ell = \mathbf{Z}$ for $\ell \geq 2$.
2. Every oriented closed 1-manifold bounds an oriented 2-manifold, since $S^1 = \partial D^2$. Therefore $\Omega_1^{SO} = 0$.

3. Every oriented 2-manifold bounds an oriented 3-manifold since any oriented 2-manifold embeds in $\mathbf{R}^3$ with one of the two complementary components compact. Thus $\Omega_2^{SO} = 0$.

4. A theorem of Rohlin states that every oriented 3-manifold bounds a 4-manifold. Thus $\Omega_3^{SO} = 0$.

5. An oriented 4-manifold has a signature in $\mathbf{Z}$, i.e. the signature of its intersection form. A good exercise using Poincaré duality (see the projects for Chapter 3) shows that this is an oriented bordism invariant, and hence defines a homomorphism $\Omega_4^{SO} \to \mathbf{Z}$. This turns out to be an isomorphism. More generally the signature defines a map $\Omega_{4k}^{SO} \to \mathbf{Z}$ for all k. This is a surjection since the signature of $\mathbf{C}P^{2k}$ is 1.

6. It is a fact that away from multiples of 4, the oriented bordism groups are torsion, i.e. $\Omega_n^{SO} \otimes \mathbf{Q} = 0$ if $n \neq 4k$.

7. For all n, Ω_n^{SO} is finitely generated, in fact, a finite direct sum of $\mathbf{Z}$'s and $\mathbf{Z}/2$'s.

Statements 5, 6, and 7 can be proven by computing $\pi_{n+\ell}(MSO_\ell)$. How does one do this? A starting point is the *Thom isomorphism theorem*, which says that for all k,

$$H_n(BSO(\ell)) \cong \tilde{H}_{n+\ell}(MSO_\ell)$$

(where $\tilde{H}$ denotes reduced cohomology). The cohomology of $BSO(n)$ can be studied in several ways, an so one can obtain information about the cohomology of MSO_ℓ by this theorem. Combining this with the Hurewicz theorem and other methods leads ultimately to a complete computation of oriented bordism (due to C.T.C. Wall), and this technique was generalized by Adams to a machine called the Adams spectral sequence. We will return to the Thom isomorphism theorem in Chapter 10.

Once the coefficients are understood, one can use the fact that bordism is a homology theory to compute $\Omega_n^{SO}(X)$. For now we just remark that there is a map $\Omega_n^{SO}(X) \to H_n(X)$ defined by taking $f : M \to X$ to the image of the fundamental class $f_*[M]$. Thus for example, the identity map on a closed, oriented manifold M^n is non-zero in $\Omega_n^{SO}(M)$.

We can also make an elementary remark about unoriented bordism, which corresponds to $G_n = O(n)$. Notice first that for any $\alpha \in \Omega_n^O(X)$, $2\alpha = 0$. Indeed, if $f : V^n \to X$ represents α, take $F : V \times I \to X$ to be $F(x,t) = f(x)$ then $\partial(V \times I, F) = 2(V, f)$. Thus $\Omega_n^O(X)$ consists only of elements of order 2. The full computation of unoriented bordism is due to Thom. We will discuss this more in Section 10.10.

Exercise 139. Show that $\Omega_0^O = \mathbf{Z}/2$, $\Omega_1^O = 0$, and $\Omega_2^O = \mathbf{Z}/2$. (Hint: for Ω_2^O use the classification theorem for closed surfaces, then show that if a surface F is a boundary of a 3-manifold, then $\dim H^1(F;\mathbf{Z}/2)$ is even.)

There are several conventions regarding notation for bordism groups; each has its advantages. Given a structure defined by a sequence $\mathbf{G} = \{G_n\}$, one can use the notation

$$\Omega_*^G(X), \quad H_*(X;\mathbf{MG}) \text{ or } MG_*(X).$$

There is a generalization of a $\mathbf{G}$-structure called a $\mathbf{B}$-structure. It is given by a sequence of commutative diagrams

$$\begin{array}{ccc} B_n & \longrightarrow & B_n \\ \Big\downarrow{\scriptstyle \xi_n} & & \Big\downarrow{\scriptstyle \xi_{n+1}} \\ BO_n & \longrightarrow & BO_{n+1} \end{array}$$

where the vertical maps are fibrations. A $\mathbf{G}$-structure in the old sense gives a $\mathbf{BG} = \{BG_n\}$-structure. A $\mathbf{B}$-structure has a Thom spectrum $\mathbf{TB} = \{T(\xi_n)\}$, where ξ_n here denotes the vector bundle pulled back from the canonical bundle over BO_n. There is a notion of a stable $\mathbf{B}$-structure on a normal bundle of an embedded M, which implies that there is a map from the (stablized) normal bundle to ξ_k. There is a Pontrjagin-Thom isomorphism

$$\Omega_n^B(X) \cong H_n(X;\mathbf{TB}).$$

For a precise discussion of $\mathbf{B}$-bordism and for further information on bordism in general, see [**30**], [**39**] and the references therein.

8.8. Generalized homology theories

We have several functors from (based) spaces to graded abelian groups: stable homotopy $\pi_n^S(X)$, bordism $\Omega_n^G(X)$, or, more generally, homology of a space with coefficients in a spectrum $H_n(X;\mathbf{K})$. These are examples of *generalized homology theories*. Generalized homology theories come in two (equivalent) flavors, *reduced* and *unreduced*. Unreduced theories apply to unbased spaces and pairs. Reduced theories are functors on based spaces. The equivalence between the two points of view is obtained by passing from (X,A) to X/A and from X to X_+.

There are three high points to look out for in our discussion of homology theories.

- The axioms of a (co)homology theory are designed for computations. One first computes the coefficients of the theory (perhaps using the

Adams spectral sequence), and then computes the homology of a CW-complex X, using excision, Mayer–Vietoris, or a generalization of cellular homology discussed in the next chapter, the Atiyah–Hirzebruch spectral sequence.

- There is a uniqueness theorem. A natural transformation of (co)-homology theories inducing an isomorphism on coefficients induces an isomorphism for all CW-complexes X.
- A (co)homology theory is given by (co)homology with coefficients in a spectrum $\mathbf{K}$.

8.8.1. Reduced homology theories. Let $\mathcal{K}_*$ be the category of compactly generated spaces with non-degenerate base points.

Definition 8.27. A *reduced homology theory* is

1. A family of covariant functors
$$h_n : \mathcal{K}_* \to \mathcal{A} \text{ for } n \in \mathbf{Z}$$
where $\mathcal{A}$ denotes the category of abelian groups. (Remark: we *do not* assume h_n is zero for $n < 0$.)
2. A family of natural transformations
$$e_n : h_n \to h_{n+1} \circ \mathcal{S}$$
where $S : \mathcal{K}_* \to \mathcal{K}_*$ is the (reduced) suspension functor.

These must satisfy the three following axioms:

A1. *(Homotopy)* If $f_0, f_1 : X \to Y$ are homotopic, then
$$h_n(f_0) = h_n(f_1) : h_n(X) \to h_n(Y)$$
A2. *(Exactness)* For $f : X \to Y$, let C_f be the mapping cone of f, and $j : Y \hookrightarrow C_f$ the inclusion. Then
$$h_n(X) \xrightarrow{h_n(f)} h_n(Y) \xrightarrow{h_n(j)} h_n(C_f)$$
is exact for all $n \in \mathbf{Z}$.
A3. *(Suspension)* The homomorphism
$$e_n(X) : h_n(X) \to h_{n+1}(SX)$$
given by the natural transformation e_n is an isomorphism for all $n \in \mathbf{Z}$.

Exercise 140. Show that ordinary singular homology defines a homology theory in this sense by taking $h_n(X)$ to be the reduced homology of X.

There are two other "nondegeneracy" axioms which a given generalized homology theory or may not satisfy.

A4. *(Additivity)* If X is a wedge product $X = \bigvee_{j\in J} X_j$, then

$$\bigoplus_{j\in J} h_n(X_j) \to h_n(X)$$

is an isomorphism for all $n \in \mathbf{Z}$.

A5. *(Isotropy)* If $f : X \to Y$ is a weak homotopy equivalence, then $h_n(f)$ is an isomorphism for all $n \in \mathbf{Z}$.

If we work in the category of based CW-complexes instead of $\mathcal{K}_*$, then A5 follows from A1 by the Whitehead theorem. Given a reduced homology theory on based CW-complexes, it extends uniquely to an isotropic theory on $\mathcal{K}_*$.

For any reduced homology theory, $h_n(\text{pt}) = 0$ for all n, since

$$h_n(\text{pt}) \to h_n(\text{pt}) \to h_n(\text{pt}/\text{pt}) = h_n(\text{pt})$$

is exact, but also each arrow is an isomorphism. Thus the reduced homology of a point says nothing about the theory; instead one makes the following definition.

Definition 8.28. The *coefficients* of a reduced homology theory are the groups $\{h_n(S^0)\}$.

A homology theory is called *ordinary* (or *proper*) if it satisfies

$$h_n(S^0) = 0 \text{ for } n \neq 0.$$

(This is the dimension axiom of Eilenberg–Steenrod.) Singular homology with coefficients in an abelian group A is an example of a ordinary theory. It follows from a simple argument using the Atiyah-Hirzebruch spectral sequence that any ordinary reduced homology theory is isomorphic to reduced singular homology with coefficients in $A = h_0(S^0)$.

If (X, A) is an NDR pair, then we saw in Chapter 6 that the mapping cone C_f is homotopy equivalent to X/A. Thus $h_n(A) \to h_n(X) \to h_n(X/A)$ is exact. Also in Chapter 6 we proved that the sequence

$$A \to X \to X/A \to SA \to SX \to S(X/A) \to \cdots$$

has each three term sequence a (homotopy) cofibration. Thus

$$h_n(A) \to h_n(X) \to h_n(X/A) \to h_n(SA) \to h_n(SX) \to \cdots$$

is exact. Applying the transformations e_n and using Axiom A3 we conclude that

$$\to h_n(A) \to h_n(X) \to h_n(X/A) \to h_{n-1}(A) \to h_{n-1}(X) \to \cdots$$

is exact. Thus to any reduced homology theory one obtains a long exact sequence associated to a cofibration.

Exercise 141. Let X be a based CW-complex with subcomplexes A and B, both of which contain the base point. Show that for any reduced homology theory h_* there is a Mayer–Vietoris long exact sequence

$$\cdots \to h_n(A\cap B) \to h_n(A)\oplus h_n(B) \to h_n(X) \to h_{n-1}(A\cap B) \to \cdots$$

8.8.2. Unreduced homology theories. We will derive unreduced theories from reduced theories, to emphasize that these are the same concept, presented slightly differently.

Let $\mathcal{K}^2$ denote the category of NDR pairs (X, A), allowing the case when A is empty. Given a reduced homology theory $\{h_n, e_n\}$ define functors H_n on $\mathcal{K}^2$ as follows (for this discussion, H_n *does not* denote ordinary singular homology!).

1. Let
$$H_n(X,A) = h_n(X_+/A_+) = \begin{cases} h_n(X/A) & \text{if } A \neq \phi, \\ h_n(X_+) & \text{if } A = \phi \end{cases}$$
2. Let $\partial_n : H_n(X, A) \to H_{n-1}(A)$ be the composite:
$$H_n(X,A) = h_n(X_+/A_+) \xrightarrow{\cong} h_n(C_i) \longrightarrow h_n(SA_+) \xrightarrow{\cong} h_{n-1}(A_+) = H_{n-1}(A)$$
where C_i is the mapping cone of the inclusion $i : A_+ \hookrightarrow X_+$, and $C_i \to SA_+$ is the quotient
$$C_i \to C_i/X_+ = SA_+.$$

Then $\{H_n, \partial_n\}$ satisfy the Eilenberg–Steenrod axioms:

A1. *(Homotopy)* If $f_0, f_1 : (X, A) \to (Y, B)$ are (freely) homotopic then
$$H_n(f_0) = H_n(f_1) : H_n(X,A) \to H_n(Y,B)$$

A2. *(Exactness)* For a cofibration $i : A \hookrightarrow X$, let $j : (X, \phi) \hookrightarrow (X, A)$, then
$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \to \cdots$$
is exact.

A3. *(Excision)* Suppose that $X = A \cup B$, with A, B closed, and suppose that $(A, A \cap B)$ is an NDR pair. Then
$$H_n(A, A\cap B) \to H_n(X,B)$$
is an isomorphism for all $n \in \mathbf{Z}$.

Exercise 142. Prove that these three properties hold using the axioms of a reduced theory.

If a reduced theory is additive and/or isotropic, the functors H_n likewise satisfy

A4. *(Additivity)* Let $X = \amalg_{j \in J} X_j,\ A \subset X,\ A_j = X_j \cap A$. Then

$$\bigoplus_{j \in J} H_n(X_j, A_j) \to H_n(X, A)$$

is an isomorphism for all $n \in \mathbf{Z}$.

A5. *(Isotropy)* If $f : X \to Y$ is a weak homotopy equivalence, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism for all $n \in \mathbf{Z}$.

Notice that if the reduced theory is ordinary, then $H_n(\text{pt}) = 0$ for $n \neq 0$.

One uses these properties to define an unreduced homology theory.

Definition 8.29. A collection of functors $\{H_n, \partial_n\}$ on $\mathcal{K}^2$ is called a *(unreduced) homology theory* if it satisfies the three axioms A1, A2, and A3. It is called *additive* and/or *isotropic* if A4 and/or A5 hold. It is called *ordinary* or *proper* if $H_n(\text{pt}) = 0$ for $n \neq 0$.

The *coefficients* of the unreduced homology theory are the groups $\{H_n(\text{pt})\}$.

One can go back and forth: an unreduced homology theory $\{H_n, \partial_n\}$ defines a reduced one by taking $h_n(X) = H_n(X, \{*\})$. The following theorem is proved in [**43**, Section XII.6].

Theorem 8.30. *These constructions set up a* $1-1$ *correspondence (up to natural isomorphism) between reduced homology theories on* $\mathcal{K}_*$ *and (unreduced) homology theories on* $\mathcal{K}^2$*. Moreover the reduced theory is additive, isotropic, or ordinary if and only if the corresponding unreduced theory is.*
□

The uniqueness theorem below has an easy inductive cell-by-cell proof in the case of finite CW-complexes, but requires a more delicate limiting argument for infinite CW-complexes.

Theorem 8.31 (Eilenberg–Steenrod uniqueness theorem)**.**

1. *Let* $T : (H_n, \partial_n) \to (H'_n, \partial'_n)$ *be a natural transformation of homology theories defined on the category of finite CW-pairs such that* $T : H_*(\text{pt}) \to H'_*(\text{pt})$ *is an isomorphism. Then* $T : H_*(X, A) \to H'_*(X, A)$ *is an isomorphism for all finite CW-pairs.*
2. *Let* $T : (H_n, \partial_n) \to (H'_n, \partial'_n)$ *be a natural transformation of additive homology theories defined on the category of CW-pairs where* $T : H_*(\text{pt}) \to H'_*(\text{pt})$ *is an isomorphism. Then* $T : H_*(X, A) \to H'_*(X, A)$ *is an isomorphism for all CW-pairs.*

8.8.3. Homology theories and spectra.

Theorem 8.32. *(Reduced) homology with coefficients in a spectrum* $\mathbf{K}$

$$\widetilde{H}_n(-;\mathbf{K}) : X \mapsto \lim_{\ell\to\infty} \pi_{n+\ell}(X \wedge K_\ell)$$
$$H_n(-;\mathbf{K}) : (X,A) \mapsto \lim_{\ell\to\infty} \pi_{n+\ell}((X_+/A_+) \wedge K_\ell)$$

is a (reduced) homology theory satisfying the additivity axiom.

One needs to prove the axioms A1, A2, A3, and A5. The homotopy axiom is of course obvious. The axiom A2 follows from the facts about the Puppe sequences we proved in Chapter 6 by passing to the limit. The suspension axiom holds almost effortlessly from the fact that the theory is defined by taking the direct limit over suspension maps. The additivity axiom follows from the fact that the image of a sphere is compact and that a compact subspace of an infinite wedge is contained in a finite wedge.

A famous theorem of E. Brown (the Brown representation theorem) gives a converse of the above theorem. It leads to a shift in perspective on the functors of algebraic topology by prominently placing spectra as the source of homology theories. Here is a precise statement.

Theorem 8.33.

1. *Let* $\{H_n, \partial_n\}$ *be an homology theory. There there exists a spectrum* $\mathbf{K}$ *and a natural isomorphism* $H_n(X,A) \cong H_n(X,A;\mathbf{K})$ *for all finite CW-pairs.*
2. *Let* $\{H_n, \partial_n\}$ *be an additive homology theory. There there exists a spectrum* $\mathbf{K}$ *and a natural isomorphism* $H_n(X,A) \cong H_n(X,A;\mathbf{K})$ *for all CW-pairs.*

We have seen several examples: an ordinary homology theory corresponds to the Eilenberg–Maclane spectrum $\mathbf{K}(A)$, stable homotopy corresponds to the sphere spectrum $\mathbf{S}$, and the bordism theories correspond to Thom spectra. Note that the Brown representation theorem shows that for any homology theory, there is a spectrum, and hence an associated generalized cohomology theory.

Exercise 143. Give a definition of a map of spectra. Define maps of spectra $\mathbf{S} \to \mathbf{K}(\mathbf{Z})$ and $\mathbf{S} \to \mathbf{MG}$ inducing the Hurewicz map $\pi_n^S(X) \to \widetilde{H}_n(X)$ and the map $\Omega_n^{\text{fr}}(X) \to \Omega_n^G(X)$ from framed to G-bordism.

8.8.4. Generalized cohomology theories. The development of cohomology theories parallels that of homology theories following the principle of reversing arrows.

Exercise 144. Define reduced and unreduced *cohomology* theories.

There is one surprise however. In order for $H^n(\quad;\mathbf{K})$ to be an additive theory (which means the cohomology of a disjoint union is a direct *product*), one must require that $\mathbf{K}$ is an Ω-*spectrum*, a spectrum so that the adjoints

$$K_n \to \Omega K_{n+1}$$

of the structure maps k_n are homotopy equivalences. Conversely, the Brown representation theorem applied to an additive cohomology theory produces an Ω-spectrum. The Eilenberg–MacLane spectrum is an Ω-spectrum while the sphere spectrum or more generally bordism spectra are not.

An important example of a generalized cohomology theory is topological K-theory. It is the subject of one of the projects at the end of this chapter. Complex topological K-theory has a definition in terms of stable equivalence classes of complex vector bundles, but we instead indicate the definition in terms of a spectrum. Most proofs of the Bott periodicity theorem (Theorem 6.51, which states that $\pi_n U \cong \mathbf{Z}$ for n odd and $\pi_n U = 0$ for n even), actually prove a stronger result, that there is a homotopy equivalence

$$\mathbf{Z} \times BU \simeq \Omega^2(\mathbf{Z} \times BU).$$

This allows the definition of the complex K-theory spectrum with

$$K_n = \begin{cases} \mathbf{Z} \times BU & \text{if } n \text{ is even,} \\ \Omega(\mathbf{Z} \times BU) & \text{if } n \text{ is odd.} \end{cases} \tag{8.1}$$

The structure maps k_n

$$S(\mathbf{Z} \times BU) \to \Omega(\mathbf{Z} \times BU)$$
$$S\Omega(\mathbf{Z} \times BU) \to \mathbf{Z} \times BU$$

are given by the adjoints of the Bott periodicity homotopy equivalence and the identity map

$$\mathbf{Z} \times BU \to \Omega^2(\mathbf{Z} \times BU)$$
$$\Omega(\mathbf{Z} \times BU) \to \Omega(\mathbf{Z} \times BU).$$

Thus the complex K-theory spectrum is an Ω-spectrum. The corresponding cohomology theory is called complex K-theory and satisfies

$$K^n(X) = K^{n+2}(X) \quad \text{for all } n \in \mathbf{Z}.$$

In particular this is a non-connective cohomology theory, where a *connective* cohomology theory is one that satisfies $H^n(X) = 0$ for all $n < n_0$. Ordinary homology, as well as bordism theories, are connective, since a manifold of negative dimension is empty.

A good reference for the basic results in the study of spectra (stable homotopy theory) is Adams' book [**2**].

8.9. Projects for Chapter 8

8.9.1. Basic notions from differential topology. Define a smooth manifold and submanifold, the tangent bundle of a smooth manifold, a smooth map between manifolds and its differential, an isotopy, the Sard theorem, transversality, the tubular neighborhood theorem, the decomposition

$$TM|_P = TP \oplus \nu(P \hookrightarrow M),$$

where $P \subset M$ is a smooth submanifold, and show that if $f : M \to N$ is a smooth map transverse to a submanifold $Q \subset N$, with $P = f^{-1}(Q)$, then the differential of f induces a bundle map $df : \nu(P \hookrightarrow M) \to \nu(Q \hookrightarrow n)$ which is an isomorphism in each fiber. A good reference is Hirsch's book [**16**].

8.9.2. Definition of K-theory. Define the complex (topological) K-theory of a space in terms of vector bundles. Indicate why the spectrum for this theory is $\{K_n\}$ given in Equation (8.1). State the Bott periodicity theorem. Discuss vector bundles over spheres. Discuss real K-theory. References for this material are the books by Atiyah [**3**] and Husemoller [**17**].

8.9.3. Spanier-Whitehead duality. Spanier-Whitehead duality is a generalization of Alexander duality which gives a geometric method of going back and forth between a generalized homology theory and a generalized cohomology theory. Suppose that $X \subset S^{n+1}$ is a finite simplicial complex, and let $Y = S^{n+1} - X$, or better, $Y = S^{n+1} - U$ where U is some open simplicial neighborhood of X which deformation retracts to X. Recall that Alexander duality implies that

$$\widetilde{H}^p(X) \cong \widetilde{H}_{n-p}(Y).$$

(See Theorem 3.26.) What this means is that the cohomology of X determines the homology of Y and vice versa.

The strategy is to make this work for generalized cohomology theories and any space X, and to remove the dependence on the embedding. The best way to do this is to do it carefully using spectra. Look at Spanier's article [**35**]. There is a good sequence of exercises developing this material in [**36**, pages 462-463]. Another reference using the language of spectra is [**39**, page 321].

Here is a slightly low-tech outline. You should lecture on the following, providing details.

Given based spaces X and Y, let

$$\{X, Y\} = \lim_{k\to\infty} [S^k \wedge X, S^k \wedge Y]_0.$$

Given a finite simplicial subcomplex $X \subset S^{n+1}$, let $D_nX \subset S^{n+1}$ be a finite simplicial subcomplex which is a deformation retract of $S^{n+1} - X$. Then SD_nX is homotopy equivalent to $S^{n+2} - X$.

For k large enough, the homotopy type of the suspension S^kD_nX depends only on X and $k+n$, and not on the choice of embedding into S^{n+1}. Moreover, for any spaces Y and Z

$$\{S^qY, D_nX \wedge Z\} = \{S^{q-n}Y \wedge X, Z\} \tag{8.2}$$

As an example, taking $Y = S^0$ and $Z = K(\mathbf{Z}, p+q-n)$, Equation (8.2) says that

$$\{S^q, D_nX \wedge K(\mathbf{Z}, p+q-n)\} = \{S^{q-n} \wedge X, K(\mathbf{Z}, p+q-n)\}. \tag{8.3}$$

Definition 8.20 says that the left side of Equation (8.3) is $\widetilde{H}_{n-p}(D_nX; \mathbf{K}(\mathbf{Z}))$. The right side is $\widetilde{H}^p(X; \mathbf{Z})$, using the fact that $[SA, K(\mathbf{Z}, k)] = [A, \Omega K(\mathbf{Z}, k)] = [A, K(\mathbf{Z}, k-1)]$.

What this means is that by combining Alexander duality, the result $H^q(X) = [X, K(\mathbf{Z}, q)]$ of obstruction theory, and Spanier-Whitehead duality (i.e. Equation (8.2)), the definition of homology with coefficients in the Eilenberg–Maclane spectrum given in Definition 8.20 coincides with the usual definition of (ordinary) homology (at least for finite simplicial complexes, but this works more generally).

This justifies Definition 8.20 of homology with coefficients in an arbitrary spectrum $\mathbf{K}$. It also gives a duality $\widetilde{H}_{n-p}(D_nX; \mathbf{K}) = \widetilde{H}^p(X; \mathbf{K})$, which could be either considered as a generalization of Alexander duality or as a further justification of the definition of (co)homology with coefficients in a spectrum.