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Algebraic Topology and Modular Forms

M. J. Hopkins*

1. Introduction

The problem of describing the homotopy groups of spheres has been fundamental to algebraic topology for around 80 years. There were periods when specific computations were important and periods when the emphasis favored theory. Many mathematical invariants have expressions in terms of homotopy groups, and at different times the subject has found itself located in geometric topology, algebra, algebraic K-theory, and algebraic geometry, among other areas.

There are basically two approaches to the homotopy groups of spheres. The oldest makes direct use of geometry, and involves studying a map $f : S^{n+k} \to S^n$ in terms of the inverse image $f^{-1}(x)$ of a regular value. The oldest invariant, the degree of a map, is defined in this way, as was the original definition of the Hopf invariant. In the 1930's Pontryagin[1] [43, 42] showed that the homotopy class of a map f is completely determined by the geometry of the inverse image $f^{-1}(B_\epsilon(x))$ of a small neighborhood of a regular value. He introduced the basics of framed cobordism and framed surgery, and identified the group $\pi_{n+k}S^n$ with the cobordism group of smooth k-manifolds embedded in $\mathbb{R}^{n+k}$ and equipped with a framing of their stable normal bundles.

The other approach to the homotopy groups of spheres involves comparing spheres to spaces whose homotopy groups are known. This method was introduced by Serre [50, 51, 16, 15] who used Eilenberg-MacLane spaces $K(A, n)$, characterized by the property

$$\pi_i K(A, n) = \begin{cases} A & i = n \\ 0 & \text{otherwise.} \end{cases}$$

By resolving a sphere into Eilenberg-MacLane spaces Serre was able to compute $\pi_{k+n}S^n$ for all $k \leq 8$.

For some questions the homotopy theoretic methods have proved more powerful, and for others the geometric methods have. The resolutions that lend themselves to computation tend to use spaces having convenient homotopy theoretic properties, but with no particularly accessible geometric content. On the other hand, the geometric methods have produced important homotopy theoretic moduli spaces and

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA. E-mail: mjh@math.mit.edu

[1]Lefschetz reported on this work at the 1936 ICM in Oslo.

relationships between them that are difficult, if not impossible, to see from the point of view of homotopy theory. This metaphor is fundamental to topology, and there is a lot of power in spaces, like the classifying spaces for cobordism, that directly relate to both geometry and homotopy theory. It has consistently proved important to understand the computational aspects of the geometric devices, and the geometric aspects of the computational tools.

A few years ago Haynes Miller and I constructed a series of new cohomology theories, designed to isolate certain "sectors" of computation. These were successful in resolving several open issues in homotopy theory and in contextualizing many others. There seemed to be something deeper going on with one of them, and in [27] a program was outlined for constructing it as a "homotopy theoretic" moduli space of elliptic curves, and relating it to the Witten genus. This program is now complete, and we call the resulting cohomology theory tmf (for *topological modular forms*). The theory of topological modular forms has had applications in homotopy theory, in the theory of manifolds, in the theory of lattices and their θ-series, and most recently seems to have an interesting connection with the theory of p-adic modular forms. In this note I will explain the origins and construction of tmf and the way some of these different applications arise.

2. Sixteen homotopy groups

By the Freudenthal suspension theorem, the value of the homotopy group $\pi_{n+k}S^n$ is independent of n for $n > k+1$. This group is k^{th} stable homotopy group of the sphere, often written $\pi_k^{\text{st}}(S^0)$, or even as $\pi_k S^0$ if no confusion is likely to result. In the table below I have listed the values of $\pi_{n+k}S^n$ for $n \gg 0$ and $k \leq 15$.

k	0	1	2	3	4	5	6	7	8
$\pi_{n+k}S^n$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

9	10	11	12	13	14	15
$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/480$

In geometric terms, the group $\pi_{n+k}S^n$ is the cobordism group of stably framed manifolds, and a homomorphism from $\pi_{n+k}S^n$ to an abelian group A is a cobordism invariant with values in A. The groups in the above table thus represent universal invariants of framed cobordism. Some, but not all of these invariants have geometric interpretations.

When $k = 0$, the invariant is simply the number of points of the framed 0-manifold. This is the geometric description of the degree of a map.

When $k = 1$ one makes use of the fact that any closed 1-manifold is a disjoint union of circles. The $\mathbb{Z}/2$ invariant is derived from the fact that a framing on S^1 differs from the framing which bounds a framing of D^2 by an element of $\pi_1 SO(N) = \mathbb{Z}/2$.

There is an interesting history to the invariant in dimension 2. Pontryagin originally announced that the group $\pi_{n+2}S^n$ is trivial. His argument made use of the classification of Riemann surfaces, and a new geometric technique, now known as framed surgery. He later [44] correctly evaluated this group, but for his corrected argument didn't need the technique of surgery. Surgery didn't reappear in again until around 1960, when it went on to play a fundamental role in geometric topology. The invariant is based on the fact that a stable framing of a Riemann surface Σ determines a quadratic function $\phi : H^1(\Sigma; \mathbb{Z}/2) \to \mathbb{Z}/2$ whose underlying bilinear form is the cup product. To describe ϕ, note that each 1-dimensional cohomology class $x \in H^1(\Sigma)$ is Poincaré dual to an oriented, embedded 1-manifold, C_x, which inherits a framing of its stable normal bundle from that of Σ. The manifold C_x defines an element of $\pi_1^{\mathrm{st}} S^0 = \mathbb{Z}/2$, and the value of $\phi(x)$ is taken to be this element. The cobordism invariant in dimension 2 is the Arf invariant of ϕ.

A similar construction defines a map

$$\pi_{4k+2}^{\mathrm{st}} S^0 \to \mathbb{Z}/2. \tag{2.1}$$

In [14] Browder interpreted this invariant in homotopy theoretic terms, and showed that it can be non-zero only for $4k+2 = 2^m - 2$. It is known to be non-zero for $\pi_2^{\mathrm{st}} S^0$, $\pi_6^{\mathrm{st}} S^0$, $\pi_{14}^{\mathrm{st}} S^0$, $\pi_{30}^{\mathrm{st}} S^0$ and $\pi_{62}^{\mathrm{st}} S^0$. The situation for $\pi_{2^m-2}^{\mathrm{st}} S^0$ with $m > 6$ is unresolved, and remains an important problem in algebraic topology. More recently, the case $k = 1$ of (2.1) has appeared in M-theory [58]. Building on this, Singer and I [26] offer a slightly more analytic construction of (2.1), and relate it to Riemann's θ-function.

Using K-theory, Adams [2] defined surjective homomorphisms (the d and e-invariants)

$$\begin{aligned}
\pi_{4n-1} S^0 &\to \mathbb{Z}/d_n, \\
\pi_{8k} S^0 &\to \mathbb{Z}/2, \\
\pi_{8k+1} S^0 &\to \mathbb{Z}/2 \oplus \mathbb{Z}/2, \\
\pi_{8k+2} S^0 &\to \mathbb{Z}/2.
\end{aligned}$$

where d_n denotes the denominator of $B_{2n}/(4n)$. He (and Mahowald [35]) showed that they split the inclusion of the image of the J-homomorphism, making the latter groups summands. A geometric interpretation of these invariants appears in Stong [53] using Spin-cobordism, and an analytic expression for the e-invariant in terms of the Dirac operator appears in the work of Atiyah, Patodi and Singer [7, 8]. The d-invariants in dimensions $(8k+1)$ and $(8k+2)$ are given by the mod 2 index of the Dirac operator [11, 10].

This more or less accounts for the all of the invariants of framed cobordism that can be constructed using known geometric techniques. In every case the geometric invariants represent important pieces of mathematics. What remains is the following list of homotopy theoretic invariants having no known geometric interpretation:

···	8	9	10	11	12	13	14	15
···	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$			$\mathbb{Z}/3$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

This part of homotopy theory is not particularly exotic. In fact it is easy to give examples of framed manifolds on which the geometric invariants vanish, while the homotopy theoretic invariants do not. The Lie groups $SU(3)$, $U(3)$, $Sp(2)$, $Sp(1) \times Sp(2)$, G_2, $U(1) \times G_2$ have dimensions 8, 9, 10, 13, 14, and 15, respectively. They can be made into framed manifolds using the left invariant framing, and in each case the corresponding invariant is non-zero. We will see that the theory of topological modular forms accounts for *all* of these invariants, and in doing so relates them to the theory of elliptic curves and modular forms. Moreover many new invariants are defined.

3. Spectra and stable homotopy

In order to explain the theory of topological modular forms it is necessary to describe the basics of stable homotopy theory.

3.1. Spectra and generalized homology

Suppose that X is an $(n-1)$-connected pointed space. By the Freudenthal Suspension Theorem, the suspension homomorphism

$$\pi_{n+k}(X) \to \pi_{n+k+1}\Sigma X$$

is an isomorphism in the range $k < 2n-1$. This is the *stable range* of dimensions, and in order to isolate it and study only and *stable homotopy theory* one works in the category of *spectra*.

Definition 3.2 (see [34, 23, 19, 20, 4]) *A* spectrum E *consists of a sequence of pointed spaces* E_n, $n = 0, 1, 2, \ldots$ *together with maps*

$$s_n^E : \Sigma E_n \to E_{n+1} \tag{3.3}$$

whose adjoints

$$t_n^E : E_n \to \Omega E_{n+1} \tag{3.4}$$

are homeomorphisms.

A map $E \to F$ of spectra consists of a collection of maps

$$f_n : E_n \to F_n$$

which is compatible with the structure maps t_n^E and t_n^F.

For a spectrum $E = \{E_n, t_n\}$ the value of the group $\pi_{n+k}E_n$ is independent of n, and is written $\pi_n E$. Note that this makes sense for any $n \in \mathbb{Z}$. More generally, for any pointed space X, the E-homology and E-cohomology groups of X are defined as

$$\begin{aligned} E^k(X) &= [\Sigma^n X, E_{n+k}], \\ E_k(X) &= \varinjlim \pi_{n+k} E_n \wedge X. \end{aligned}$$

Any homology theory is represented by a spectrum in this way, and any map of homology theories is represented (not necessarily uniquely) by a map of spectra. For example, the spectrum HA with HA_n the Eilenberg-MacLane space $K(A, n)$ represents ordinary homology with coefficients in an abelian group A.

3.2. Suspension spectra and Thom spectra

In practice, spectra come about from a sequence of spaces X_n and maps $t_n : \Sigma X_n \to X_{n+1}$. If each of the maps t_n is a closed inclusion, then the collection of spaces

$$(LX)_k = \varinjlim \Omega^n X_{n+k}$$

forms a spectrum. In case $X_n = S^n$, the resulting spectrum is the *sphere spectrum* and denoted S^0. By construction

$$\pi_k S^0 = \pi_k^{\mathrm{st}} S^0 = \pi_{n+k} S^n \qquad n \gg 0.$$

In case $X_n = \Sigma^n X$, the resulting spectrum is the *suspension spectrum* of X, denoted $\Sigma^\infty X$ (or just X when no confusion with the space X is likely to occur). Its homotopy groups are given by

$$\pi_k \Sigma^\infty X = \pi_k^{\mathrm{st}} X = \pi_{n+k} \Sigma^n X \qquad n \gg 0,$$

and referred to as the *stable homotopy groups of X*.

Another important class of spectra are *Thom spectra*. Let $BO(n)$ denote the Grassmannian of n-planes in $\mathbb{R}^\infty$, and $MO(n)$ the Thom complex of the universal n-plane bundle over $BO(n)$. The natural maps

$$\Sigma MO(n) \to MO(n+1)$$

lead to a spectrum MO, the unoriented bordism spectrum. This spectrum was introduced by Thom [54], who identified the group $\pi_k MO$ with the group of cobordism classes of k-dimensional unoriented smooth manifolds. Using the complex Grassmannian instead of the real Grassmannian leads to the *complex cobordism* spectrum MU. The group $\pi_k MU$ can be interpreted as the group of cobordism classes of k-dimensional stably almost complex manifolds [39]. More generally, a Thom spectrum X^ζ is associated to any map

$$\zeta : X \to BG$$

from a space X to the classifying space BG for stable spherical fibrations.

The groups $\pi_k MO$ and $\pi_k MU$ have been computed [54, 39], as many other kinds of cobordism groups. The spectra representing cobordism are among the few examples that lend themselves to both homotopy theoretic and geometric investigation.

3.3. Algebraic structures and spectra

The set of homotopy classes of maps between spectra is an abelian group, and in fact the category of abelian groups makes a fairly good guideline for contemplating the general structure of the category of spectra. In this analogy, spaces correspond to sets, and spectra to abelian groups. The smash product of pointed spaces

$$X \wedge Y = X \times Y/(x,*) \sim (*,y)$$

leads to an operation $E \wedge F$ on spectra analogous to the tensor product of abelian groups. Using this "tensor structure" one can imitate many constructions of algebra in stable homotopy theory, and form analogues of associative algebras (A_∞-ring spectra), commutative algebras (E_∞-ring spectra), modules, etc. The details are rather subtle, and the reader is referred to [20] and [29] for further discussion.

The importance of refining common algebraic structures to stable homotopy theory has been realized by many authors [38, 20, 55, 56], and was especially advocated by Waldhausen.

The theory of topological modular forms further articulates this analogy. It is built on the work of Quillen relating formal groups and complex cobordism. In [45], Quillen portrayed the complex cobordism spectrum MU as the universal cohomology theory possessing Chern classes for complex vector bundles (a *complex oriented* cohomology theory). These generalized Chern classes satisfy a Cartan formula expressing the Chern classes of a Whitney sum in terms of the Chern classes of the summands. But the formula for the Chern classes of a tensor product of line bundles is more complicated than usual one. Quillen showed [45, 4] that it is as complicated as it can be. If E is a complex oriented cohomology theory, then there is a unique power series

$$F[s,t] \in \pi_* E[[s,t]]$$

with the property that for two complex line bundles L_1 and L_2 one has

$$c_1(L_1 \otimes L_2) = F[c_1(L_1), c_1(L_2)].$$

The power series $F[s,t]$ is a formal group law over $\pi_* E$. Quillen showed that when $E = MU$, the resulting formal group law is universal in the sense that if F is any formal group law over a ring R, then there is a unique ring homomorphism $MU_* \to R$ classifying F. In this way the complex cobordism spectrum becomes a topological model for the moduli space of formal group laws.

3.4. The Adams spectral sequence

There are exceptions, but for the most part what computations can be made of the stable and unstable homotopy groups involve approximating a space by the spaces of a spectrum E whose homotopy groups are known, or at least qualitatively understood. A mechanism for doing this was discovered by Adams [1] in the case $E = H\mathbb{Z}/p$, and later for a general cohomology theory by both Adams [3, 5] and Novikov [41, 40]. The device is known as the *E-Adams spectral sequence for X*, or, in the case $E = MU$, the *Adams-Novikov spectral sequence for* X (or, in case $X = S^0$, just the Adams-Novikov spectral sequence).

The Adams-Novikov spectral sequence has led to many deep insights in algebraic topology (see, for example, [47, 48] and the references therein). It is usually displayed in the first quadrant, with the groups contributing to $\pi_k S^0$ all having x-coordinate k. The y-coordinate is the MU-Adams filtration, and can be described as follows: a stable map $f : S^k \to S^0$ has filtration $\geq s$ if there exists a factorization

$$S^k = X_0 \to X_1 \to \cdots \to X_{s-1} \to X_s = S^0$$

with the property that each of the maps $MU_*X_n \to MU_*X_{n+1}$ is zero. There is a geometric interpretation of this filtration: a framed manifold M has filtration $\geq s$ if it occurs as a codimension n corner in a manifold N with corners, equipped with suitable almost complex structures on its faces (see [32]). The Adams-Novikov spectral starts with the purely algebraic object

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*)\,.$$

The quotient of the subgroup of $\pi^k S^0$ consisting of elements of Adams-Novikov filtration at least s, by the subgroup of those of filtration at least $(s+1)$ is a sub-quotient of the group $\operatorname{Ext}^{s,t}$ with $(t-s) = k$.

3.5. Asymptotics

For a number k, let $g(k) = s$ be the largest integer s for which $\pi_k S^0$ has a non-zero element of Adams-Novikov filtration s. The graph of g is the *MU-vanishing curve*, and the main result of [18] is equivalent to the formula

$$\lim_{k\to\infty} \frac{g(k)}{k} = 0.$$

This formula encodes quite a bit of the large scale structure of the category of spectra (see[18, 28, 48]), and it would be very interesting to have a more accurate asymptotic expression. This is special to complex cobordism. In the case of the original Adams spectral sequence for a finite CW complex X (based on ordinary homology with coefficients in $\mathbb{Z}/p$), it can be shown [25] that

$$\lim_{k\to\infty} \frac{g_X^{H\mathbb{Z}/p}(k)}{k} = \frac{1}{2(p^m - 1)},$$

for some m. This integer m is an invariant of X know as the “type” of X. It coincides with the largest value m for which the Morava K-group $K(m)_*X$ is non-zero. For more on the role of this invariant in algebraic topology, see [28, 22].

Now the E_2-term of the Adams-Novikov spectral sequence is far from being zero above the curve $g(n)$, and a good deal of what happens in spectral sequence has to do with getting rid of what is up there. A few years ago, Haynes Miller, and I constructed a series of spectra designed to classify and capture the way this happens. We were motivated by connective KO-theory, whose Adams-Novikov spectral sequence more or less coincides with the Adams-Novikov spectral for the sphere above a line of slope $1/2$, and is very easy to understand below that line (and in fact connective KO can be used to capture everything above a line of slope $1/5$ [37, 33, 36]). By analogy we called these cohomology theories EO_n. These spectra were used to solve several problems about the homotopy groups of spheres.

The theory we now call tmf was originally constructed to isolate the "slope $1/6^{\text{th}}$-sector" of the Adams Novikov spectral sequence, and in [**?**], for the reasons mentioned above, it was called eo_2. In the next section the spectrum tmf will be constructed as a topological model for the moduli space (stack) of generalized elliptic curves.

4. tmf

4.1. The algebraic theory of modular forms

Let C be the projective plane curve given by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{4.5}$$

over the ring

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6].$$

Let

$$A_* = \bigoplus_{n\in\mathbb{Z}} A_{2n},$$

be the graded ring with

$$A_{2n} = H^0(C; (\Omega^1_{C/A})^{\otimes n}).$$

If $u \in A_2$ is the differential

$$u = \frac{dx}{2y + a_1x + a_3},$$

then

$$A_* \approx A[u^{\pm 1}].$$

The A-module A_2 is free over A of rank 1, and is the module of sections of the line bundle

$$\omega := H^0\left(\mathcal{O}_C(-e)/\mathcal{O}_C(-2e)\right) \approx p_*\Omega^1 C.$$

In this expression $p : C \to \operatorname{Spec} A$ is the structure map, and $e : \operatorname{Spec} A \to C$ is the point at ∞.

Let G be the algebraic group of projective transformations

$$\begin{aligned} x &\mapsto \lambda^2 x + r, \\ y &\mapsto \lambda^3 y + sx + t. \end{aligned}$$

Such a transformation carries C to the curve C' defined by an equation

$$y^2 + a_1'xy + a_3'y = x^3 + a_2'x^2 + a_4'x + a_6',$$

for some a_i'. This defines an action of G on A_*. The ring of invariants

$$H^0(G; A_*)$$

is the ring of *modular forms over* $\mathbb{Z}$.

The structure of $H^0(G; A_*)$ was worked out by Tate (see Deligne [17]). After inverting 6 and completing the square and the cube, equation (4.5) can be put in the form

$$\tilde{y}^2 = \tilde{x}^3 + \tilde{c}_4\, \tilde{x} + \tilde{c}_6 \qquad \tilde{c}_4, \tilde{c}_6 \in A[\tfrac{1}{6}],$$

with

$$\tilde{x} = x + \frac{a_1^2 + 4a_2}{12} \qquad \tilde{y} = y + \frac{a_1 x + a_3}{2}.$$

The elements

$$c_4 = 48\, u^4 \tilde{c}_4,$$
$$c_6 = 864\, u^6 \tilde{c}_6,$$

lie in A_*, and

$$H^0(G; A_*) = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = 1728\Delta).$$

We'll write

$$M_n = H^0(G; A_{2n})$$

for the homogeneous part of degree $2n$. It is the group of *modular forms of weight* n *over* $\mathbb{Z}$.

4.2. The topological theory of modular forms

In [27, 24] it is shown that this algebraic theory refines from rings to *ring spectra*, leading to a topological model for the theory of elliptic curves and modular forms. Here is a rough idea of how it goes.

The set of regular points of C has a unique group structure in which the point at ∞ is the identity element, and in which collinear points sum to zero. Expanding the group law in terms of the coordinate $t = x/y$ gives a formal group law

$$C^f[s, t] \in A[[s, t]]$$

over A, which, by Quillen's theorem (see §3.3.) is classified by a graded ring homomorphism

$$MU_* \to A_*.$$

The functor

$$X \mapsto MU_*(X) \otimes_{MU_*} A_*$$

is not quite a cohomology theory, but it becomes one after inverting c_4 or Δ.

Based on this, a spectrum E_A can be constructed with

$$\pi_* E_A = A_*,$$

and representing a complex oriented cohomology theory in which the formula for the first Chern class of a tensor product of complex line bundles is given by

$$c_1(L_1 \otimes L_2) = u^{-1}\, C^f\, (u\, c_1(L_1), u\, c_1(L_2))\, .$$

A spectrum E_G can be constructed out of the affine coordinate ring of G in a similar fashion, as can an "action" of E_G on E_A. The spectrum tmf is defined to be the (-1)-connected cover of the homotopy fixed point spectrum of this group action.

To actually carry this out requires quite a bit of work. The difficulty is that the theory just described only defines an action of E_G on E_A up to homotopy, and this isn't rigid enough to form the homotopy fixed point spectrum. In the end it can be done, and there turns out to be an unique way to do it.

4.3. The ring of topological modular forms

The spectrum tmf is a homotopy theoretic refinement of the ring $H^0(G; A_*)$, there is a spectral sequence

$$H^s\,(G; A_t) \Rightarrow \pi_{t-s}\,\mathrm{tmf}\, .$$

The ring $\pi_*\,\mathrm{tmf}$ is the ring of *topological modular forms*, and the group $\pi_{2n}\,\mathrm{tmf}$ the group of *topological modular forms of weight n*. The edge homomorphism of this spectral sequence is a homomorphism

$$\pi_{2n}\,\mathrm{tmf} \to M_n.$$

This map isn't quite surjective, and there is the following result of myself and Mark Mahowald

Proposition 4.6 *The image of the map $\pi_{2*}\,\mathrm{tmf} \to M_*$ has a basis given by the monomials*

$$a_{i,j,k}\, c_4^i c_6^j \Delta^k \qquad i, k \geq 0, j = 0, 1$$

where

$$a_{i,j,k} = \begin{cases} 1 & i > 0, j = 0 \\ 2 & j = 1 \\ 24/\gcd(24, k) & i, j = 0. \end{cases}$$

In the table below I have listed the first few homotopy groups of tmf

k	0	1	2	3	4	5	6	7	8
$\pi_k\,\mathrm{tmf}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$

9	10	11	12	13	14	15
$(\mathbb{Z}/2)^2$	$\mathbb{Z}/6$	0	$\mathbb{Z}$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

The homotopy homomorphism induced by the unit $S^0 \to \mathrm{tmf}$ of the ring tmf is the "tmf-degree," a ring homomorphism

$$\pi_* S^0 \to \pi_* \,\mathrm{tmf}\,.$$

The tmf-degree is an isomorphism in dimensions ≤ 6, and it take non-zero values on each of the classes represented by the Lie groups $SU(3)$, $U(3)$, $Sp(2)$, $Sp(1)\times Sp(2)$, G_2, $U(1)\times G_2$, regarded as framed manifolds via their left invariant framings. Thus, combined with the Hopf-invariant and the invariants coming from KO-theory, the tmf-degree accounts for all of $\pi_* S^0$ for $* \leq 15$. In fact the Hopf-invariant and the invariants coming from KO can also be described in terms of tmf and nearly all of $\pi_* S^0$ for $* < 60$ can be accounted for.

5. $\boldsymbol{\theta}$-series

5.1. Cohomology rings as rings of functions

Consider the computation

$$H^*(CP^\infty;\mathbb{Z}) = \mathbb{Z}[x].$$

On one hand this tells us something about the cell structure of complex projective space; the cohomology class x^n is "dual" to the cell in dimension $2n$. On the other hand, a polynomial is a function on the affine line, and the elements of $H^*(CP^\infty)$ tell us something about the affine line. Combining these, the prospect presents itself, of using the cell structure of one space to get information about the function theory of another.

We will apply this not to ordinary cohomology, but to the cohomology theory E_A. Before doing so, more of the function theoretic aspects of E_A need to be spelled out. By construction, the ring $E_A^0(CP^\infty)$ is the ring of functions on the formal completion of C at the point e at ∞. The ring $E_A^0(HP^\infty)$ is the ring of functions which are invariant under the involution

$$\begin{aligned} \tau(x) &= x \\ \tau(y) &= -y - a_1 x - a_3 \end{aligned} \tag{5.7}$$

given by the "inverse" in the group law. The map

$$E_A^0(CP^\infty) \to E_A^0(\mathrm{pt})$$

corresponds to evaluation at e, and the reduced cohomology group $\tilde{E}_A^0(CP^\infty)$ to the ideal of formal function vanishing at e. Note that this is consistent with the definition of A_2 as sections of the line bundle ω:

$$A_2 = \pi_2 E_A = \tilde{E}_A^0(S^2) = I/I^2 = H^0\left(\mathcal{O}_C(-e)/\mathcal{O}_C(-2e)\right).$$

Now $\tilde{E}^0_A(CP^\infty)$ is the cohomology group of the Thom complex $E^0_A(CP^{\infty L})$. More generally, there is an additive correspondence

$$\{\text{virtual representations of } U(1)\} \leftrightarrow \{\text{divisors on } C\}\,,$$

under which a virtual representation V corresponds to a divisor D for which

$$E^0\left((CP^\infty)^V\right) = H^0\left(C^f; \mathcal{I}(D) \otimes \left(\Omega^1\right)^{\otimes \mu(V)}\right),$$

where $\mu(V)$ is the multiplicity of the trivial representation in V. There is a similar correspondence between even functions with divisors of the form $D + \tau^* D$, virtual representations V of $SU(2)$ and $E^0_A\left((HP^\infty)^V\right)$ (with τ the involution (5.7)).

5.2. The Hopf fibration and the Weierstrass $\mathcal{P}$-function

Consider the function x in the Weierstrass equation (4.5). This function has a double pole at e. We now ask if there is a "best" x to choose, ie, a function with a double pole at e which is invariant under the action of r, s and t. Such a function will be an eigenvector for λ with eigenvalue λ^2. It is more convenient to search for an quantity which is invariant under λ as well, so instead we search for a quadratic differential on C, i.e. a section

$$x' \in H^0\left(C; \Omega^1(e)^{\otimes 2}\right)$$

which is invariant under r, s, t and λ. Now the space $H^0(C; \Omega^1(e)^2)$ has dimension 2, and sits in a short exact sequence of vector spaces over Spec A

$$0 \to \omega^2 \to H^0(C; \Omega^1(e)^2) \to \mathcal{O}_A \to 0, \tag{5.8}$$

where ω is the line bundle of invariant differentials on C, and the second map is the "residue at e". This sequence is G-equivariant, and defines an element of

$$\operatorname{Ext}^1(\mathcal{O}, \omega^2) = H^1(G; A_4).$$

The obstruction to the existence of an x' with residue 1 is the Yoneda class ν of this extension. Completing the square and cube in (4.5), gives the G-invariant expression

$$\left(12x + a_1^2 + 4a_2\right) u^2,$$

so that $12\nu = 0$. In fact the group $H^1(G; A_4)$ is cyclic of order 12 with ν as a generator. The group $\pi_3\,\mathrm{tmf} = \mathbb{Z}/24$ is assembled from $H^1(G; A_4)$ and $H^3(G; A_6) = Z/2$, and sits in an exact sequence

$$0 \to H^3(G; A_6) \to \pi_3\,\mathrm{tmf} \to H^1(G; A_4) \to 0.$$

This 12 can also be seen transcendentally. Over the complex numbers, a choice of x is given by the Weierstrass $\mathcal{P}$-function:

$$\mathcal{P}(z, \tau) = \frac{1}{z^2} + \sum_{0 \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2}.$$

The Fourier expansion of $\mathcal{P}(z,\tau)\,dz^2$ is (with $q = e^{2\pi i \tau}$ and $u = e^{2\pi i z}$)

$$\mathcal{P}(z,\tau)\,dz^2 = \left(\sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} + \frac{1}{12} - 2\sum_{n\geq 1} \frac{q^n}{(1-q^n)^2} \right) \left(\frac{du}{u} \right)^2 .$$

Note that all of the Fourier coefficients of $\mathcal{P}$ are integers, except for the constant term, which is $1/12$. This is the same 12.

Under the correspondence between divisors and Thom complexes, the differential x' corresponds to a G-invariant element

$$x'_{\mathrm{top}} \in E^0_A \left(HP^{(2-V)} \right),$$

with V the defining representation of $SU(2)$. Now the spectrum HP^{2-V} has a (stable) cell decomposition

$$HP^{2(1-L)} = S^0 \cup_\nu e^4 \cup \dots$$

with one cell in every real dimension $4k$. The 4-cell is attached to the 0-cell by the stable Hopf map $\nu : S^7 \to S^4$, which generates $\pi_3(S^0) = \mathbb{Z}/24$. The restriction of the quadratic differential x'_{top} to the zero cell is given by the residue at e, and the obstruction to the existence of an G-invariant x'_{top} with residue k is the image of k under the connecting homomorphism

$$H^0(G;A) \to H^1\left(G; E^0_A\left(HP^{2-V}/S^0\right)\right). \tag{5.9}$$

To evaluate (5.9), note that the map

$$H^1\left(G; E^0_A\left(HP^{2-V}/S^0\right)\right) \to H^1\left(G; E^0_A(S^4)\right) = H^1\left(G; A_4\right)) = \mathbb{Z}/12$$

is projection onto a summand, and the image of (5.9) is contained in this summand. Thus the obstruction to the existence of the quadratic differential x_{top} is the same $k \in \mathbb{Z}/12$. In this way, the theory of topological modular forms relates the Hopf map ν to the constant term in the Fourier expansion of the Weierstrass $\mathcal{P}$-function, and to the existence of a certain quadratic differential on the universal elliptic curve.

5.3. Lattices and their θ-series

There is a slightly more sophisticated application of these ideas to the theory of even unimodular lattices. Suppose that L is a positive definite, even unimodular lattice of dimension $2d$. The *theta function* of L, θ_L is the generating function

$$\begin{aligned} \theta_L(q) &= \sum_{\ell\in L} q^{\frac{1}{2}\langle \ell,\ell\rangle} \\ &= \sum_{n\geq 0} L_n q^n, \\ L_n &= \#\{\ell \mid \langle l,l\rangle = 2n\}. \end{aligned}$$

It follows from the Poisson summation formula that $\theta_L(q)$ is the q-expansion of a modular form over $\mathbb{Z}$ of weight d, and so lies in the ring

$$\begin{aligned}\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta) &\subset Z[\![q]\!], \\ c_4 &= 1 + 240\sum_{n>0}\sigma_3(n)q^n, \\ c_6 &= 1 - 504\sum_{n>0}\sigma_5(n)q^n, \\ \Delta &= q\prod_{n=1}^{\infty}(1-q^n)^{24}.\end{aligned}$$

Since the group of modular forms of a given weight is finitely generated, the first few L_n determine the rest. This leads to many restrictions on the distributions of lengths of vectors in a positive definite, even unimodular lattice.

The θ-series of L is the value at $z = 0$ of the θ-function

$$\theta(z,\tau) = \sum_{\ell\in L} e^{\pi i\langle \ell,\ell\rangle\tau + 2\pi i\langle\ell,z\rangle}, \qquad z \in \mathbb{C}\otimes L, \quad q = e^{2\pi i\,\tau},$$

which, under the correspondence between divisors and representations has the following topological interpretation. Let V be any d-dimensional (complex) virtual representation of $U(1)\otimes L$ with the property that $c_1(V) = 0$ and $c_2(V)$ corresponds to the quadratic form, under the isomorphism

$$H^2\left(B(U(1)\otimes L);\mathbb{Z}\right) = \operatorname{Sym}^2 L^*.$$

For example, if $A = (a_{ij})$ is the matrix of the quadratic form with respect to some basis, then V could be taken to be

$$V = \mathbb{C}^d + \frac{1}{2}\sum_{i,j} a_{ij}(1-L_i)(1-L_j),$$

where L_i is the character of $U(1)\otimes L$ dual to the i^{th} basis element. The series $\theta(z,\tau)$ corresponds to a G-invariant element

$$\theta^{\text{top}} \in E_A^0((BU(1)\otimes L)^V).$$

The restriction of θ^{top} to $\{\text{pt}\}^V = S^{2n}$ is an element of $H^0(G; A_{2d})$, i.e. an algebraic modular form of weight d. This modular form is θ_L.

Now the Thom spectrum $(BU(1)\otimes L)^V$ has a stable cell decomposition

$$S^{2d} \cup \bigvee^{2d} e^{2d+2} \cup \bigvee^{d(2d+1)} e^{2d+4} \cup \cdots.$$

Since $c_1(V) = 0$, the cells of dimension $2d+2$ are not attached to the $(2d)$-cell. The assumptions on the quadratic form, and on $c_2(V)$ imply that one of the $(2d+4)$-cells is attached to the $(2d)$-cell by (a suspension of) the stable Hopf map ν. The presence of this attaching map implies the following mod 24 congruence on θ_L.

Theorem 5.10 *Suppose L is a positive definite, even unimodular lattice of dimension $24k$. Write*

$$\theta_L(q) = c_4^{3k} + x_1 c_4^{3(k-1)}\Delta + \cdots + x_k\Delta^k.$$

Then

$$x_k \equiv 0 \mod 24. \tag{5.11}$$

The above result was originally proved by Borcherds [12] as part of his investigation into infinite product expansions for automorphic forms on certain indefinite orthogonal groups. The above topological proof can be translated into the language of complex function theory. The details are in the next section.

The congruence of Theorem 5.10 together with Proposition 4.6 give the following

Proposition 5.12 *Suppose L is a positive definite, even unimodular lattice of dimension $2d$. There is an element $\theta_L^{top} \in \mathrm{tmf}^0(S^{2d})$ whose image in M_d is θ_L.*

It can also be shown that the G-invariant

$$\theta^{\mathrm{top}} \in H^0(G; E_A^0(B(U(1)\otimes L)^V))$$

is truly topological in the sense that it is the representative in the E_2-term of the spectral sequence

$$H^0(G; E_A^0(B(U(1)\otimes L)^V)) \Rightarrow \mathrm{tmf}^0(B(U(1)\otimes L)^V),$$

of an element in $\mathrm{tmf}^0(B(U(1)\otimes L)^V)$. I don't know of a direct construction of these truly topological theta series.

5.4. An analytic proof of Theorem 5.10

The analytic interpretation of the proof of Theorem 5.10 establishes the result in the form

$$\mathrm{Res}_{\Delta=0}\frac{\theta_L(q)}{\Delta^k}\frac{d\Delta}{\Delta} \equiv \mathrm{Res}_{q=0}\frac{\theta_L(q)}{\Delta^k}\frac{dq}{q} \equiv 0 \mod 24. \tag{5.13}$$

The equivalence of (5.11) with (5.13) follows easily from the facts

$$c_4 \equiv 0 \mod 24,$$
$$\frac{d\Delta}{\Delta} \equiv \frac{dq}{q} \mod 24.$$

Set $u = e^{2\pi i z}$, and for a vector $\mu \in L$ let

$$\phi_\mu(z,\tau) = \frac{\sum_{\ell\in L} q^{\frac{1}{2}\langle \ell,\ell\rangle} u^{\langle \mu,\ell\rangle}}{\sigma(q,u)^{\langle \mu,\mu\rangle}},$$

where

$$\sigma(q,u) = u^{\frac{1}{2}}(1-u^{-1})\prod_{n=1}^{\infty}\frac{(1-q^n u)(1-q^n u^{-1})}{(1-q^n)^2}$$

is the Weierstrass σ-function. It is immediate from the definition that

$$\phi_\mu(-z,\tau) = \phi_\mu(z,\tau),$$

and it follows from the modular transformation formula for θ that for

$$m,n \in \mathbb{Z}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(Z),$$

the function $\phi_\mu(z,\tau)$ satisfies

$$\phi_\mu(z + m\tau + n, \tau) = \phi_\mu(z,\tau), \tag{5.14}$$

$$\phi_\mu\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{d/2}\,\phi_\mu(z,\tau). \tag{5.15}$$

These identities are equivalent to saying that if we write, with $x = 2\pi i z$,

$$\phi_\mu(z,\tau) = x^{-\langle\mu,\mu\rangle}\left(\phi_\mu^{(0)} + \phi_\mu^{(2)}x^2 + \dots\right),$$

then $\phi_\mu^{(2k)}$ is a modular form of weight $d/2 + 2k$.

It is the term $\phi_\mu^{(2)}$ that stores the information about the $(2d+4)$-cells of the spectrum $B(U(1) \otimes L)^V$. A little computation shows that

$$\phi_\mu^{(2)} = \theta_\mu - \frac{\langle\mu,\mu\rangle}{24}\theta_L, \tag{5.16}$$

where

$$\theta_\mu = \sum_{\ell \in L} q^{\frac{1}{2}\langle\ell,\ell\rangle}\frac{\langle\ell,\mu\rangle^2}{2} - \langle\mu,\mu\rangle\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}.$$

Because of (5.15), the expression

$$\frac{\phi_\mu^{(2)}}{\Delta^k}\,\frac{dq}{q}$$

is a meromorphic differential on the projective j-line. It's only pole is at $q=0$. It follows that[2]

$$\mathrm{Res}_{q=0}\,\frac{\phi_\mu^{(2)}}{\Delta^k}\,\frac{dq}{q} = 0,$$

and hence

$$\mathrm{Res}_{q=0}\,\frac{\theta_\mu}{\Delta^k}\frac{dq}{q} = \frac{\langle\mu,\mu\rangle}{24}\,\mathrm{Res}_{q=0}\,\frac{\theta_L}{\Delta^k}\frac{dq}{q}. \tag{5.17}$$

To deduce (5.13), define

$$p : L \otimes \mathbb{Z}/2 \to \mathbb{Z}/2$$

[2] I learned this trick from Borcherds, who told me he learned it from a referee.

by letting $p(\mu)$ be the value of (5.17) reduced modulo 2. The left hand side of (5.17) shows that p does in fact takes its values in $\mathbb{Z}/2$. Making use of the symmetry $\ell \mapsto -\ell$, it also shows that p is linear. The right hand side shows that p is quadratic with underlying bilinear form

$$p(\mu_1 + \mu_2) - p(\mu_1) - p(\mu_2) = \langle \mu_1, \mu_2 \rangle \frac{1}{12} \operatorname{Res}_{q=0} \frac{\theta_L}{\Delta^k} \frac{dq}{q}.$$

It follows that

$$\langle \mu_1, \mu_2 \rangle \frac{1}{12} \operatorname{Res}_{q=0} \frac{\theta_L}{\Delta^k} \frac{dq}{q} \equiv 0 \mod 2.$$

Since L is unimodular, there are vectors μ_1 and μ_2 with

$$\langle \mu_1, \mu_2 \rangle = 1$$

and so

$$\frac{1}{12} \operatorname{Res}_{q=0} \frac{\theta_L}{\Delta^k} \frac{dq}{q} \equiv 0 \mod 2.$$

This is (5.13).

6. Topological modular forms as cobordism invariants

6.1. The Atiyah-Bott-Shapiro map

As described in §2, the part of the homotopy groups of spheres that is best understood geometrically is the part that is captured by KO-theory. The geometric interpretations rely on the fact that all of the corresponding framed cobordism invariants can be expressed in terms of invariants of Spin-cobordism. From the point of view of topology what makes this possible is the factorization

$$S^0 \to M\mathrm{Spin} \to KO$$

of the unit in KO-theory, through the Atiyah-Bott-Shapiro map $M\mathrm{Spin} \to KO$. The Atiyah-Bott-Shapiro map is constructed using the representations of the spinor groups, and relies on knowing that for a space X, elements of $KO^0(X)$ are represented by vector bundles over X. It gives a KO-theory Thom isomorphism for Spin-vector bundles, and is topological expression for the index of the Dirac operator. Until recently it was not known how to produce this map by purely homotopy theoretic means.

The framed cobordism invariants coming from tmf cannot be expressed directly in terms of Spin-cobordism. One way of seeing this is to note that the group $\pi_3\, M\mathrm{Spin}$ is zero. The fact that the map

$$\pi_3 S^0 \to \pi_3\, \mathrm{tmf}$$

is an isomorphism prevents a factorization of the form

$$S^0 \to M\mathrm{Spin} \to \mathrm{tmf}\,.$$

In some sense this is all that goes wrong. Let $BO\langle 8\rangle$ be the 7-connected cover of BSpin, and $MO\langle 8\rangle$ the corresponding Thom spectrum. We will see below that there is a factorization of the unit

$$S^0 \to MO\langle 8\rangle \to \mathrm{tmf}\,.$$

There is not yet a geometric interpretation of $\mathrm{tmf}^0(X)$, so the construction of a map $MO\langle 8\rangle \to \mathrm{tmf}$ must be made using homotopy theoretic methods. The key to doing this is to exploit the E_∞-ring structures on the spectra involved.

6.2. E_∞-maps

The spectra $MO\langle 8\rangle$, MSpin, tmf and KO are all E_∞-ring spectra, and from the point of view of homotopy theory it turns out easier to construct E_∞ maps

$$M\mathrm{Spin} \to KO \quad \text{and} \quad MO\langle 8\rangle \to \mathrm{tmf}$$

than merely to produce maps of spectra. In fact the homotopy type of the spaces of E_∞-maps

$$E_\infty\left(MO\langle 8\rangle, \mathrm{tmf}\right) \quad \text{and} \quad E_\infty\left(M\mathrm{Spin}, KO\right)$$

can be fairly easily identified using homotopy theoretic methods. In this section I will describe their sets of path components.

A map $\phi : M\mathrm{Spin} \to KO$ determines a Hirzebruch genus with an even characteristic series $K_\phi(z) \in \mathbb{Q}[[x]]$. Define the *characteristic sequence* of ϕ to be the sequence of rational numbers

$$(b_2, b_4, \cdots)$$

given by

$$\log\left(K_\phi(z)\right) = -2\sum_{n>0} b_n\,\frac{x^n}{n!}.$$

We use this sequence to form the *characteristic map*

$$\pi_0 E_\infty\left(M\mathrm{Spin}, KO\right) \to \{(b_2, b_4, \cdots) \mid b_{2i} \in \mathbb{Q}\}$$

from the set of homotopy classes of E_∞ maps $M\mathrm{Spin} \to KO$ to the set of sequences of rational numbers.

Let B_n denote the n^{th} Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$

The following result is due to Matthew Ando, myself, and Charles Rezk:

Theorem 6.18 *The characteristic map*

$$\pi_0 E_\infty (M\mathrm{Spin}, KO) \to \{(b_2, b_4, \cdots) \mid b_{2i} \in \mathbb{Q}\}$$

gives an isomorphism of $\pi_0 E_\infty (M\mathrm{Spin}, KO)$ *with the set of sequences*

$$(b_2, b_4, b_6, \dots)$$

for which

i) $b_n \equiv B_n/2n \mod \mathbb{Z}$*;*

ii) *for each odd prime* p *and each* p*-adic unit* c*,*

$$m \equiv n \mod p^k(p-1)$$
$$\implies (1-c^n)(1-p^{n-1})b_n \equiv (1-c^m)(1-p^{m-1})b_m \mod p^{k+1};$$

iii) *for each* 2*-adic unit* c

$$m \equiv n \mod 2^k$$
$$\implies (1-c^n)(1-2^{n-1})b_n \equiv (1-c^m)(1-2^{m-1})b_m \mod 2^{k+2}.$$

Remark 6.19 By the Kummer congruences, the sequence with $b_{2n} = B_{2n}/(4n)$ comes from an E_∞-map $M\mathrm{Spin} \to KO$. The associated characteristic series is

$$\frac{x}{e^{x/2} - e^{-x/2}} = \frac{x/2}{\sinh(x/2)}$$

and the underlying map of spectra coincides with the one constructed by Atiyah-Bott-Shapiro. Theorem 6.18 therefore gives a purely homotopy theoretic construction of this map.

We now turn to describing the set of homotopy classes of E_∞ maps

$$MO\langle 8\rangle \to \mathrm{tmf}\,.$$

Associated to a multiplicative map $MO\langle 8\rangle \to \mathrm{tmf}$ is a characteristic series of the form

$$K_\phi(z) = \sum a_{2n} z^{2n} \qquad a_{2n} \in \mathbb{Q}[\![q]\!]$$

which is well-defined up to multiplication by the exponential of a quadratic function in z. We associate to such a series, a sequence

$$(g_4, g_6 \cdots) \quad g_{2n} \in \mathbb{Q}[\![q]\!] \tag{6.20}$$

according to the rule

$$\log\left(K_\phi(z)\right) = -2 \sum_{n>0} g_n \frac{x^n}{n!}.$$

For $n > 2$ the terms g_n is independent of the quadratic exponential factor, and is the q-expansion of a modular form of weight n (see [27, 6]). This defines the *characteristic map*

$$\pi_0 E_\infty\left(MO\langle 8\rangle, \mathrm{tmf}\right) \to \{(g_2, g_4, \cdots) \mid g_{2n} \in M_{2n} \otimes \mathbb{Q}\}.$$

Following Serre [52] let G_{2k} denote the (un-normalized) Eisenstein series of weight $2k$

$$G_{2k} = -\frac{B_{2k}}{4k} + \sum_{n>0} \sigma_{2k-1}(n) q^n, \qquad \sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1},$$

and for a prime p, let G^*_{2k} be the (un-normalized) p-adic Eisenstein series

$$G^*_{2k} = -(1-p^{2k-1})\frac{B_{2k}}{4k} + \sum_{n>0} \sigma^*_{2k-1}(n) q^n, \qquad \sigma^*_{2k-1}(n) = \sum_{\substack{d|n \\ (d,p)=1}} d^{2k-1}.$$

We will also need the Atkin (U) and Verschiebung (V) operators on p-adic modular forms of weight k (See [52, §2.1]). For a p-adic modular form

$$f = \sum_{n=0}^{\infty} a_n q^n$$

of weight k, one defines

$$f|U = \sum_{n=0}^{\infty} a_{pn} q^n \qquad f|V = \sum_{n=0}^{\infty} a_n q^{pn}.$$

Finally, we set

$$f^* = f - p^{k-1} f|V.$$

(This gives two meanings to the symbol G^*_{2k}, which are easily checked to coincide.)

Proposition 6.21 *The image of the characteristic map*

$$\pi_0 E_\infty\left(MO\langle 8\rangle, \mathrm{tmf}\right) \to \{(g_2, g_4, \cdots) \mid g_{2n} \in M_{2n} \otimes \mathbb{Q}\}$$

is the set of sequences (g_{2n}) *satisfying*

i) $g_{2n} \equiv G_{2n} \mod \mathbb{Z}$

ii) *For each odd prime p and each p-adic unit c,*

$$m \equiv n \mod p^k(p-1) \implies (1-c^n)g^*_n \equiv (1-c^m)g^*_m \mod p^{k+1},$$

iii) *For each 2-adic unit c,*

$$m \equiv n \mod 2^k \implies (1-c^n)g^*_n \equiv (1-c^m)g^*_m \mod 2^{k+2}.$$

iv) *For each prime p,* $g_m^*|U = g_m^*$.

The characteristic map is a principle A-bundle over its image, where A is a group isomorphic to a countably infinite product of $\mathbb{Z}/2$*'s.*

Remark 6.22 The group A occurring in the final assertion of Proposition 6.21 can be described explicitly in terms of modular forms. The description is somewhat technical, and has been omitted.

By the Kummer congruences, the sequence of Eisenstein series $(G_4, G_6, \cdots)$ satisfy the conditions of Proposition 6.21. The corresponding characteristic series can be taken to be

$$\frac{z/2}{\sinh(z/2)} \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n e^z)(1-q^n e^{-z})},$$

and the genus is the Witten genus ϕ_W [57, 49, 21]. This gives the following corollary, which was the main conjecture of [**?**].

Corollary 6.23 *There is an* E_∞*-map* $MO\langle 8\rangle \to \mathrm{tmf}$ *whose underlying genus is the Witten genus.*

Remark 6.24 This refined Witten genus is only specified up to action by an element of the group A occurring in Proposition 6.21. It looks as if a more careful analysis could lead to specifying a single E_∞-map $MO\langle 8\rangle \to \mathrm{tmf}$, but this has not yet been carried out.

6.3. The image of the cobordism invariant

The existence of a refined Witten genus has an application to the theory of even unimodular lattices. The following two results are due to myself and Mark Mahowald.

Theorem 6.25 *Let* $MO\langle 8\rangle \to \mathrm{tmf}$ *be any multiplicative map whose underlying genus is the Witten genus. Then the induced map of homotopy groups* $\pi_* MO\langle 8\rangle \to \pi_* \mathrm{tmf}$ *is surjective.*

Combining this with Proposition 5.12 then gives

Corollary 6.26 *Let L be a positive definite, even unimodular lattice of dimension* $2d$*. There exists a* 7*-connected manifold* M_L *of dimension* $2d$*, whose Witten genus is the* θ*-function of L, i.e.*

$$\phi_W(M) = \theta_L.$$

In case L is the Leech lattice, the existence of M_L gives an affirmative answer to Hirzebruch's "Prize Question" [21].

6.4. Spectra of units and E_∞-maps

The proofs of Theorems 6.18 and 6.21 come down to understanding the structure of the group of units in $KO^0(X)$ and $\mathrm{tmf}^0(X)$.

For a ring spectrum R, let $Gl_1(R)$ be the classifying space for the group of units in R-cohomology:

$$[X, Gl_1(R)] = R^0(X)^\times.$$

If $R = \{R_n, t_n\}$, then $Gl_1(R)$ is part of the homotopy pullback square

$$\begin{array}{ccc} Gl_1(R) & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ \pi_0 R^\times & \longrightarrow & \pi_0 R. \end{array} \tag{6.27}$$

When R is an A_∞-ring spectrum, $Gl_1(R)$ has a classifying space $BGl_1(R)$. When R is E_∞, then $Gl_1(R)$ is an infinite loop space, i.e. there is a spectrum $gl_1(R)$ with $gl_1(R)_0 = Gl_1(R)$ (and $gl_1(R)_1 = BGl_1(R)$). The space $BGl_1(S^0)$ is the classifying space for unoriented stable spherical fibrations, and the map

$$BO\langle 8\rangle \to BGl_1(S^0) \tag{6.28}$$

whose associated Thom spectrum is $MO\langle 8\rangle$, is an infinite loop map. More specifically, let $bo\langle 8\rangle$ be the 7-connective cover of the spectrum KO. Then

$$\left(\Sigma^{-1} bo\langle 8\rangle\right)_1 = BO\langle 8\rangle,$$

and there is a map of spectra

$$\Sigma^7 bo\langle 8\rangle \to gl_1(S^0)$$

for which the induced map $\left(\Sigma^{-1} bo\langle 8\rangle\right)_1 \to \left(gl_1(S^0)\right)_1$ becomes (6.28).

The following result is what makes it easier to construct E_∞-maps than merely maps of spectra.

Proposition 6.29 *The space $E_\infty(M\mathrm{Spin}, KO)$ is canonically homotopy equivalent to the space of factorizations*

$$\begin{array}{ccc} gl_1(S^0) & \longrightarrow & gl_1(S^0) \cup C\Sigma^{-1} bo\langle 4\rangle \\ \downarrow & \swarrow & \\ gl_1(KO), & & \end{array}$$

and the space $E_\infty(MO\langle 8\rangle, \mathrm{tmf})$ is canonically homotopy equivalent to the space of factorizations

$$\begin{array}{ccc} gl_1(S^0) & \longrightarrow & gl_1(S^0) \cup C\Sigma^{-1} bo\langle 8\rangle \\ \downarrow & \swarrow & \\ gl_1(\mathrm{tmf}). & & \end{array}$$

6.5. The Atkin operator and the spectrum of units in tmf

Proposition 6.29 emphasizes the important role played by the spectrum of units $gl_1(\mathrm{tmf})$ and $gl_1(KO)$. Getting at the homotopy type of these spectra uses work of Bousfield [13] and Kuhn [31]. Fix a prime p let $K(n)$ denote the n^{th} Morava K-theory at p, and

$$L_{K(n)} : \mathrm{Spectra} \to \mathrm{Spectra}$$

the localization functor with respect to $K(n)$ (see [46, 48]). Bousfield (in case $n = 1$) and Kuhn (in case $n > 1$) construct a functor

$$\Sigma^{K(n)} : \mathrm{Spaces} \to \mathrm{Spectra}$$

and a natural equivalence of $\Sigma^{K(n)}(E_0)$ with $L_{K(n)}E$. The spectrum $\Sigma^{K(n)}(X)$ depends only on the connected component of X containing the basepoint. In the special case of an E_∞-ring spectrum E it gives (because of (6.27)) a canonical equivalence

$$L_{K(n)}\, gl_1\, E \approx L_{K(n)}E,$$

which, when composed with the localization map $gl_1\, E \to L_{K(n)}\, gl_1\, E$, leads to a "logarithm"

$$\log^E_{K(n)} : gl_1\, E \to L_{K(n)}E.$$

Bousfield showed that for KO, the logarithm

$$\log^{KO}_{K(1)} : gl_1\, KO \to L_{K(1)}KO = KO_p$$

becomes an equivalence after completing at p and passing to 2-connected covers. Charles Rezk has recently shown that for a space X, the map

$$\log^{KO}_{K(1)} : KO^0(X)^\times \to KO^0_p(X)$$

is given by the formula

$$\frac{1}{p} \log\left(\frac{\psi_p(x)}{x^p}\right).$$

This equivalence between the multiplicative and additive groups of K-theory was originally observed by Sullivan, and proved by Atiyah-Segal [9].

In the case of tmf, Paul Goerss, Charles Rezk and I have shown that the Bousfield-Kuhn logarithms leads to a commutative diagram

$$\begin{array}{ccc} gl_1\, \mathrm{tmf} & \xrightarrow{\log^{\mathrm{tmf}}_p} & \mathrm{tmf}_p \\ \downarrow & & \downarrow \\ L_{K(1)}\,\mathrm{tmf} & \xrightarrow[1-U]{} & L_{K(1)}\,\mathrm{tmf} \end{array} \tag{6.30}$$

which becomes homotopy Cartesian after completing at p and passing to 3-connected covers. The spectrum tmf_p is the p-adic completion of tmf, and the map $\log^{\mathrm{tmf}}_p$ has a description in terms of modular forms, but to describe it would take us outside the scope of this paper. The spectrum $L_{K(1)}$ tmf is the topological analogue of the theory of p-adic modular forms of Serre [52] and Katz [30], and the map U is the topological Atkin operator. One noteworthy feature of this square is that it locates the Atkin operator in the theory of all modular forms (and not just p-adic modular forms). Another is that it connects the failure of $\log_p^{\mathrm{tmf}}$ to be an isomorphism with the spectrum of U. For example, let F denote the fiber of

$$gl_1\, \mathrm{tmf}_p \xrightarrow{\log_p^{\mathrm{tmf}}} \mathrm{tmf}_p\,.$$

By the square (6.30)

$$\pi_{23}F = \begin{cases} \mathbb{Z}_p & p = 691 \\ \mathbb{Z}_p \oplus \mathbb{Z}_p/(\tau(p)-1) & p \neq 691, \end{cases}$$

where τ is the Ramanujan τ function, defined by

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{k=1}^{\infty}\tau(k)q^k.$$

For $p \neq 691$ the torsion subgroup of $\pi_{23}F$ has order determined by the p-adic valuation of $(\tau(p)-1)$. The only primes less that $35,000$ for which $\tau(p) \equiv 1 \mod p$ are 11, 23, and 691. It is not known whether or not $\tau(p) \equiv 1 \mod p$ holds for infinitely many primes.

The fiber of $\log_p^{\mathrm{tmf}}$ seems to store quite a bit of information about the spectrum of the Atkin operator and p-adic properties of modular forms. Investigating its homotopy type looks like interesting prospect for algebraic topology.

References

[1] J. F. Adams, *On the structure and applications of the Steenrod algebra*, Commentarii Mathematici Helvetici **32** (1958), 180–214.

[2] ———, *On the groups J(X), IV*, Topology **5** (1966), 21–71.

[3] J. F. Adams, *Lectures on generalised cohomology*, Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three), Springer, Berlin, 1969, 1–138. MR 40 #4943

[4] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, 1974.

[5] J. Frank Adams, *Stable homotopy theory*, Springer-Verlag, Berlin, 1969. MR 40 #2065

[6] M. Ando, M. J. Hopkins, and N. P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. **146** (2001), no. 3, 595–687. MR 2002g:55009

[7] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432. MR 53 #1655b

[8] ———, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 71–99. MR 53 #1655c

[9] M. F. Atiyah and G. B. Segal, *Exponential isomorphisms for λ-rings*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 371–378. MR 45 #344

[10] M. F. Atiyah and I. M. Singer, *Index theory for skew-adjoint Fredholm operators*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 37, 5–26. MR 44 #2257

[11] M. F. Atiyah and I. M. Singer, *The index of elliptic operators: V*, Annals of Mathematics **93** (1971), 139–149.

[12] Richard E. Borcherds, *Automorphic forms on $\mathrm{o}_{s+2,2}(\mathbf{r})$ and infinite products*, Invent. Math. **120** (1995), no. 1, 161–213. MR 96j:11067

[13] A. K. Bousfield, *Uniqueness of infinite deloopings for K-theoretic spaces*, Pacific J. Math. **129** (1987), no. 1, 1–31. MR 89g:55017

[14] W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Annals of Mathematics **90** (1969), 157–186.

[15] Henri Cartan and Jean-Pierre Serre, *Espaces fibrés et groupes d'homotopie. I. Constructions générales*, C. R. Acad. Sci. Paris **234** (1952), 288–290. MR 13,675a

[16] ———, *Espaces fibrés et groupes d'homotopie. II. Applications*, C. R. Acad. Sci. Paris **234** (1952), 393–395. MR 13,675b

[17] P. Deligne, *Courbes elliptiques: formulaire (d'après J. Tate)*, Modular Functions of One Variable III, Lecture Notes in Mathematics, vol. 476, Springer-Verlag, 1975, 53–73.

[18] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241. MR 89m:55009

[19] A. D. Elmendorf, I. Kříž, M. A. Mandell, and J. P. May, *Modern foundations for stable homotopy theory*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, 213–253. MR 97d:55016

[20] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR 97h:55006

[21] F. Hirzebruch, T. Berger, and R. Jung, *Manifolds and modular forms*, Aspects of Mathematics, vol. E 20, Friedr. Vieweg & Sohn, Braunschweig, 1994.

[22] M. J. Hopkins, *Global methods in homotopy theory*, Proceedings of the 1985 LMS Symposium on Homotopy Theory (J. D. S. Jones and E. Rees, eds.), 1987, 73–96.

[23] M. J. Hopkins and P. Goerss, *Multiplicative stable homotopy theory*, book in preparation.

[24] M. J. Hopkins, M. Mahowald, and H. R. Miller, *Elliptic curves and stable homotopy theory I*, in preparation.

[25] M. J. Hopkins, J. H. Palmieri, and J. H. Smith, *Vanishing lines in generalized Adams spectral sequences are generic*, Geom. Topol. **3** (1999), 155–165 (electronic). MR 2000c:55017

[26] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and*

M-theory, math.AT/0211216.

[27] Michael J. Hopkins, *Topological modular forms, the Witten genus, and the theorem of the cube*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, 554–565. MR 97i:11043

[28] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49. MR 99h:55009

[29] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208. MR 2000h:55016

[30] Nicholas M. Katz, *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, 69–190. Lecture Notes in Mathematics, Vol. 350. MR 56 #5434

[31] Nicholas J. Kuhn, *Morava K-theories and infinite loop spaces*, Algebraic topology (Arcata, CA, 1986), Springer, Berlin, 1989, 243–257. MR 90d:55014

[32] Gerd Laures, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5667–5688 (electronic). MR 1 781 277

[33] W. Lellman and M. E. Mahowald, *The bo-adams spectral sequence*, Transactions of the American Mathematical Society **300** (1987), 593–623.

[34] L. G. Lewis, J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, New York, 1986.

[35] M. E. Mahowald, *The order of the image of the J-homomorphism*, Bulletin of the American Mathematical Society **76** (1970), 1310–1313.

[36] ———, *bo-resolutions*, Pacific Journal of Mathematics **192** (1981), 365–383.

[37] ———, *The image of J in the EHP sequence*, Annals of Mathematics **116** (1982), 65–112.

[38] J. Peter May, *E_∞ ring spaces and E_∞ ring spectra*, Springer-Verlag, Berlin, 1977, With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave, Lecture Notes in Mathematics, Vol. 577. MR 58 #13008

[39] J. W. Milnor, *On the cobordism ring Ω^* and a complex analogue, Part I*, American Journal of Mathematics **82** (1960), 505–521.

[40] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951. MR 36 #4561

[41] ———, *Rings of operations and spectral sequences of Adams type in extraordinary cohomology theories. U-cobordism and K-theory*, Dokl. Akad. Nauk SSSR **172** (1967), 33–36. MR 36 #884

[42] L. S. Pontryagin, *A classification of continuous transformations of a complex into a sphere. 1*, Doklady Akad. Nauk SSSR (N.S.) **19** (1938), 361–363.

[43] ———, *A classification of continuous transformations of a complex into a sphere. 2*, Doklady Akad. Nauk SSSR (N.S.) **19** (1938), 147–149.

[44] ———, *Homotopy classification of mappings of an $(n+2)$-dimensional sphere on an n-dimensional one*, Doklady Akad. Nauk SSSR (N.S.) **19** (1950), 957–959, Russian.

[45] D. G. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bulletin of the American Mathematical Society **75** (1969), 1293–1298.

[46] D. C. Ravenel, *Localization with respect to certain periodic homology theories*,

American Journal of Mathematics **106** (1984), 351–414.

[47] ———, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, New York, 1986.

[48] Douglas C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Princeton University Press, Princeton, NJ, 1992, Appendix C by Jeff Smith. MR 94b:55015

[49] G. Segal, *Elliptic cohomology*, Séminaire Bourbaki 1987/88, Astérisque, vol. 161-162, Societe Mathematique de France, Féverier 1988, 187–201.

[50] Jean-Pierre Serre, *Sur la suspension de Freudenthal*, C. R. Acad. Sci. Paris **234** (1952), 1340–1342. MR 13,675d

[51] ———, *Sur les groupes d'Eilenberg-MacLane*, C. R. Acad. Sci. Paris **234** (1952), 1243–1245. MR 13,675c

[52] ———, *Formes modulaires et fonctions zêta p-adiques*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), Springer, Berlin, 1973, 191–268. Lecture Notes in Math., Vol. 350. MR 53 #7949a

[53] R. E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, 1968.

[54] R. Thom, *Quelques propriétés globales des variétés differentiables*, Commentarii Mathematici Helvetici **28** (1954), 17–86.

[55] Friedhelm Waldhausen, *Algebraic K-theory of topological spaces. I*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, Amer. Math. Soc., Providence, R.I., 1978, 35–60. MR 81i:18014a

[56] ———, *Algebraic K-theory of topological spaces. II*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Springer, Berlin, 1979, 356–394. MR 81i:18014b

[57] E. Witten, *The index of the Dirac operator in loop space*, Elliptic Curves and Modular Forms in Algebraic Topology (New York) (P. S. Landweber, ed.), Lecture Notes in Mathematics, vol. 1326, Springer-Verlag, 1988, 161–181.

[58] Edward Witten, *Five-brane effective action in M-theory*, J. Geom. Phys. **22** (1997), no. 2, 103–133, hep-th/9610234.