Notes on Gel'fand–Fuks Cohomology and Characteristic Classes

by

M. Mostow and J. Perchik

These notes were prepared in response to
a series of lectures given by
Raoul Bott
at the Eleventh Holiday Symposium
at New Mexico State University,
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Preface

These notes make up the written record of the Eleventh Holiday Symp sium at New Mexico State University in Las Cruces, New Mexico on 27- December 1973. Professor Raoul Bott of Harvard University was the princi lecturer and the notes from his lectures comprise the first (and major) port of these Proceedings. In addition to Professor Bott's lectures, there were s sions for contributed papers and versions of these talks ranging from abstra to actual texts comprise the balance of the Proceedings. There is also appenc a list of participants.

Financial support for this Symposium was supplied in part by New Mex State University and to a large extent by the National Science Foundati To these institutions the Symposium Committee wishes to record its than Since all Symposium activities (including board and lodging of the participan were held at the Holy Cross Retreat of the Franciscan Friars, the Symposi Committee also thanks them for their warm hospitality.

Robert J. Wisner
Warren M. Krueger
New Mexico State University
May 1975

Foreword

The following notes were written by Mark Mostow and Jim Perchik in response to my lectures at the 1973 Holiday Symposium in Las Cruces, New Mexico.

My aim during this highly concentrated Symposium – two lectures a day for five days by the main speaker as well as two or three more by interested participants – was to explain to nonexperts some of the phenomena which I associate with the work of Gel'fand–Fuks on the Lie algebra of vector fields of a smooth manifold.

In particular I tried to outline: (1) a "new" proof of the finite dimensionality of the G.F. cohomology along lines devised by G. Segal and myself; (2) the relationship of this cohomology to characteristic classes of various sorts, and especially the ones encountered in Foliation theory à la Haefliger.[†]

My lectures were mainly informal with theorems illustrated by example, rather than proved in detail, and this of course makes note taking in the strong sense very difficult. Rather, Jim and Mark have each worked out a section of the material according to their own tastes and have supplied proofs, background material and precise references where they most felt the need for them.

By and large I am very pleased with what they have wrought and I hope that nonexperts in some parts of the material will also find this most unBourbaki-like account of a piece of mathematics refreshing.

Since these lectures were delivered the field has moved forward considerably: At this date, I have in hand independent outlines of the proof of my main conjecture (pg. 175), one by G. Segal and one by A. Haefliger. Quite recently, P. Trauber and Stasheff–Anderson have announced an affirmative solution. In the opposite direction I should point out that during the write up of these notes Jim Perchik pointed out some technical difficulties in our original argument to the G.F. spectral sequence, but this was cleared up by G. Segal after an anxious S.O.S. from us! Also, Mark Mostow pointed out that much greater care has to be taken in the geometric realization of simplicial spaces than I originally did and he has explained and clarified these matters in his write up.

Granting the truth of the conjecture – and I still haven't truly digested any of the versions – the most challenging next step in this general area would seem to me to be a proof that $F\Gamma_q$ (that is $B\overline{\Gamma}_q$) is $2q$ connected. (Thurston shows it to be $(q+1)$ connected.) Also of course, the case of Hamiltonian manifolds is still quite unexplored, and in particular, the finite dimensionality in that case is completely untouched.

Raoul Bott
October 9, 1974

† I acknowledge with thanks that part of this research was supported by NSF grant GP4361.

Contents

Notes on Gel'fand–Fuks Cohomology and Characteristic Classes

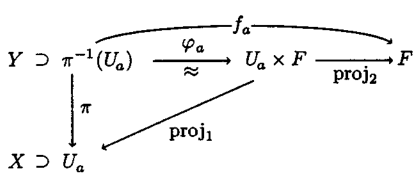
M. Mostow and J. Perchik

1. Bundles, classifying spaces, and characteristic classes

We regard a bundle as a twisted product. (See [41] and [64] for more details on bundles.)

Definition. Let X, Y, and F be topological spaces. Let G be a topological group which is a subgroup of $\mathrm{Aut}(F) = \{\text{homeomorphisms of } F\}$ (i.e., G acts continuously on F on the left). Let $\pi : Y \to X$ be a continuous surjection such that F is homeomorphic to $\pi^{-1}(x)$ for all $x \in X$. Then the combination $\{Y, \pi, X\}$ is a *bundle* (also a *G-bundle*) with *base space X*, *total space Y*, *fiber F*, and *structural group G*, if it satisfies the following local triviality and gluing conditions:

1) There exists an open covering $\{U_a\}$ of X and maps (called *horizontal projections*)
$$f_a : \pi^{-1}(U_a) \longrightarrow F$$
such that for all indices a, the map
$$\varphi_a = (\pi, f_a) : \pi^{-1}(U_a) \longrightarrow U_a \times F$$
is a homeomorphism (see diagram).



2) For all pairs of indices a, b, there is a continuous map
$$g_{ba} : U_a \cap U_b \longrightarrow G$$
(called a *transition function*) which relates f_a and f_b. Specifically, for all y in $\pi^{-1}(U_a \cap U_b)$, if $x = \pi(y)$, then
$$f_b(y) = g_{ba}(x) \cdot f_a(y)$$

Note. By abuse of notation one often refers to the total space Y as bundle.

Definition. Let all notation be as above, and suppose $F = G$. Let equation
$$f_b(y) = g_{ba}(x) \cdot f_a(y)$$
(in the definition of bundle) denote multiplication in G. Then the bundle i *principal G-bundle.* The multiplication map
$$F \times G \longrightarrow F$$
defines a *right* action of G on Y which preserves each fiber. Specifically, in coordinates, for $g \in G$ and $y \in \pi^{-1}(U_a)$,
$$(\pi(y), f_a(y)) \cdot g = (\pi(y), f_a(y) \cdot g),$$
and this definition is independent of coordinates.

Definition. A bundle with fiber $F = \mathbb{R}^n$ or $\mathbb{C}^n$ and structure gr $G = GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, resp., acting in the usual way on F, is calle *vector bundle* or an *n-plane bundle.*

The transition maps g_{ba} describe the twisting of a bundle. For exampl bundle is *trivial*, i.e., a product $X \times F$, if and only if the maps g_{ba} can all taken to be
$$g_{ba} : U_a \cap U_b \longrightarrow id \in G.$$
One observes that the maps g_{ba} satisfy a compatibility condition, ca the *cocycle condition*, on triple intersections. Namely,
$$g_{ab}(x) \cdot g_{bc}(x) = g_{ac}(x) \quad \text{for all} \quad x \in U_a \cap U_b \cap U_c.$$
In fact, the maps $\{g_{ab}\}$ contain all the information necessary to reconstruct bundle, if we are also given the action of G on the fiber F; and given any se maps $\{g_{ab}\}$ satisfying the cocycle condition (such a set $\{g_{ab}\}$ is called a *coc on X with values in G*), one can construct a bundle having $\{g_{ab}\}$ as transit functions. One simply takes the disjoint union
$$\coprod_a U_a \times F$$
and divides by the relations
$$(x, f) \in U_a \times F \sim (x, g_{ba}(x) \cdot f) \in U_b \times F.$$
A *bundle map* is a commutative diagram
$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \pi' \downarrow & & \downarrow \pi \\ X' & \longrightarrow & X \end{array}$$

where $\pi : Y \to X$ and $\pi' : Y' \to X'$ are G-bundles with the same fiber F, and

$$Y' \longrightarrow Y$$

restricted to any fiber is a homeomorphism belonging to G. Two bundles with the same base space X are *isomorphic* if there is a bundle map

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \pi' \downarrow & & \downarrow \pi \\ X' & \xrightarrow{=} & X \end{array}$$

In terms of cocycles, the condition for two bundles over X to be isomorphic is the *coboundary condition.* Namely, if there exist maps,

$$h_a : U_a \longrightarrow G$$

such that

$$h_b(x) \cdot g_{ba}(x) = g'_{ba}(x) \cdot h_a(x) \quad \text{for all} \quad x \in U_a \cap U_b,$$

then the cocycles $\{g_{ba}\}$ and $\{g'_{ba}\}$ define isomorphic bundles.

One defines $H^1(X;G)$ to be the set of isomorphism classes of G-bundles over X. Roughly speaking, $H^1(X;G)$ is the set of cocycles $\{\{g_{ab}\}\}$ modulo the set of coboundaries $\{\{h_a\}\}$. In case G is the whole group $\mathrm{Aut}(F)$, $H^1(X;G)$ is the set of isomorphism classes of fiber bundles over X with fiber F.

Now G-bundles pull back under continuous maps. Specifically, if

$$\pi : Y \longrightarrow X$$

is a G-bundle with cocycle $\{g_{ab}\}$ relative to the cover $\{U_a\}$ of X, and if

$$f : X' \longrightarrow X$$

is a continuous map, then by defining $f^{-1}(Y) = Y'$ to be

$$X' \, x_{f,\pi} \, Y,$$

the fiber product of X' and Y [41], we have a commutative diagram

$$\begin{array}{ccc} f^{-1}(Y) = Y' & \longrightarrow & Y \\ \pi' \downarrow & & \downarrow \pi \\ X' & \underset{f}{\longrightarrow} & X. \end{array}$$

$\pi' : Y' \to X'$ is a G-bundle (denoted $f^{-1}(Y)$, or $f^{-1}(\xi)$ if $\xi =$ the bundle $\{Y, \pi, X\}$), defined relative to the open cover $\{f^{-1}(U_a)\}$ of X' by the cocycle $\{g_{ab} \circ f\}$. Thus the above diagram is a G-bundle map.

Let Q now be the functor from the category *Spaces* (= *Topological spaces*) to *Sets* which takes a space X to

$$Q(X) = H^1(X;G) = \{\text{isomorphism classes of } G\text{-bundles over } X\},$$

and a map $f : X' \to X$ into the map

$$Q(f) : Q(X) \to Q(X')$$

defined by

$$Q(f)(\overline{Y}) = \overline{f^{-1}(Y)},$$

where $\pi : Y \to X$ is a bundle and $\overline{Y}$ its isomorphism class.

Q is a contravariant functor from *Spaces* to *Sets*, and is in fact *homoto* *invariant*, i.e., if f and g are homotopic maps from X' to X, then

$$Q(f) = Q(g) \quad \text{as maps:} \quad Q(X) \to Q(X').$$

This follows from the standard results of bundle theory [41].

Let us now fix a G-bundle

$$\pi : E \longrightarrow Z.$$

Let

$$[X, Z] = \{\text{homotopy classes of maps:} \quad X \to Z\}.$$

Then $[\cdot, Z]$ is a homotopy invariant contravariant functor from *Spaces* to *S* By the homotopy invariance of the functor Q, there is a natural transformati of functors from $[\cdot, z]$ to Q. Namely, to each space X, we assign the map

$$[X, Z] \longrightarrow Q(X)$$

given by

$$(f : X \longrightarrow Z) \mapsto \overline{f^{-1}(E)}.$$

If the map $[X, Z] \to Q(X)$ is an isomorphism for each X, we say that

$$[\cdot, Z] \longrightarrow Q$$

is an *isomorphism of functors*, and that Z is a *classifying space* for the func Q (in our case, Z is a classifying space for G-bundles, but the definitions valid for *any* homotopy invariant contravariant functor Q : *Spaces*→ *Sets*). [

There are several ways of finding a classifying space Z[11,6,41,64]. By universal classifying property, Z is unique up to homotopy type if it exists.

Definition. BG will always denote any classifying space Z for G-bundl If $\pi : E \to BG$ is a principal G-bundle, then E will be denoted EG.

$$\pi : EG \longrightarrow BG$$

is called a *universal principal G-bundle.*

The following characterization of the classifying spaces is due to Steenr ([64], p. 102).

Theorem 1.1. *Let G be a topological subgroup, and let $\pi : E \to Z$ a principal G-bundle. Then Z is a classifying space for G-bundles on fin simplicial complexes if and only if $\pi_i(E) = 0$ for all i.*

We have deliberately been rather vague about what the fiber of any particular G-bundle is, because we are most interested in its cocycle $\{g_{ab}\}$. This vagueness is justified by the following. As we saw above, given any cocycle $\{g_{ab}\}$ we can construct a bundle with any given fiber F (where F is a space upon which G acts on the left), having $\{g_{ab}\}$ as its cocycle. Alternatively, suppose we have a *principal* G-bundle

$$\pi : Y \longrightarrow X$$

and want to find a bundle with the same cocycle $\{g_{ab}\}$ but with fiber F.

Exercise. Show that such a bundle is given by

$$Y \times_G F \longrightarrow X,$$

where $Y \times_G F = Y \times F$ modulo the action of G; i.e.,

$$(yg, f) \sim (y, gf) \quad \text{for} \quad y \in Y,\, f \in F,\, g \in G.$$

In particular,

$$EG \times_G F \longrightarrow BG$$

is a *universal G-bundle with fiber F*; notice that the base space BG does not depend on F.

In general, to represent any functor Q via a classifying space, one must restrict the category of spaces, for example to finite polyhedra or paracompact spaces. Later on we shall represent functors such as

$$Q = \{\text{homotopy classes of foliations on manifolds}\}$$

and

$$Q = (\text{Gel'fand–Fuks cohomology of a manifold})$$

in terms of appropriate classifying spaces (though the classification of Gel'fand–Fuks cohomology will be different from the type of classification being discussed now).

Assuming we have found a classifying space Z for any functor Q (for example $\{G\text{-bundles}\}$), the obstructions to trivializing or extending a *Q-structure* (i.e., an element of $Q(X)$, where X is some space), can be studied by standard obstruction theory [64]. Specifically, let

$$h \in Q(X)$$

be a Q-structure on a space X. Then h is classified by the homotopy class of a map

$$f_h : X \longrightarrow Z.$$

h is a *trivial Q-structure* if and only if f_h is homotopic to a trivial map. For example, in the case

$$Q = \{\text{isomorphism classes of } G\text{-bundles}\}$$

and $Z = BG$, as before, we have the diagram

$$\begin{array}{ccc} f_h^{-1}(EG) & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f_h} & BG = Z \end{array}$$

and the bundle $h = (f_h^{-1}(EG) \to X)$ is *trivial* (i.e., a product G-bundle $X \times$ if and only if f_h is a nullhomotopic map.

The obstructions to homotoping f_h to a trivial map lie in

$$H^q(X; \pi_q(Z)), \quad q = 0, 1, 2, \dots ;$$

the "universal obstructions" lie in $\pi_q(Z)$, $q = 0, 1, \dots$. One can regard elements of $H^*(Z)$ as another measurement of the nontriviality of the func Q, since if

$$s \in H^*(Z),\ f_h : X \to Z \quad \text{classifies } h \in Q(X), \text{ and } f_h^* s \neq 0 \text{ in } H^*(X),$$

then f_h is not nullhomotopic, so h is not a trivial Q-structure on X.

Whether or not a functor

$$Q : \textit{Spaces} \longrightarrow \textit{Sets}$$

has a classifying space Z, we define a *characteristic class* $s = \{s_h\}$ of the func Q to be a choice

$$s_h \in H^*(X) \qquad \text{for each } h \in Q(X) \text{ for each space } X,$$

such that if

$$f : X' \longrightarrow X$$

and

$$h' \in Q(X') \quad \text{and} \quad h \in Q(X)$$

are such that

$$h' = Q(f)(h),$$

then

$$s_{h'} = f^* s_h,$$

where

$$Q(f) : Q(X) \longrightarrow Q(X')$$

and

$$f^* : H^*(X) \longrightarrow H^*(X').$$

In other words, a characteristic class is a choice of cohomology classes natu with respect to morphisms of Q-structures [53]. For example, a characteris class for G-bundles is a choice of

$$s_h \in H^*(X)$$

for each G-bundle

$$h = (\pi : Y \longrightarrow X),$$

such that given a G-bundle map

$$h'\left\{\begin{array}{ccc} Y' & \longrightarrow & Y \\ \pi'\downarrow & & \downarrow\pi \\ X' & \underset{f}{\longrightarrow} & X \end{array}\right\} h,$$

$$f^* s_h = s_{h'}, \in H^*(X').$$

If the functor Q does have a classifying space Z, let s be an element of $H^*(Z)$. If for each $h \in Q(X)$ with classifying map

$$f_h : X \longrightarrow Z$$

we assign

$$s_h \underset{def.}{=} f_h^*(s) \in H^*(X),$$

then $\{s_h\}$ is a characteristic class for Q. For example, if $Q = \{G\text{-bundle isom. classes}\}$ and

$$s \in H^*(BG),$$

then for each bundle $h = (Y \to X)$, classified by

$$h\left\{\begin{array}{ccc} Y & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \underset{f_h}{\longrightarrow} & BG \end{array}\right.$$

we have

$$s_h = f_h^*(s) \in H^*(X).$$

Exercise. For any homotopy invariant contravariant functor Q: *Spaces* $\to$ *Sets* with classifying space Z, the map of sets

$$H^*(Z) \longrightarrow \{\text{characteristic classes of } Q\}$$

defined by

$$s \longrightarrow \{s_h\}$$

is a 1-1 correspondence.

Note. Because of this result, we shall usually let s designate the characteristic class $\{s_h\}$.

Examples of classifying spaces and characteristic classes for G-bundles

Example 1. Let G be a discrete group, i.e., any group with the discrete topology imposed on its underlying set. Let $K = K(G,1)$ be an Eilenberg–MacLane space, i.e., a CW complex with

$$\pi_1(K) = G \qquad \pi_i(K) = 0,\ i \neq 1.$$

Such spaces always exist [20]. Let $\widetilde{K}$ be the universal covering space of
Then

$$\pi_i(\widetilde{K}) = 0 \quad \text{for all} \quad i,$$

so $\widetilde{K}$ is contractible. G acts freely and discretely on $\widetilde{K}$, and

$$K = \widetilde{K}/G.$$

Exercise. $\widetilde{K} \to K$ is a G-bundle.

Thus by Theorem 1.1,

$$BG = K, \quad EG = \widetilde{K}.$$

In this case $\pi_i(BG) = G$ for $i = 1$ and 0 otherwise, but $H^*(BG)$ is hard compute for most discrete groups G.

Example 1a. Let $G = \mathbb{Z}_2$. Then $BG = \mathbb{R}P_\infty = S^\infty/\mathbb{Z}_2$ = infin dimensional real projective space.

Example 1b. $G = \mathbb{Z}$. Then $BG = S^1 = \mathbb{R}/\mathbb{Z}$. Since

$$H^i(S^1) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 1 \\ 0 & \text{otherwise,} \end{cases}$$

$\mathbb{Z}$-bundles have characteristic classes only in dimensions 0 and 1.

Example 1c. G non-Abelian and discrete. Here BG is hard to fi explicitly.

Example 2. $G = \pi_1(S)$, where S is a Riemann surface of genus > 1. Th

$$BG = S.$$

Existence of BG for all Lie groups G

For all Lie groups G, in particular for $G = GL(n,\mathbb{R}), GL(n,\mathbb{C}), O(n), U($ and other matrix groups, BG exists ([3], also cf. infra §6). In contrast to t discrete case, here $H^*(BG)$ is relatively easy to compute, while $\pi_i(BG)$ is ha to find.

By Theorem 1.1 a principal G bundle $E \to Z$ is universal if and only if is contractible. Thus a strategy to find BG is to look for a contractible spa E upon which G acts freely, let

$$Z = E/G,$$

and then prove that $E \to Z$ is a bundle, which proves that $Z = BG$. (The hard part is to show that Z has open cover $\{U_a\}$ which trivializes the bundle. This can usually be done for Lie groups using topological arguments.)

Example 1. Embed G as a subgroup of $GL(n)$ for suitably large n, via a representation of G. Let

$$\begin{aligned} E &= \{n\text{-frames in } \mathbb{R}^\infty\} \\ &= \{(v_1, \ldots, v_n) | v_i \in \mathbb{R}^\infty \\ &\quad \text{and } v_1, \ldots, v_n \text{are linearly independent}\}, \end{aligned}$$

where

$$\mathbb{R}^\infty = \{(a_1, a_2, \ldots) | a_i \in \mathbb{R} \quad \text{and the sequence is eventually} \quad 0\}.$$

Exercise. Prove E is contractible [41].

Now $GL(n, \mathbb{R})$ acts freely on E (via multiplication of an n-tuple of vectors by an $n \times n$ matrix), so $G \subset GL(n)$ acts freely on E. Thus we can take

$$BG = E/G.$$

Definition. Let E be as in Example 1, and let $G = GL(n, \mathbb{R})$. Then E/G is called the *Grassmannian* space of n-planes in $\mathbb{R}^\infty$.

Example 2. One constructs Grassmannians from quotients $U(n + m)/U(m)$ for large m.

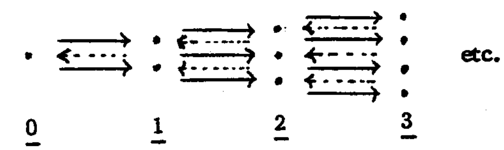
2. Semi-simplicial objects

There is a very general kind of construction, called a semi-simplicial object, which will appear again and again in these notes. (See [21], [56], and [50] for general references.) Philosophically, one can regard a semi-simplicial space as a generalization of a space, or a semi-simplicial module as a generalization of a module, in much the same way that a chain or cochain complex is a generalization of a module. In fact, one can even construct a semi-simplicial space which is a "free resolution" of a manifold, in the sense that a chain complex is a free resolution of a module. This will be used later to prove the finite dimensionality of the Gel'fand–Fuks cohomology of a compact manifold.

Semi-simplicial objects can be defined most concisely using the language of category theory [56]. Let *Ord* be the category whose objects are finite sets ordered by a transitive $\leq$ relation, and whose morphisms are order-preserving set maps. The objects of *Ord* are called 0,1,2, etc. where $0 = \{0\} = \cdot$, $1 = \{0, 1\} =:$, $2 = \{0, 1, 2\} = \vdots$. The ordering of the set n is $0 \leq 1 \leq \ldots \leq n$. Of

all the morphisms in *Ord*, we focus attention on the injections from n to $n+$ which we call ∂_i or *face maps*, and the surjections from $n + 1$ to n which call ϵ_i or *degeneracy maps*. (In fact, all morphisms in *Ord* are compositions face and degeneracy maps.)

The category *Ord* is represented by the picture



In this picture, the sets of vertically placed dots are the objects 0,1,2,.. *Ord*. The solid arrows are the morphisms ∂_i and the dashed arrows are t morphisms ϵ_i. These morphisms are defined by the following:

$$\partial_i : n \to n+1, \quad i = 0, 1, \ldots, n+1$$

$$\partial_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq 1 \end{cases} \qquad \text{for } j \in n = \{0, 1, \ldots, n\}.$$

That is, ∂_i maps n to $n+1$ injectively, preserving order and omitting $i \in n$ - from its range.

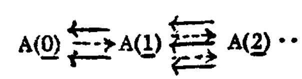
$$\epsilon_i : n \longrightarrow n-1, \quad i = 0, \ldots, n-1$$

$$\epsilon_i(j) = \begin{cases} j & \text{if} \quad j \leq i \\ j-1 & \text{if} \quad j > i \end{cases} \qquad \text{if } j \in n = \{0, 1, \ldots, n\}.$$

That is, ϵ_i maps n to $n - 1$ surjectively, preserving order and mapping i a $i + 1 \in n$ to $i \in n - 1$.

Observe that there are $n+2$ face maps ∂_i from n to $n+1$, and n degenera maps ϵ_i from n to $n - 1$.

Now let C be any category, for example *Sets, Topological spaces,* C^∞ *m ifolds, Modules.* Let A be a contravariant functor from *Ord* to C. Then is called a semi-simplicial object (abbreviated s.s. object) in C, i.e., a se simplicial space, manifold, module, etc. If B is a covariant functor from (to C, then B *co-semi-simplicial* (co-s.s.) space, module, etc. (The nomenc ture reflects the fact that semi-simplicial objects were studied before co-se simplicial objects.) Explicitly, let $A(n)$ be the object in C associated to t object n in *Ord*. Then for each face map $\partial_i : n \to n + 1$, there is a m $A(\partial_i) : A(n + 1) \to A(n)$. Designating $A(\partial_i)$ by a solid arrow as before, write the semi-simplicial object A as



The co-semi-simplicial object B is written

$$B(\underline{0}) \rightleftarrows B(\underline{1}) \;\substack{\longrightarrow\\ \leftarrow\\ \longrightarrow\\ \leftarrow\\ \longrightarrow}\; B(\underline{2}) \cdots .$$

Example. Given a polyhedron P with an ordering on its vertices, let $\sum(n)$ be the set of abstract n-simplexes of P, i.e., the set of ordered $(n+1)$-tuples $(v_0, \ldots, v_n)$ of vertices v_i such that $(v_0, \ldots, v_n)$ is a face of P and $v_0 \leq v_1 \leq \ldots \leq v_n$ in the given ordering. Then $\sum$ is a semi-simplicial set with face maps $\sum(\partial_i)$ and degeneracy maps $\sum(\epsilon_i)$ (which we call ∂_i and ϵ_i by abuse of notation, even though $\sum$ is a contravariant functor) defined by

$$\partial_i(v_0, \ldots, v_n) = (v_0, \ldots, \hat{v}_i, \ldots, v_n), \quad i = 0, \ldots, n$$
$$\epsilon_i(v_0, \ldots, v_n) = (v_0, \ldots, v_i, v_i, \ldots, v_n), \quad i = 0, \ldots, n.$$

Note that the union of the images of all the ϵ_i's is exactly the set of degenerate simplexes, i.e. simplexes with a vertex repeated.

Now there are three important general constructions done on semi-simplicial objects. The first of these is geometric realization of a semi-simplicial space.

Let X be any semi-simplicial space, i.e.

$$X : \mathit{Ord} \longrightarrow \mathit{Top.\ spaces}.$$

(As a special case we may allow X to be an s.s. set, since we may make any set into a topological space by giving it the discrete topology). We write X as

$$X(\underline{0}) \;\substack{\longleftarrow\\ \rightarrow\\ \longleftarrow}\; X(\underline{1}) \;\substack{\longleftarrow\\ \rightarrow\\ \longleftarrow\\ \rightarrow\\ \longleftarrow}\; X(\underline{2}) \cdots .$$

Define a co-semi-simplicial space $\Delta =$

$$\Delta^0 \;\substack{\longrightarrow\\ \leftarrow\\ \longrightarrow}\; \Delta^1 \;\substack{\longrightarrow\\ \leftarrow\\ \longrightarrow\\ \leftarrow\\ \longrightarrow}\; \Delta^2 \cdots$$

where Δ^n is the Euclidean n-simplex, and

$$\partial_i = \Delta(\partial_i) : \Delta^{n-1} \longrightarrow \Delta^n$$

and

$$\epsilon_i = \Delta(\epsilon_i) : \Delta^n \longrightarrow \Delta^{n-1}$$

are the standard face and degeneracy maps of singular homology theory. In barycentric coordinates,

$$\Delta(\partial_i)(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

and

$$\Delta(\epsilon_i)(t_0, \ldots, t_n) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_n).$$

Now $|X|$, the *geometric realization* of X, is defined to be

$$\left[\coprod_n (X(n) \times \Delta^n)\right] / \sim,$$

the quotient space of the disjoint union (topological sum) of the spaces

$$X(n) \times \Delta^n$$

under the relations

$$(x, \Delta(f)(t)) \sim (X(f)(x), t),$$

or in abuse of notation

$$(x, f(t)) \sim (f(x), t)$$

for $x \in X(m)$, $t \in \Delta^n$, and $f : n \longrightarrow m$ any morphism in *Ord*. (These relatior are generated by the relations with $f = \partial_i$ or $f = \epsilon_i$.)

Example. Let P be a polyhedron with $\sum$ its associated abstract simplici complex as before. Then

$$|\Sigma| = \coprod_n \Sigma(n) \times \Delta^n$$

modulo the relations

$$[(v_0, \ldots, v_n), (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)]$$
$$\sim [(v_0, \ldots, \hat{v}_i, \ldots, v_n), (t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)]$$

and

$$(v_0, \ldots, v_i, v_i, \ldots, v_n), (t_0, \ldots, t_{n+1})]$$
$$\sim [(v_0, \ldots, v_i, \ldots, v_n), (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1})].$$

Writing $[(v_0, \ldots, v_n), (t_0, \ldots, t_n)]$ as $t_0 v_0 + \ldots + t_n v_n$, we recognize $|\Sigma|$ as tl usual geometric realization of Σ, namely $|\Sigma| = P$.

The second construction that arises with semi-simplicial objects is makir a chain complex out of a semi-simplicial module (or a cochain complex out a co-semi-simplicial module). Let M be a semi-simplicial module, $M =$

$$M(\underline{0}) \;\substack{\longleftarrow\\ \rightarrow\\ \longleftarrow}\; M(1) \;\substack{\longleftarrow\\ \rightarrow\\ \longleftarrow\\ \rightarrow\\ \longleftarrow}\; M(2) \cdots .$$

Define a boundary map

$$\delta : M(n) \longrightarrow M(n-1)$$

by taking the alternating sum of the face maps from $M(n)$ to $M(n-1)$, i.e.

$$\delta = \sum_{i=0}^{n} (-1)^i M(\partial_i),$$

where $\partial_i : n-1 \longrightarrow n$. The fact that M is a functor implies that $\delta^2 = 0$ that

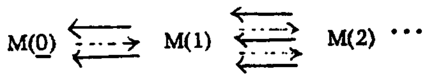
$$M(0) \underset{\delta}{\leftarrow} M(1) \underset{\delta}{\leftarrow} M(2) \cdots$$

is a chain complex, sometimes denoted $k \circ M$; i.e. k:*Semi-simplicial modules-Chain complexes* is the functor which takes alternating sums of face map Similarly, if N is a co-semi-simplicial module, then $k \circ N$ is a cochain comple

Note. By abuse of notation, we shall often write $H_\delta(M(^*))$ or $H_\delta(M)$ to mean $H_\delta(ko)$ the homology of M, or refer to a semi-simplicial module as a chain complex.

In computing the homology of the chain complex $k \circ M$ or of the cochain complex $k \circ N$, it is often useful to work with a normalized complex in which the degenerate elements have been quotiented out. More precisely, one proceeds as follows. (See [18] for details and proofs.)

Let $M =$



be an s.s. module. For each module M_n, we define the submodules

$$M'_n = \bigcap_{i=1}^{n} \ker(\partial_i : M_n \to M_{n-1}) \quad \text{(N.B. } \partial_0 \text{ is omitted)}$$

$$DM_n = \sum_{i=0}^{n-1} \text{image} \quad (\epsilon_i : M_{n-1} \to M_n) \ \text{(all } \epsilon_i \text{ are included).}$$

Exercise. $\{M'_n\}$ is a subcomplex of $k \circ M$ (It is called the *normal subcomplex*). The boundary map $\sum_{i=0}^{n}(-1)^i\partial_i$ reduces to ∂_0 on this subcomplex. $\{DM_n, \sum_{i=0}^{n}(-1)^i\partial_i\}$ is a subcomplex of $k \circ M$ (called the *subcomplex of degenerate elements*).

Theorem 2.1. (Normalization Theorem of Eilenberg–MacLane. See [18], pp. 218–225)

1. $k \circ M \cong M' \oplus DM$, i.e.

$$M_n \cong M'_n \oplus DM_n \ \forall n.$$

2. *The inclusion $M' \to k \circ M$ of the normal subcomplex into the full complex is a homotopy equivalence of chain complexes. DM is null-homotopic.*

Definition. M/DM is called the *normalized* (chain) *complex* of M.

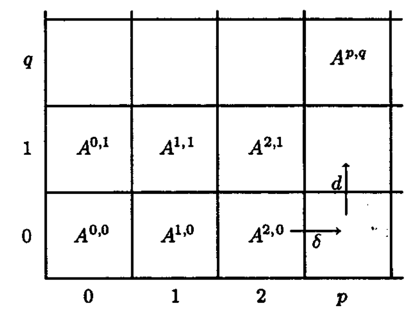
Corollary 2.1.1. *The composition $M' \to kM \to M/DM$ is an isomorphism between the normal subcomplex and the normalized complex.*

Corollary 2.1.2. $H_\delta(k \circ M) \to H_\delta(M/DM)$ *is an isomorphism, where H_δ denotes homology of chain complexes under the boundary map $\delta = \sum_{i=0}^{n}(-1)^i\partial_i$.*

Note. Analogous results exist for a co-semi-simplicial module N. In this case we define the degenerate complex by $DN = N/\bigcap \ker \epsilon_i$ and the normal complex by $N' = N/\sum_{i>0} \text{image } \partial_i$. ([18], pp. 283–290).

3. Spectral sequences and double complexes

We recall some basic notions from spectral sequences as they appl computing the (co) homology of a double complex [61,58]. Assume we ha double complex $A = \oplus_{p,q} A^{p,q}$ of R-modules (R is any ring).



A is bigraded by p and q, and graded by $p+q$. Let there be given bound maps δ and d on A such that $\delta : A^{p,q} \to A^{p+1,q}, d : A^{p,q} \to A^{p,q+1}$, and $d\delta =$ (One could also consider the cases $\delta : A^{p,q} \to A^{p-1,q}$, etc.) We assume t for each fixed n, $A^{p,n-p}$ is 0 for all but a finite number of p. This is true, example, if $A^{p,q}$ is a first quadrant complex as in the diagram, i.e., if $A^{p,q}$ if p or q is negative. We define a boundary map

$$D = d \pm \delta = d + (-1)^q\delta$$

on A. Then A is a cochain complex with respect to D, and we seek $H_D(A)$

To find $H_D(A)$ using spectral sequences, we filter A as

$$A = F_0 \supset F_1 \supset F_2 \supset \cdots,$$

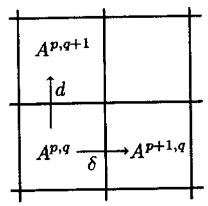
where

$$F_i = \oplus_{p \geq i} A^{p,q}.$$

The associated graded module $\oplus_p F_p/F_{p+1}$ of the filtration is called E_0. E bigraded by the filter degree p and the complementary degree q, and graded $p+q$. In our case

$$E_0^{p,q} \cong A^{p,q}.$$

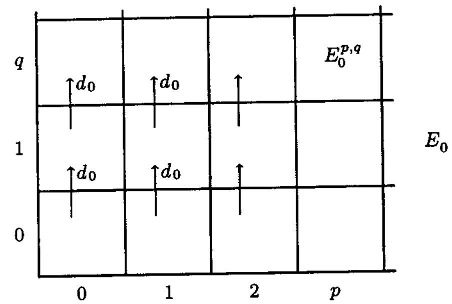
Thus the original picture of A above is also a picture of E_0. Now the map D on $A^{p,q}$ is $d \pm \delta$,



and this induces the map $d : E_0^{p,q} \to E_0^{p,q+1}$ since in $E_0^{p,q}$ we are working modulo $A^{p+1,q}$. We denote by d_0 the map d on $E_0^{p,q}$, and define

$$E_1 = H_{d_0}(E_0),$$

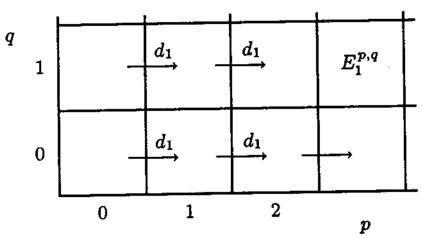
i.e., E_1 is the homology of the complex



under the maps d_0. Hence

$$E_1 = H_d(A).$$

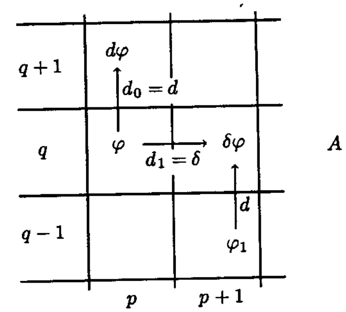
D induces a map $d_1 = \delta$ on E_1:



We define

$$E_2 = H_{d_1}(E_1) = H_\delta(H_d(A)).$$

What does it mean to say $\varphi \in E_2^{p,q}$?

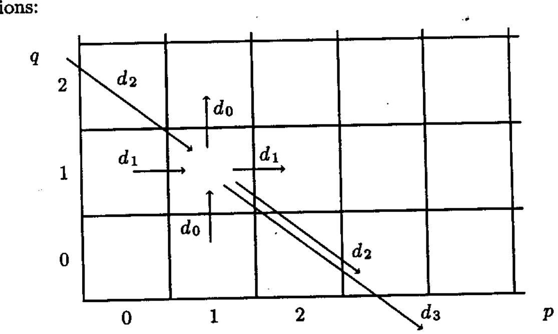


We can consider $\varphi \in A^{p,q}$ with the following properties. First, $0 = d_0\varphi = d$ Second $0 = d_1\varphi = \delta\varphi$ in $E_1^{p+1,q}$. This means that $\delta\varphi$ is a boundary in E_0^{p+} with respect to d_0, i.e., that there exists φ_1 in $A^{p+1,q-1}$ such that $d\varphi_1 = \delta$ Thus $D(\varphi \pm \varphi_1)$ has entry 0 in $A^{p,q+1}$ and in $A^{p+1,q}$. Thus $E_2^{p,q}$ consists elements of $A^{p,q}$ which can be continued downward one step towards bei D-cocycles. Actually $E_2^{p,q}$ is the quotient of these partial cocycles by parti coboundaries.

The spectral sequence continues:

$$H_{d_2}(E_2) = E_3, \ldots, H_{d_n}(E_n) = E_{n+1},$$

where $d_2, d_3, \ldots$ are maps induced by D. The maps d_i point in the followi directions:

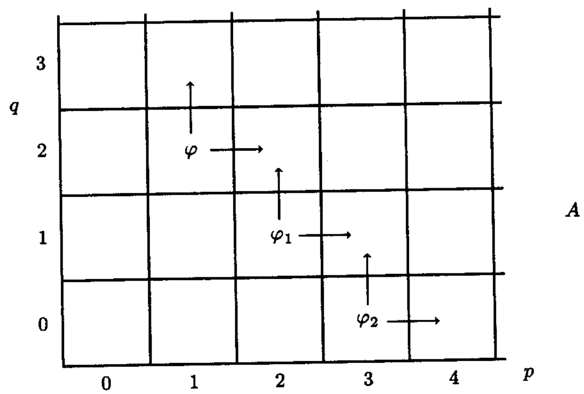


In general, d_i increases the filter degree (p) by i, and increases total degree 1. As the diagram shows, d_i eventually points to or comes from boxes whi are 0, so the spectral sequence eventually stabilizes in each box (p, q), and wh it does, we write $E_n^{p,q} = E_\infty^{p,q}$.

The result of spectral sequence theory is that

$\oplus_{p+q=n} E_\infty^{p,q}$ *is the associated graded module of* $H_D^n(A)$
which is isomorphic as a vector space to $H_D^n(A)$ *if* R *is a field.*

We note that any non-zero element $E_\infty^{p,q}$ can be represented by an element φ of $A^{p,q}$ which can be completed downward to form a D-cocycle which is not a D-coboundary.



For example, if $\varphi \in A^{1,2}$ is non-zero in $E_\infty^{1,2}$, then there exist φ_1, φ_2 as shown so that $D(\varphi + \varphi_1 + \varphi_2) = 0$ and $(\varphi + \varphi_1 + \varphi_2)$ is not a D-coboundary.

Recall that filtering by p gave $E_1 = H_d(A), E_2 = H_\delta(H_d(A))$ and completed D-cocycles downward. Similarly, we could filter A by q. This would give a spectral sequence with

$$E_1 = H_\delta(A),\ E_2 = H_d(H_\delta(A)),$$

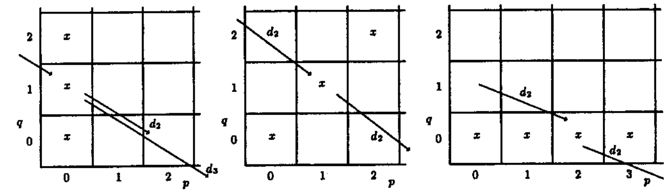
and in which cocycles are completed upward. Both filtrations give the same answer for $H_D(A)$, so in both cases (if R is a field)

$$\oplus_{p+q=n} E_\infty^{p,q}$$

is the same, though $E_\infty^{p,q}$ need not be the same.

In many important cases E_2 "collapses," i.e., so many terms $E_2^{p,q}$ are 0 that all succeeding maps $d_2, d_3, \ldots$ point to or come from boxes with 0's in them, so that $E_\infty^{p,q} = E_2^{p,q}$. The following configurations are examples of such E_2 terms:

(blank boxes denote 0, x's denote non-zero entries)



(The d_2 maps are drawn assuming filtering was by p, but $E_\infty = E_2$ also filtering had been by q.)

4. Cohomology of a semi-simplicial space

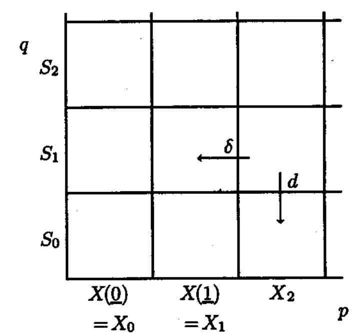
Let X: *Ord* $\to$ *Spaces* be a s.s space, and let $|X|$ be its geometric realizatio Let S_q: *Spaces* $\to$ *Groups* be the functor which assigns to each space its grou of singular q-chains with coefficients in $\mathbb{Z}$. Then

$$S_qX : Ord \longrightarrow Groups$$

is a contravariant functor, hence an s.s. group. Applying the alternating su functor k of Section 2, we make S_qX into a chain complex:

$$S_q(X)(0) \xleftarrow{\delta} S_qX(1) \xleftarrow{\delta} \cdots$$

At the same time we have the singular boundary maps $d : S_qX(\underline{n}) \to S_{q-1}X(\underline{n}$ d and δ commute, so there is a double complex S_*X:



$$S_*X = \bigoplus_{p,q} S_qX(\underline{p})$$

Let $D = d + (-1)^q\delta$, and grade S_*X by $p + q$. Then (S_*X, D) is a chai complex, so $H_D(S_*X)$ is defined.

Theorem 4.1 (Moore). *There is a natural chain map $f : (S_*X, D) \to (S_*(|X|), d)$. f is a chain equivalence (hence induces isomorphism in homology and cohomology with any coefficients) if X is "good" in the following sense:*

Definition. X is *good* if each X_n is compactly generated (meaning that X_n is Hausdorff and that $C \subset X_n$ is closed whenever $C \cap K$ is closed for every compact subspace $K \subset X_n$), and if each degeneracy map $\epsilon_i : X_{n-1} \to X_n$ (which is automatically an inclusion, since $\partial_i \epsilon_i = id : X_{n-1} \to X_{n-1}$) is a closed cofibration (cofibration means that the pair $(X_n, \epsilon_i(X_{n-1}))$ has the homotopy extension property with respect to any space).

Proof. We define the unnormalized geometric realization of X, denoted $\|X\|$, to be the space

$$\coprod_n X_n \times \Delta^n / \sim,$$

where $\sim$ means to mod out only by face maps ∂_i (rather than by face *and* degeneracy maps as in the construction of $|X|$ (see Section 2)). There is a natural map $i : \|X\| \to |X|$, and if X is good then i is a homotopy equivalence [Segal, 1974]. Hence the theorem will follow from

Theorem 4.2. *For any s.s. space X, there is a natural chain equivalence*

$$f : (S_*X, D) \longrightarrow (S_*(\|X\|), d).$$

Proof. Let Y and Z denote any two topological spaces. By acyclic models, Eilenberg and Zilber showed there exists a chain equivalence

$$B : S_*(Y) \otimes S_*(Z) \longrightarrow S_*(Y \times Z)$$

which is functorial in Y and Z. We shall use a particular choice of B called the *shuffle homomorphism* [Greenberg, p. 208] defined by

$$B(\partial \otimes \tau) = \sum \pm(\epsilon_{j_q} \cdots \epsilon_{j_1} \partial,\ \epsilon_{i_p} \cdots \epsilon_{i_1} \tau)$$

where ∂ is a singular p-simplex of Y, τ a singular q-simplex of Z, ϵ_k is the k-th degeneracy map, the sum is over all permutations $(i_1, \ldots, i_p, j_1, \ldots, j_p)$ of $(0, \ldots, p+q-1)$ such that $i_1 < i_2 < \ldots < i_p$ and $j_1 < \ldots < j_q$, and the sign is the signature of the permutation.

Let Δ^p be the Euclidean p-simplex, and let $\partial_p \in S_p(\Delta^p)$ be the identity map of Δ^p. Define a map

$$f : S_q(X_p) \longrightarrow S_{p+q}(\|X\|)$$

by the composition

$$S_q(X_p) \xrightarrow{\otimes \partial_p} S_q(X_p) \otimes S_p(\Delta^p) \xrightarrow{B} S_{p+q}(X_p \times \Delta^p) \xrightarrow[\pi_*]{} S_{p+q}(\|X\|)$$

where $\pi : X_p \times \Delta^p \to \|X\|$ is the canonical map. To show that f is a ch[a]in map

$$f : (S_*X, D) \longrightarrow (S_*(\|X\|), d),$$

it suffices to show that

$$\begin{array}{ccc} S_q(X_p) & \xrightarrow{f} & S_{p+q}(\|X\|) \\ \Big\downarrow D = d + (-1)^q \delta & & \Big\downarrow d \\ S_{q-1}(X_p) \oplus S_q(X_{p-1}) & \xrightarrow{f} & S_{p+q-1}(\|X\|) \end{array}$$

commutes. Let $a \in S_q(X_p)$. Then

$$\begin{aligned} df(a) &= d\pi_* B(a \otimes \partial_p) = \pi_* Bd(a \otimes \partial_p) \\ &= \pi_* B(da \otimes \partial_p + (-1)^q a \otimes d\partial_p) \end{aligned}$$

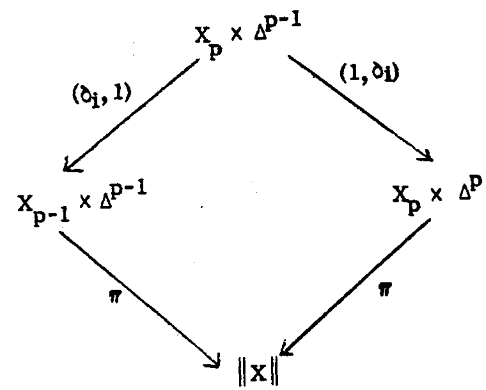
while

$$fD(a) = f(da + (-1)^q \delta a) = \pi_* B(da \otimes \partial_p) + (-1)^q \pi_* B(\delta a \otimes \partial_{p-1}).$$

Thus to show that $df = fD$ it suffices to show that

$$\pi_* B(a \otimes d\partial_p) = \pi_* B(\delta a \otimes \partial_{p-1}).$$

Now consider the diagram



which commutes by the construction of $\|X\|$. Applying the functor S_* to the diagram and choosing $B(a \otimes \partial_{p-1}) \in S_{p+q-1}(X_p \times \Delta^{p-1})$ shows that

$$\pi_* B(\partial_i a \otimes \partial_{p-1}) = \pi_* B(a \otimes \partial_i \partial_{p-1}).$$

(This uses the naturality of B, i.e., $(\partial_i, 1)_* B = B(\partial_i, 1)_*$ and similarly for $(1, \partial_i)$.) But by definition,

$$\sum_i (-1)^i \partial_i a = \delta a \qquad \sum_i (-1)^i \partial_i \partial_{p-1} = d\partial_p$$

so

$$\pi_* B(\delta a \otimes \partial_{p-1}) = \pi_* B(a \otimes d\partial_p),$$

as was to be shown.

To prove that f is a chain equivalence, it suffices, since it is a morphism of chains of free Abelian groups, to show that f induces isomorphism in homology [Spanier, p. 192]. We prove isomorphism by filtering both chain complexes and showing that the induced map of spectral sequences is an isomorphism on the E^1 level (see [61] for more details on spectral sequences).

We filter (S_*X, D) by

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \ldots, \cup_p F_p = S_*X,$$

where

$$F_p = \oplus_{i=0}^{p} S_* X_i.$$

In $\|X\|$, let $Y_p = \|X\|_p$ be the image of $X_p \times \Delta^p$ (under the map π). Then

$$\phi = Y_{-1} \subset Y_0 \subset Y_1 \subset \ldots, \cup_p Y_p = \|X\|$$

induces a filtration $F'_p = S_*(Y_p)$,

$$0 = F'_{-1} \subset F'_0 \subset F'_1 \subset \ldots$$

of $(S_*(\|X\|), d)$. One detail must be proved:

Lemma 4.3. $\cup_{p=0}^{\infty} F'_p = S_*(\|X\|)$.

Proof. We have to prove that any map

$$\Delta^n \longrightarrow \|X\|$$

factors

$$\Delta^n \longrightarrow Y_p \xrightarrow{\pi} \|X\|$$

for some p. Since the image of Δ^n is compact, it suffices to prove that any compact $K \subset \|X\|$ lies in Y_p for some p. We use a modified form of Whitehead's proof that a compact subspace of a CW complex lies in some p-skeleton [73]; the modifications are necessary because $\|X\|$, unlike a CW complex, need not be Hausdorff or even $T1$. (In fact, this can cause this lemma to be false if we use $|X|$ in place of $\|X\|$).

Let Z be the trivial s.s. set with Z_p =(point) for all p. Then there is a unique s.s. map $g : X \to Z$, which induces $h : \|X\| \to \|Z\|$. $\|Z\|$ contain one copy of Δ^n for each n. Now h maps $\|X\|_p$ to $\|Z\|_p = \pi(pt. \times \Delta^p)$ an furthermore, $h^{-1}(\|Z\|_p) = \|X\|_p$. Thus to be done it suffices to prove tha any compact $K \subset \|Z\|$ lies in $\|Z\|_p$ for some p. Now we can use Whitehead' argument, since points are closed in $\|Z\|$. Suppose that K is not contained i any $\|Z\|_p$. Then we can pick a sequence of distinct points $z_1, z_2, \ldots$ in K suc that only finitely many z_i's lie in $\|Z\|_p$ for any p. Let $W = \{z_i\}_{i=1}^{\infty}$. Now W is closed in $\|Z\|$, since $\pi^{-1}(W) \subset \coprod_p (pt. \times \Delta^p)$ is the union of finitely man points in each $(pt. \times \Delta^p)$. By the same argument, every subset of W is close in $\|Z\|$. Hence W is discrete. Since $W \subset K$ with W closed and K compact W is compact. Thus W is compact, discrete, and infinite, which is impossible Q.E.D.

Returning to the proof of the theorem, we see that

$$f : (S_*X, D) \longrightarrow (S_*(\|X\|), d)$$

is filtration preserving, i.e., $f(F_p) \subset F'_p$. Hence f induces maps $f^i : E^i \to E^i$, $i = 0, 1, \ldots, \infty$ of the respective spectral sequences. By [61], Theorem 9.1.3 if f^i is an isomorphism for some $i \geq 1$, then $f_* : H_D(S_*X) \to H_d(S_*(\|X\|))$ i an isomorphism. Thus we shall be done when we show that f^1 is an isomor phism.

On the E^0 level we have

$$E^0_p = F_p/F_{p-1} = S_*X_p; \quad E^0_{p,q} = S_q X_p$$
$$d_0 : E^0_p \to E^0_p \text{ is the map } \ d : S_q X_p \to S_{q-1} X_p.$$
$$'E^0_p = F'_p/F'_{p-1} = S_*Y_p/S_*Y_{p-1} = S_*(Y_p, Y_{p-1})$$

$$'E^0_{p,q} = S_{p+q}(Y_p, Y_{p-1}).$$
$$d'_0 : \ 'E^0_p \to \ 'E^0_p \quad \text{is the map}$$
$$d : S_{p+q}(Y_p, Y_{p-1}) \to S_{p+q-1}(Y_p, Y_{p-1}).$$
$$f^0 : E^0_{p,q} \to \ 'E^0_{p,q} \quad \text{is the composition}$$

$$S_q X_p \xrightarrow{\otimes \partial_p} S_q X_p \otimes S_p \Delta^p \xrightarrow{B} S_{p+q}(X_p \times \Delta^p) \xrightarrow{\pi_*} S_{p+q}(Y_p, Y_{p-1}).$$

Letting $\dot{\Delta}^p$ denote the boundary of Δ^p, the functoriality of the maps show that f^0 can also be written as the composition

$$S_q X_p \xrightarrow{\otimes \partial_p} S_q(X_p) \otimes S_p(\Delta^p, \dot{\Delta}^p) \xrightarrow{B'} S_{p+q}(X_p \times (\Delta^p, \dot{\Delta}^p))$$
$$\xrightarrow{\pi_*} S_{p+q}(Y_p, Y_{p-1}) S_{p+q}(X_p \times \Delta^p, X_p \times \dot{\Delta}^p)$$

where we now regard $\partial_p \in S_p(\Delta^p, \dot{\Delta}^p)$, and B' can be defined from B since E is natural. Furthermore, B' is a chain equivalence ([17], pp. 180-1).

The three maps $\otimes\partial_p$, B', and π_* each commute with d. Taking H_d of each term above gives the map $f^1 : E^1_{p,q} \to {}'E^1_{p,q}$; it is the composition

$$E^1_{p,q} = H_q(X_p) \xrightarrow{(\otimes\partial_p)_*} H_{p+q}(S_qX_p \otimes S_p(\Delta^p, \dot\Delta^p))$$
$$\xrightarrow{B'_*} H_{p+q}(X_p \times \Delta^p, X_p \times \dot\Delta^p) \xrightarrow{\pi_*} H_{p+q}(Y_p, Y_{p-1}) = {}'E^1_{p,q}.$$

By the Künneth theorem ([17], VI. 9. 13), together with the fact

$$\begin{cases} H_p(\Delta^p, \dot\Delta^p) &= Z, \quad \text{generated by } \partial_p \\ H_n(\Delta^p, \dot\Delta^p) &= 0, \quad n \neq p \end{cases}$$

We see that $H_{p+q}(S_qX_p \otimes S_p(\Delta^p, \dot\Delta^p)) = H_qX_p \otimes H_p(\Delta^p, \dot\Delta^p)$ and that $(\otimes\partial_p)_*$ is therefore an isomorphism. B'_* is an isomorphism since B' is a chain equivalence. Thus to show that f^1 is an isomorphism, which will finish the proof, it suffices to show that

$$\pi_* : H_n(X_p \times \Delta^p, X_p \times \dot\Delta^p) \longrightarrow H_n(Y_p, Y_{p-1})$$

is an isomorphism for all n.

Pick a neighborhood U of $\dot\Delta^p$ in Δ^p and a deformation

$$h_t : U \to U; h_0 = id, \quad h_1 : U \to \dot\Delta^p, \quad h_t|\dot\Delta^p = id \quad \text{for all } t.$$

Then $X_p \times \dot\Delta^p$ is a strong deformation retract of $X_p \times U$, and Y_{p-1} is a s.d. retract of $\pi(X_p \times U)$. The latter follows from the fact that Y_p has the topology quotient of $\coprod_{i=0}^p X_p \times \Delta^p$ (i.e., this topology coincides with the topology of Y_p as a subspace of $\|X\|$).

Thus it suffices to show that

$$\pi_* : H_n(X_p \times \Delta^p,\ X_p \times U) \xrightarrow{\cong} H_n(Y_p, \pi(X_p \times U)).$$

Now excising $X_p \times \dot\Delta^p$ from $X_p \times U$ and Y_{p-1} from $\pi(X_p \times U)$, it suffices to show

$$\pi_* : H_n(X_p \times (\Delta^p - \dot\Delta^p),\ X_p \times (U - \dot\Delta^p)) \xrightarrow{\cong} H_n(Y_p - Y_{p-1}, \pi(X_p \times U) - Y_{p-1}).$$

But

$$\pi : (X_p \times (\Delta^p - \dot\Delta^p)) \longrightarrow (Y_p - Y_{p-1})$$

is a homeomorphism, as is

$$\pi : X_p \times (U - \dot\Delta^p) \to \pi(X_p \times U) - Y_{p-1} = \pi(X_p \times (U - \dot\Delta^p)),$$

so π_* is indeed an isomorphism. Q.E.D.

Remarks. The hypothesis of "goodness" in Theorem 4.1 is fulfilled if X is an s.s. set or s.s. CW complex, or if it is an s.s. manifold with degeneracy maps s_i which embed X_{n-1} as a submanifold of X_n. If X is *not* good, we can

either use $\|X\|$ instead of $|X|$, or we can use one of several other geometri realizations, such as the *unwinding* $|X_Z|$, defined by

$$|X_Z| = \coprod_{n,(i_0,\ldots,i_n)} X_n \times \Delta^n / \sim,$$

where $0 \leq i_0 < i_1 < \cdots < i_n$ are integers, and $\sim$ means all relations

$$(x; d_jt; i_0, \ldots, i_n) \sim (\partial_j x; t; i_0, \ldots, \hat{i_j}, \ldots, i_n),\ x \in X_n,\ t \in \Delta^{n-1}.$$

Theorem 4.4. *The map* $|X_Z| \to \|X\|$ *defined by*

$$(x; t; i_0, i_1, \ldots, i_n) \longrightarrow (x; t) \quad (x \in X_n,\ t \in \Delta^n)$$

induces isomorphism in homology.

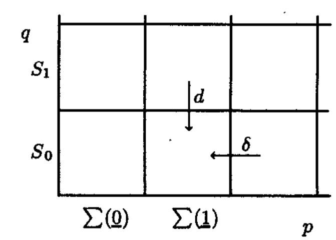
The proof, which we omit, is algebraic and uses a modification of the nor malization procedure for an s.s. module [18].

The major use of the unwinding is in constructing classifying spaces fo topological groups or categories, especially those with "bad" topologies (se [6], Stasheff's appendices B and C).

Example of Theorem 4.1. As in Section 2, let P be a polyhedron wit ordered vertices, and $\sum(n)$ the set of its n-simplices. $\sum$ is an s.s. set, an $|\Sigma| = P$. By Theorem 4.1 there is a chain equivalence

$$f : (S_*\Sigma, D) \longrightarrow (S_*(|\Sigma|), d) = (S_*P, d).$$

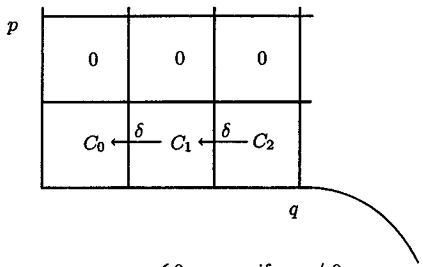
Now $(S_*\Sigma, D)$ is the double complex



Filtering $S_*\Sigma$ by p, we find $E^1_{p,q} = H^d_q(S_* \sum(p)) = H_q(\sum(p))$. But since $\sum(p$ is a discrete space,

$$H_q(\textstyle\sum(p)) = \begin{cases} 0 & \text{if} \quad q \neq 0 \\ C_p & \text{if} \quad q = 0 \end{cases}$$

where C_p is the free Abelian group generated by the elements of $\sum(p)$. Thus $E^1 =$



and

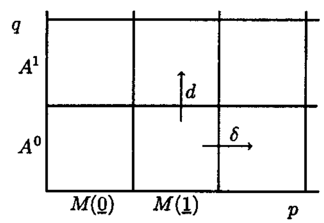
$$E^2_{p,q} = \begin{cases} 0 & \text{if} \quad q \neq 0 \\ H^\delta_p(C_*) & \text{if} \quad q = 0. \end{cases}$$

Since the spectral sequence collapses, we find

$$E^2_{p,0} = E^\infty_{p,0} = H_p(S_*\Sigma).$$

Thus by Theorem 4.1, $H^\delta_p(C_*) = H_p(S_*P) = H_p(P)$. Analogous results hold for cohomology or for other coefficients. Thus in this special case, Theorem 4.1 reduces to the familiar result that the (co)homology of a (geometric) simplicial complex equals the (co)homology of the free chain complex (resp. dual of the free chain complex) on its simplices.

Now let M be an s.s. manifold. Let A^q: *Manifolds*→ $\mathbb{R}$-*Modules* be the functor which assigns to each manifold the vector space of its differential q-forms, and let d be the DeRham differential from A^q to A^{q+1}. Then A^qM is a co-s.s. module, and in the usual way we form the double complex $AM = \oplus A^qM(p) =$



Now if N is a manifold we call a singular q-simplex $\partial \in S_q(N)$ *smooth* if $\partial : \Delta^q \to N$ can be extended to a smooth map defined on a neighborhood of Δ^q in $\mathbb{R}^q$. Let $S^*_{sm.}(N)$ denote the module of singular cochains of N which are only defined on smooth simplices. There is a restriction map $r : S^*N \to S^*_{sm.}N$ which induces isomorphism in cohomology by the DeRham theorem (one approximates cycles by smooth cycles to prove this.) Now integrating q-forms over smooth q-cycles gives a natural map of cochain complexes

$$\varphi : AN \to S^*_{sm.}N$$

of DeRham theory into smooth singular theory. By the DeRham theorem ι induces isomorphism in cohomology. Composing this with the inverse of r gives a natural isomorphism

$$H_d(AN) \xrightarrow{\sim} H_d(S^*N).$$

Applying these facts when $N = M(p)$, $p = 0, 1, \ldots$ and using the naturalit to show that r and φ commute with δ, we see that there are maps of doubl complexes

$$AM \xrightarrow{\varphi} S^*_{sm.}M \xleftarrow{r} S^*M;$$

filtering each complex by p and taking H_d gives us the $E^1_{p,q}$ terms

$$H^q_d(A^*M(p)) \underset{\varphi^*}{\xrightarrow{\sim}} H^q_d(S^*_{sm.}M(p)) \underset{r^*}{\xleftarrow{\sim}} H^q_d(S^*M(p)).$$

Since φ^* and r^* induce isomorphism of E^1 terms, they induce isomorphisms

$$H_D(AM) \underset{\sim}{\xrightarrow{\varphi^*}} H_D(S^*_{sm.}M) \underset{\sim}{\xleftarrow{r^*}} H_D(S^*M).$$

Combining this with Theorem 4.1 we have proved

Theorem 4.5. *There is a natural isomorphism*

$$H_D(AM) \xrightarrow{\sim} H_D(S^*M) = H^*(|M|) \quad \text{(coefficients in } \mathbb{R}\text{)}.$$

Thus we can compute the cohomology of $|M|$ using differential forms (on tł $M(p)$), even though $|M|$ is *not* a manifold!

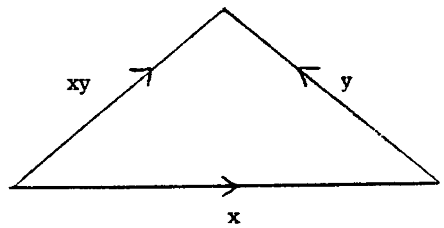
5. A semi-simplicial construction of BG for a discrete group G

Given a discrete group G, there are several constructions of a classifyir space BG (which, since G is discrete, is an Eilenberg–MacLane space; se Section 1).

We start one such construction [20] by choosing a point to be the 0-skeletc of BG, and adding an oriented loop for each element $x \in G$. One then attach 2-simplices to kill each relation of the form

$$xy = z \qquad \text{for} \quad x, y, z \in G.$$

For example, if $x, y \in G$, then one fills in the triangle



(which has all 3 vertices identified to a point) with a 2 simplex corresponding to the pair (x, y). Similarly, one kills the relation

$$x_1 x_2 \cdots x_n = y$$

by suitably attaching an n-simplex, corresponding to the n-tuple $(x_1, \ldots, x_n)$. The CW complex thus obtained is a $K(G, 1)$, hence is a BG (see Section 1).

The above construction can be described easily in semi-simplitandarcial terms. Let NG be the semi-simplicial set, that is, functor from *Ord* to *Sets*, with

$$\begin{aligned} NG(0) &= * = \text{a point}, \\ NG(1) &= G, \\ NG(2) &= G \times G, \\ NG(p) &= G^p \underset{def}{=} G \times \cdots \times G (p \text{ factors}). \end{aligned}$$

The maps ∂_i and ϵ_i (actually $NG(\partial_i)$ and $NG(\epsilon_i)$) are defined by:

$$\begin{aligned} &\partial_i : G^p \longrightarrow G^{p-1}, \quad i = 0, \ldots, p \\ &\partial_0(x_1, \ldots, x_p) = (x_2, \ldots, x_p) \\ &\partial_p(x_1, \ldots, x_p) = (x_1, \ldots, x_{p-1}) \\ &\partial_i(x_1, \ldots, x_p) = (x_1, \ldots, x_i x_{i+1}, \ldots, x_p), \quad 1 \le i \le p-1. \end{aligned}$$

$$\begin{aligned} &\epsilon_i : G^{p-1} \longrightarrow G^p, \quad i = 0, \ldots, p-1 \\ &\epsilon_0(x_1, \ldots, x_{p-1}) = (1, x_1, \ldots, x_{p-1}) \\ &\epsilon_i(x_1, \ldots, x_{p-1}) = (x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_{p-1}), \quad 1 \le i \le p-1. \end{aligned}$$

For example,

$$\begin{aligned} \partial_0(x) &= \partial_1(x) = *, \quad x \in G \\ \partial_0(x, y) &= y \quad (x, y) \in G \times G \\ \partial_1(x, y) &= xy \\ \partial_2(x, y) &= x \quad . \end{aligned}$$

We now define

$$BG = |NG| = \text{geometric realization of } NG \quad [\text{cf. Section 2}].$$

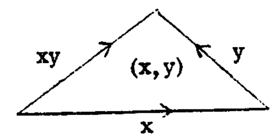
That is,

$$NG = \coprod_{p=0}^{\infty} G^p \times \Delta^p / \sim,$$

where $\sim$ means to identify via face and degeneracy maps.

$$\begin{aligned} BG &= \text{point} \cup (\text{a 1-simplex for each } x \in G) \\ &\cup \quad (\text{a 2-simplex for each pair } (x, y) \in G \times G) \cup \cdots, \end{aligned}$$

with the 2-simplex (x, y) glued to the 1-skeleton by



as we described above.

We can now compute $H^*(BG)$ by applying Theorem 4.1.

Theorem 5.1. $H^*(BG) = H_D(S^*NG)$*(for G a discrete group).*

Since NG is discrete, like $\sum$ in the example after Theorem 4.4, $H^p_D(S^*N$ reduces to $H^p_\delta(C^*)$, where

$$\begin{aligned} C^p &= \text{free module generated by } NG(p) \\ &= \text{Maps} \quad (G^p, R) \quad (\text{where } R \text{ is the coefficient group}) \\ C^* &= C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} \cdots \end{aligned}$$

We recognize C^* as the complex used to compute $H^*_{EM}(G; R)$, the Eilenbe MacLane cohomology of groups, hence

$$H^*_{spaces}(BG) = H^*_{EM}(G)$$

[19].

Of course, we could have avoided using Theorem 4.1 by constructing directly as the cochain complex generated by the cells of BG, a CW comp The map

$$\delta = \sum (-1)^i \partial_i,$$

an alternating sum of face maps, is just the boundary map in the coch complex of the CW complex.

We have proven

Theorem 5.2. *If G is a discrete group, $H^*(BG;R)$ = cohomology of the complex*

$$\text{Maps }(*,R) \xrightarrow{\delta} \text{Maps }(G,R) \xrightarrow{\delta} \text{Maps }(G\times G,R) \longrightarrow \cdots.$$

Theorem 5.2 gives us a combinatorial way to compute $H^*(BG)$. (We can find $H_*(BG)$ similarly). Unfortunately, the combatinorics are very difficult in all but the simplest cases.

6. Construction of BG for a topological group G

The same semi-simplicial construction used to construct BG for a discrete group G carries over almost verbatim when G is a topological group (cf. supra Section 5; also [56]). As before, we construct a semi-simplicial space $NG =$

$$* \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; G \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; G\times G \;\cdots$$

with the same ∂_i and ϵ_i as in Section 5, and let $BG = |NG| =$

$$(*\times\Delta^0)\coprod(G\times\Delta^I)\coprod(G\times G\times\Delta^2)\coprod\cdots$$

modulo gluing via the maps ∂_i and ϵ_i. The only difference from the case of G discrete is that now we topologize G with its given topology, not the discrete topology, so that $G\times G\times\Delta^2$, for example, has the product topology inherited from the topology of G and the standard topology of Δ^2. Thus BG is not canonically a simplicial complex or CW complex as it is when G is discrete. (Observe, by the way, that ∂_i and ε_i just involve multiplication in G and identity maps, which are continuous.) The construction of BG as $|NG|$ is a variation of Milnor's joint construction. Other constructions of BG have been made by Dold–Lashof, Stasheff, Steenrod–Rothenber, Milgram, Segal, and Dyer. The various constructions of BG are compared in Stasheff's appendix to [6].

We prove now that $BG = |NG|$ is really a classifying space for G-bundles.

Theorem 6.1. *Let G be a Lie group. There exists a semi-simplicial space PG and a map*

$$PG \longrightarrow NG$$

such that $PG\to NG$ and the induced map

$$|PG| \longrightarrow |NG|$$

are fibrations with fiber G, and such that $|PG|$ is contractible. Hence

$$\begin{array}{ccc} |PG| & = & EG \\ \downarrow & & \\ |NG| & = & BG \end{array}$$

is a universal principal G-bundle.

Proof. We define the semi-simplicial space $PG =$

$$G \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; G\times G \;\substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow\\ \dashrightarrow\\ \longleftarrow}\; G\times G\times G \;\cdots$$

$$PG(\underline{n}) = G^{n+1},$$

where the maps

$$\partial_i : G^{p+1}\longrightarrow G^p,\quad i=0,\dots,p$$

are given by

$$\partial_i(x_0,\dots,x_p) = (x_0,\dots,\hat{x}_i,\dots,x_p),$$

and

$$\epsilon_i : G^{p+1}\longrightarrow G^{p+2},\quad i=0,\dots,p,$$

is given by

$$\epsilon_i(x_0,\dots,x_p) = (x_0,\dots,x_i,x_i,\dots,x_p).$$

Now there is a map

$$\pi : PG\longrightarrow NG$$

given by

$$\begin{array}{ccccccc} PG = (G & \substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow} & G\times G & \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} & G\times G\times G\;\cdots) \\ \pi\downarrow\quad \pi\downarrow & & \pi\downarrow & & \pi\downarrow \\ NG = (* & \substack{\longleftarrow\\ \dashrightarrow\\ \longleftarrow} & G & \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} & G\times G\quad\cdots), \end{array}$$

$$\pi(x_0,\dots,x_p) = (x_0x_1^{-1},x_1x_2^{-1},x_2x_3^{-1},\dots,x_{p-1}x_p^{-1}).$$

To see that the above diagram commutes, hence that π is a morphism of s simplicial spaces, we must check that

$$\pi\partial_i = \partial_i\pi \quad \text{for all maps}\quad \partial_i$$
$$\pi\epsilon_i = \epsilon_i\pi \quad \text{for all}\quad \epsilon_i.$$

For example, for $i\neq 0,p$, we compute

$$\begin{aligned}\pi\partial_i(x_0,\dots,x_p) &= \pi(x_0,\dots,\hat{x}_i,\dots,x_p)\\ &= (x_0x_1^{-1},\dots,x_{i-1}x_{i+1}^{-1},\dots,x_{p-1}x_p^{-1})\\ &= (x_0x_1^{-1},\dots,(x_{i-1}x_i^{-1})(x_ix_{i+1}^{-1}),\dots)\\ &= \partial_i\pi(x_0,\dots,x_p).\end{aligned}$$

We omit the straightforward computations that show that π commutes the other ∂_i and ϵ_i.

G acts freely on the right on PG in a manner that preserves the fibers of π, by

$$(x_0, \ldots, x_p) \cdot x = (x_0 x, \ldots, x_p x) \quad \text{for} \quad (x_0, \ldots, x_p) \in PG(p) \quad \text{and} \quad x \in G.$$

Exercise. $\pi : PG \to NG$ is a principal G-bundle.

It can be shown [56] that $|PG|$ is contractible, and that

$$|\pi| : |PG| \longrightarrow |NG|$$

is a principal G-bundle (at least if G is a topological group with a reasonable topology, for example if G is a Lie group). By Theorem 1.1, $|\pi| : |PG| \to |NG|$ is a universal principal G-bundle.

Remark. By different methods one can show that $|NG_Z|$ classifies G-bundles when G is any topological group or category. $|NG_Z|$ denotes the unwinding defined in Section 4. See Stasheff's appendix to [6], and [56].

H^*BG : Results.

H^*BG is known for almost all Lie groups G [3]. For example, for

$$G = U(n) \quad \text{or} \quad O(n),$$

H^*BG can be computed using the Serre spectral sequence and the induction on n [3]. Without going into this and other computation methods, we give the results [cf. infra Section 9; also 53].

1. $H^*(BU(n)) = \mathbb{Z}[c_1, \ldots, c_n]$, a polynomial ring (with no relations) in $c_1, \ldots, c_n$, with $c_i \in H^{2i}(BU(n))$. c_i is called the *i-th Chern class.* By our discussion in Section 1, these must be all the characteristic classes for bundles with structure group $U(n)$.
2. $BGL(n, \mathbb{C})$ is homotopy equivalent to $BU(n)$, as can be seen in two ways: a) $U(n)$ has the same homotopy type as $GL(n, \mathbb{C})$, since $GL(n, \mathbb{C})$ is the product of $U(n)$ and an affine space [13].
   b) Any complex vector bundle, i.e., bundle with fiber $\mathbb{C}^n$, and structure group $GL(n, \mathbb{C})$ acting in the usual way on $\mathbb{C}^n$ can be given a Hermitian metric on the fibers. This reduces the structure group to $U(n)$. Thus $H^*BGL(n, \mathbb{C}) = H^*BU(n) = \mathbb{Z}[c_1, \ldots, c_n]$, so the Chern classes are also the characteristic classes for $GL(n, \mathbb{C})$-bundles, and in particular, for $\mathbb{C}^n$-vector bundles.
3. By similar arguments $BGL(n, \mathbb{R})$ and $BO(n)$ have the same homotopy type. With coefficients in $\mathbb{Z}_2$,

$$H^*(BGL(n, \mathbb{R}); \mathbb{Z}_2) = H^*(BO(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n],$$

where $w_i \in H^i(BO(n); \mathbb{Z}_2)$ is the *i-th Stiefel–Whitney class.* These the characteristic classes with coefficients in $\mathbb{Z}_2$ for unoriented $\mathbb{R}^n$-pl bundles (real vector bundles).

4. With coefficients in $\mathbb{Q}$ (similar results obtain if we use $\mathbb{R}$ instead)

$$H^*(BGL(n, \mathbb{R}); \mathbb{Q}) = H^*(BO(n); \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_{[n/2]}],$$

where $p_i \in H^{4i}(BO(n); \mathbb{Q})$ is the *i-th Pontryagin class*, and $[n/2]$ me the greatest integer $\leq n/2$. The Pontryagin classes are the characteri classes with coefficients in $\mathbb{Q}$ (or $\mathbb{R}$) for unoriented $\mathbb{R}^n$-plane bundles.

Note. The conceptual way to understand Pontryagin classes is that cert singular cocycles $c_1, \ldots, c_n$ can be defined on $BO(n)$ with dim $c_i = 2i$. Th c_i are like Chern classes but the odd ones $c_1, c_3, c_5, \ldots$ are torsion eleme hence vanish in $H^*(BO(n); \mathbb{Q})$. The cohomology classes of the even "Ch classes" $c_2, c_4, \ldots$ are the Pontryagin classes $p_1, p_2, \ldots$.

5. If we look at *oriented* $\mathbb{R}^n$-plane bundles, the structure group is $SO($ With coefficients in $\mathbb{Q}$, the characteristic classes are the elements $H^*(BSO(n); \mathbb{Q})$. These turn out to be the Pontryagin classes $p_1, \ldots, p_{[}$ as for $O(n)$-bundles, but if n is even, we also get the *Euler class*

$$\chi \in H^n(BSO(n); \mathbb{Q})$$

which satisfies the single relation

$$\chi^2 = p_{[n/2]}.$$

$H^*(BG)$: Methods of computation

Using our semi-simplicial machinery, we find $H^*(BG)$ by applying Theo 4.1 to the semi-simplicial space NG to get

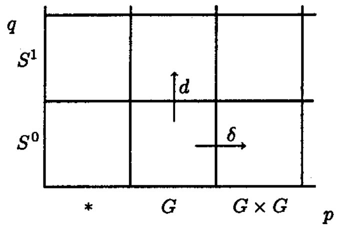
Theorem 6.2. *Let G be a Lie group. Then there are natural isomorphi*

$$H_D(S_*NG) \xrightarrow{\cong} H_*(BG) \quad \text{and} \quad H^*(BG) \xrightarrow{\cong} H_D(S^*NG).$$

Remark 1. This theorem is also true if G is any topological group category, if we take $BG = |NG_Z|$. The proof uses Theorems 4.2 and 4.4.

Remark 2. Theorem 6.2 is the same as Theorem 5.1 when G is a disc group. Of course Theorem 5.2 does not generalize to topological groups, si $H^q(G^p)$ need not be 0 when $q \neq 0$.

Graphically, $H^*(BG)$ is the cohomology of the double complex $S^*NG =$



with respect to the grading $p + q$ and coboundary $D = d \pm \delta$, where $S^i =$ singular i-cochains, $\delta : S^q(G^p) \to S^q(G^{p+1})$ is the alternating sum of the face maps

$$S^q(\partial_i) : S^q(G^p) \longrightarrow S^q(G^{p+1})$$

and d is the coboundary of singular cohomology. This construction shows explicitly that $H^*(BG)$ depends on the topology of G and multiplication in G.

To compute $H_D(S^*NG)$, we can filter S^*NG either horizontally or vertically to obtain a spectral sequence converging to H^*BG.

Filtering by p, we get

$$E_1^{p,q} = H_d^q(S^*NG(p)) = H^q(G^p) \quad \text{(singular cohomology of } G^p).$$

Filtering by q,

$$E_1^{p,q} = H_\delta^p(S^qNG),$$

a harder object to compute. (See Section 9.)

Example. Find H_*BG for $G = U(1) = S^1$. By Theorem 6.2, $H_*BG = H_D(S_*NG)$. We filter S_*NG by p to get $E_1^{p,q} = H_q(G^p)$. Now the q-th row of the double complex is

$$H_q(pt.) \longleftarrow H_q(G) \longleftarrow H_q(G^2) \cdots \qquad (*)$$

where $\delta = \sum_{i=0}^n (-1)^i \partial_i$. By Corollary 2.1.2, this complex has the same δ-homology as its normalized complex, which is obtained by dividing each $H_q(G^p)$ by the submodule of degenerate elements, i.e., by the images of $H_q(G^{p-1})$ under the degeneracy maps ϵ_i. G^p is a p-torus, and the maps ϵ_i are the inclusions $\epsilon_i : G^{p-1} \to G^p$ defined by

$$(x_1, \ldots, x_{p-1}) \longrightarrow \begin{cases} (x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_{p-1}), & i = 1, \ldots, p-1 \\ (1, x_1, \ldots, x_{p-1}), & i = 0. \end{cases}$$

Now $H_*(G^p) \cong \underbrace{H_*(G) \otimes \cdots \otimes H_*(G)}_{p \text{ factors}}$ and $H_*(G) = H_*(S^1) = E(w) =$ ext algebra generated by w in dimension 1. Thus

$$H_*(G^p) = E(w_1, \ldots, w_p), \quad w_i \in H_1(G^p)$$

and

$$\epsilon_i : H_*(G^{p-1}) \longrightarrow H_*(G^p)$$

is the map generated by

$$\epsilon_i(w_j) = w_j, \quad 1 \le j \le i$$
$$\epsilon_i(w_j) = w_{j+1}, \quad i < j \le p-1.$$

Thus every element of $H_*(G^p)$ of the form $w_{i_1} w_{i_2} \cdots w_{i_k}$ is in the image of ϵ_i, except for the element

$$w_1 w_2 \cdots w_p \in H_p(G^p).$$

(Geometrically, all elements of $H_q(G^p)$ for $q < p$ are generated by the inclu of lower dimensional tori via degeneracy maps).

Conclusion. After division by degeneracies, the chain complex (*) duced to $\mathbf{Z}$ in the q-th position. Thus

$$E_2^{p,q} = H_\delta(H_q(G^p)) = \begin{cases} \mathbf{Z}, & \text{if } p = q \\ 0, & \text{otherwise.} \end{cases}$$

$E_2^{p,q} =$

q \ p	0	1	2
2	0	0	$\mathbf{Z}$
1	0	$\mathbf{Z}$	0
0	$\mathbf{Z}$	0	0

Therefore $E_2 = E_\infty = H_D(S_*NG) = H_*BG$. Therefore

$$H_q(BU(1)) = \begin{cases} \mathbf{Z} & \text{if } q \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

(One can show that in cohomology, $H^*(BU(1)) = \mathbf{Z}[c_1], c_1 \in H^2(BU(1))$

If G is a Lie group, we can apply Theorem 4.5 to compute $H^*(BG)$ by using differential forms on pt., G, $G \times G, \cdots$.

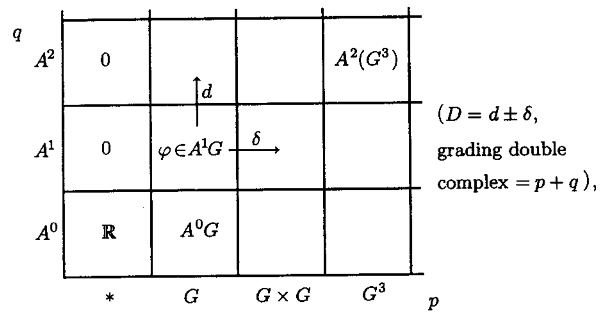
Theorem 6.3. *The DeRham map*

$$H_D(ANG) \longrightarrow H_D(S^*NG)$$

is an isomorphism, and both terms equal

$$H^*(|NG|) = H^*(BG;\mathbb{R}).$$

ANG is the double complex $\oplus A^q(G^p) =$



where $A^q(G^p) = \{\text{differential } q\text{-forms on } G^p\}$, d is the DeRham derivative on differential forms, and $\delta = \Sigma(-1)^i A^q(\partial_i)$ is the alternating sum of face maps on

$$A^q(\text{pt.}) \rightrightarrows A^q(G) \;\equiv\; A^q(G \times G) \quad \cdots.$$

For example, an element $\varphi \in A^1(G)$ is a D 2-cochain in ANG. In order that φ be a 2-cocycle on ANG, we must have $D\varphi = 0$, i.e., $d\varphi = 0$ and $\delta\varphi = 0$. Now $d\varphi = 0$ just means that φ is a closed 1-form on G. The map δ equals $\sum_{i=0}^{2}(-1)^i A^1(\partial_i)$, so since

$$H^*(G \times G;\mathbb{R}) = H^*(G;\mathbb{R}) \otimes H^*(G;\mathbb{R}),$$

it follows that

$$\delta\varphi = 1 \otimes \varphi - \mu^*\varphi + \varphi \otimes 1,$$

where

$$\mu : G \times G \longrightarrow G$$

is multiplication, $\mu(x,y) = xy$. Hence $\delta\varphi = 0$ if and only if $\mu^*\varphi = 1 \otimes \varphi + \varphi \otimes 1$, and the latter is just the definition that φ be a *primitive form* on G. In general,

a Lie group need not have any primitive forms, so that any non-zero D-co of ANG will be sums of entries in more than one box. But in special cas does have primitive forms.

Example 1. $G = S^1 = \mathbb{R}/2\pi\,\mathbb{Z}$ with θ as the coordinate of $\mathbb{R}/2\pi\mathbb{Z}$. $\mu : G \times G \to G$ is given by $\mu(\theta_1, \theta_2) = \theta_1 + \theta_2$, so

$$\mu^* d\theta = d(\theta_1 + \theta_1) = d\theta_1 + d\theta_2.$$

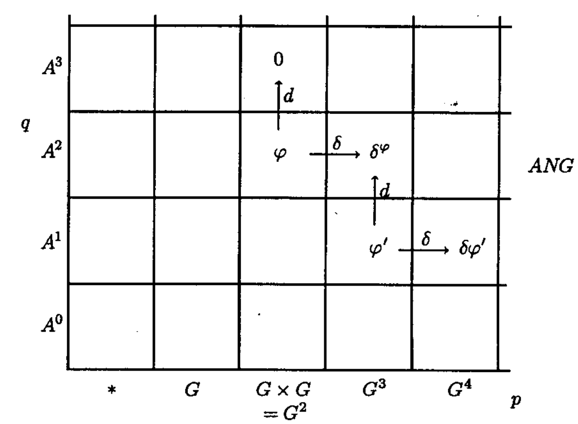
Therefore $d\theta$ is a closed primitive form on S^1.

Example 2. $G = \mathbb{C}^*$ with coordinate z. Then $\frac{dz}{z}$ is primitive bec $\mu(w,y) = wy$, so that

$$\mu^*\frac{dz}{z} = \frac{d(wy)}{wy} = \frac{wdy}{wy} + \frac{ydw}{wy} = \frac{dy}{y} + \frac{dw}{w}$$

Under the map exp: $S^1 \to C^*$ given by exp $(\theta) = e^{i\theta}$, we have exp* ($\frac{d}{z}$ $\frac{d(e^{i\theta})}{e^{i\theta}} = \frac{ie^{i\theta}d\theta}{e^{i\theta}} = id\theta$.

As mentioned above, in general a D-cocycle of ANG has entries in se boxes of ANG. For example, to build a D 4-cocycle on ANG, we might by choosing $\varphi \in A^2(G \times G)$ so that $d\varphi = 0$.



Now $d(\delta\varphi) = \delta(d\varphi) = 0$, so $\delta\varphi \in A^2(G^3)$ is a closed form. We next t find $\varphi' \in A^1(G^3)$ such that $d\varphi' = -\delta\varphi$ (of course φ' need not exist in gen

Then

$$D(\varphi+\varphi') = (d+(-1)^q\delta)(\varphi+\varphi') = (d+\delta)\varphi + (d-\delta)\varphi'$$
$$= 0 + \delta\varphi + d\varphi' - \delta\varphi' = -\delta\varphi' \in A^1(G^4).$$

Proceeding in this way, one can build a D-cocycle $\varphi + \varphi' + \cdots$, which proceeds from φ downwards and to the right. One could also work upwards and to the left. Observe that building D-cocycles (modulo D-coboundaries) step by step in this way is equivalent to computing the spectral sequence of the double complex [cf. Section 3].

7. Construction of characteristic classes for G-bundles by semi-simplicial methods

In Section 6 we showed that H^*BG can be represented by differential forms on $NG =$

$$* \longleftarrow G \longleftarrow G \times G \ \cdots .$$

Given a space X and a G-bundle η on X, classified by a map

$$f_\eta : X \longrightarrow BG,$$

the characteristic classes of η are just

$$f_\eta^*(H^*BG) \subset H^*X.$$

The trouble is, how do we know what the map f_η^* is in terms of our semi-simplicial setup?

Let us take the most naive approach to characteristic classes for bundles over manifolds. Given an open covering $\{U_a\}$ of a manifold M, a G-bundle η over M is determined by a cocycle of transition functions [cf. Section 1]

$$g_{ab} : U_a \cap U_b \longrightarrow G.$$

Given a form $\varphi_1 \in AG$, we can pull it back via the maps g_{ab} to get forms

$$g_{ab}^*\varphi_1 \in A(U_a \cap U_b).$$

On each triple intersection

$$U_{abc} \underset{def.}{=} U_a \cap U_b \cap U_c$$

we have the map

$$(g_{ab}, g_{bc}) : U_{abc} \longrightarrow G \times G,$$

Since the cocycle condition says that

$$g_{ac} = g_{ab} \cdot g_{bc},$$

the map (g_{ab}, g_{bc}) gives us enough information to recover g_{ac}. Given a form

$$\varphi_2 \in A(G \times G),$$

we can pull it back via the map (g_{ab}, g_{bc}) to a form

$$(g_{ab}, g_{bc})^*\varphi_2 \in A(U_{abc}) \quad \text{for each} \quad a, b, c.$$

Thus given a general element

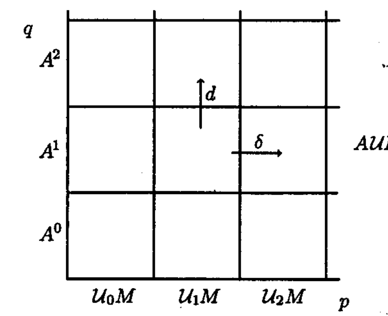
$$\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_k \in ANG,$$

we can pull it back via the maps g_{ab} to a form $\rho(\varphi)$ which lives in

$$(\oplus_{a,b} A(U_a \cap U_b)) \oplus (\oplus_{a,b,c} A(U_{abc})) \oplus \cdots .$$

However, this still does not live in AM.

To get forms in AM, we recall André Weil's proof of the isomorphism Čech and DeRham cohomology [71]. (Cf. infra Section 8 for the rest of t proof. See [6] for a proof not using spectral sequences.) Given $(M, \mathcal{U})$, whe M is a manifold and $\mathcal{U}$ an open covering $\{U_a\}$ of M such that the index set the cover is ordered by $\leq$, we define the *Čech–DeRham double complex*



with p, q term $A^q(\mathcal{U}_pM)$.

Here $U_0M = \coprod_a U_a$ (disjoint union)

$$\mathcal{U}_1M = \coprod_{a \leq b} U_a \cap U_b$$
$$\mathcal{U}_2M = \coprod_{a \leq b \leq c} U_a \cap U_b \cap U_c, \quad etc.,$$
$$d : A^q(\mathcal{U}_pM) \longrightarrow A^{q+1}(\mathcal{U}_pM)$$

is the ordinary DeRham coboundary map, and δ is a Čech coboundary m defined by

$$\delta : A^q(\mathcal{U}_{p-1}M) \longrightarrow A^q(\mathcal{U}_pM);$$

for $\varphi \in A^q(\mathcal{U}_{p-1}M)$,

$$(\delta\varphi)(U_{a_0} \cap \cdots \cap U_{a_p}) = \sum_{i=0}^{p} (-1)^i \varphi(U_{a_0} \cap \cdots \cap \widehat{U}_{a_i} \cap \cdots \cap U_{a_p})|(U_{a_0} \cap \cdots \cap U_{a}$$

(Notes: | means restriction of a differential form to a subset. The formula for δ is the one used in defining cohomology of M with values in the sheaf of germs of differential q-forms on M).

Theorem 7.1. *Let $D = d \pm \delta$ in the double complex $A\mathcal{U}M$. Then* $H_D^*(A\mathcal{U}M) = H_{DeRham}^*(M)$.

Proof. Filter $A\mathcal{U}M$ by q. Then $E_1^{p,q} = H_\delta^p(A^q U_* M) =$ p-th cohomology of the cochain complex

$$(0 \to A^q\mathcal{U}_* M) = (0 \to A^q\mathcal{U}_0 M \xrightarrow{\delta} A^q\mathcal{U}_1 M \xrightarrow{\delta} A^q\mathcal{U}_2 M \cdots).$$

Lemma 7.2.

$$H_\delta^p(A^q\mathcal{U}_* M) = \begin{cases} 0 & \text{if} \quad p \neq 0 \\ A^q M & \text{if} \quad p = 0. \end{cases}$$

In other words, the complex

$$0 \longrightarrow A^q M \xrightarrow{\delta} A^q\mathcal{U}_0 M \xrightarrow{\delta} A^q\mathcal{U}_1 M \cdots$$

is exact.

(The map $A^q M \xrightarrow{\delta} A^q U_0 M$ is defined by $\varphi \in A^q M \xrightarrow{\delta} \{\varphi|U_a\} \in A^q U_0 M$.)

Proof of Lemma. Define homotopy operators $h : A^q U_p M \to A^q U_{p-1} M$ as follows. Let $\{\lambda_a\}$ be a partition of unity on M subordinate to the covering $\{U_a\}$. Let $\varphi \in A^q\mathcal{U}_p M$. Then define $h\varphi \in A^q U_{p-1} M$ by

$$(h\varphi)(U_{a_0} \cap \cdots \cap U_{a_{p-1}}) = (-1)^p \sum_a \lambda_a \varphi(U_{a_0} \cap \cdots \cap U_{a_{p-1}} \cap U_a).$$

(We define $\varphi(U_{a_0} \cap \cdots \cap U_{a_{p-1}} \cap U_a)$ as $\pm\varphi(U_{b_0} \cap \cdots \cap U_{b_p})$, where $(b_0, \ldots, b_p)$ is an ordered (by $\leq$) permutation of $(a_0, \ldots, a_{p-1}, a)$, and $\pm$ is the sign of the permutation.) For $\varphi \in A^q\mathcal{U}_0 M$, define $h\varphi \in A^q M$ by

$$h\varphi = \sum_a \lambda_a \varphi(U_a).$$

Exercise. $h\delta + \delta h = id$. Q.E.D. for lemma.

Returning to the proof of the theorem, we have shown that $E_1^{p,q} =$

q \ p	0	1	2
2	A^2M	0	0
	$\uparrow d$		
1	A^1M	0	0
	$\uparrow d$		
0	A^0M	0	0

with $d_1 = d : A^q M \to A^{q+1} M$. Therefore

$$E_2^{p,q} = H_d(E_1) = H_d^q(AM) = H_{DeRham}^q(M).$$

Because E_2 is 0 outside the left column,

$$H_{DeRham}^* M = E_2 = E_\infty = H_D^*(AUM).$$

Q.E.D. (Theorem 7

Comment. If $\varphi \in A^q\mathcal{U}_0 M$, then

$$\delta\varphi(U_a \cap U_b) = \varphi(U_b)\ |(U_a \cap U_b) - \varphi(U_a)|\ (U_a \cap U_b),$$

so $\delta\varphi = 0$ if and only if $\varphi(U_a)$ and $\varphi(U_b)$ agree on $U_a \cap U_b$, which is true if a only if φ can be patched into a q-form defined on all of M.

The appearance of the double complex $A\mathcal{U}M$ is reminiscent of our se simplicial machinery, and in fact, Theorem 7.1 can be stated and proved us semi-simplicial objects. We do this now. Define $\mathcal{U}M$ to be the semi-simpli manifold

$$\mathcal{U}_0 M \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathcal{U}_1 M \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathcal{U}_2 M \quad \cdots,$$

where, as before, $\{U_a\}$ is an open cover of M with ordered indices a, a $\mathcal{U}_p M = \coprod_{a_0 \leq a_1 \leq \cdots \leq a_p} U_{a_0} \cap \cdots \cap U_{a_p}$. (For shorthand we also write $U_{a_0 \cdots a_p}$

$U_{a_0} \cap \cdots \cap U_{a_p}$). The maps ∂_i and ϵ_i are defined as follows:

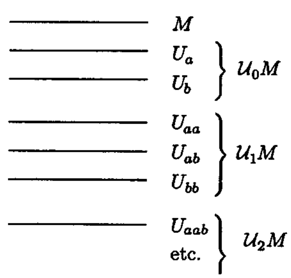
$$\partial_i : \mathcal{U}_p M \longrightarrow \mathcal{U}_{p-1} M, \quad i = 0, \dots, p$$
$$\partial_i : \mathcal{U}_{a_0} \cap \cdots \cap \mathcal{U}_{a_p} \longrightarrow \mathcal{U}_{a_0} \cap \cdots \cap \widehat{\mathcal{U}}_{a_i} \cap \cdots \mathcal{U}_{a_p}$$
$$\partial_i : x \longmapsto x$$
$$\epsilon_i : \mathcal{U}_p M \longrightarrow \mathcal{U}_{p+1} M, \quad i = 0, \dots, p$$
$$\epsilon_i : \mathcal{U}_{a_0} \cap \cdots \cap \mathcal{U}_{a_p} \longrightarrow \mathcal{U}_{a_0} \cap \cdots \cap \mathcal{U}_{a_i} \cap \mathcal{U}_{a_i} \cap \cdots \cap \mathcal{U}_{a_p}$$
$$\epsilon_i : x \longmapsto x$$

The geometric realization $|\mathcal{U}M|$ is

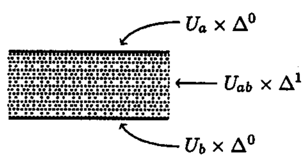
$$\mathcal{U}_0 M \times \Delta^0 \coprod \mathcal{U}_1 M \times \Delta^1 \coprod \mathcal{U}_2 M \times \Delta^2 \cdots$$

modulo gluing via ϵ_i and ∂_i.

Example. For $M = (0,1), \{U_a\} = \{U_a = (0, 3/4), U_b = (1/4, 1)\}$, with $a < b$, we have



and $|\mathcal{U}M| =$ (after shrinking degenerate simplices via ϵ_i)



There is an obvious map

$$g : |\mathcal{U}M| \longrightarrow M$$

defined by

$$U_{a_0} \cap \cdots \cap U_{a_p} \times \Delta^p \longrightarrow U_{a_0} \cap \cdots \cap U_{a_p} \subset M.$$

We construct a map

$$f : M \longrightarrow |\mathcal{U}M|$$

by choosing a partition of unity λ_a on M, subordinate to the cover $\{U$
For $x \in M$, let $a_0 < \cdots < a_p$ be the indices for which $\lambda_a(x) > 0$. Tl
$t = (\lambda_{a_0}(x), \cdots, \lambda_{a_p}(x))$ is a point in Δ^p, and we define

$$f(x) = (x, t) \in U_{a_0 \cdots a_p} \times \Delta^p \to |\mathcal{U}M|.$$

Exercises

1. f and g are well-defined (i.e., are compatible with identifications via ∂_i a ϵ_i)
2. $M \xrightarrow{f} |\mathcal{U}M| \xrightarrow{g} M$ =identity
3. $|\mathcal{U}M| \to M \to |\mathcal{U}M|$ is homotopic to the identity. Hint: Construct a lin homotopy.
   This proves

Theorem 7.3. *$M \xrightarrow{f} |\mathcal{U}M|$ and $|\mathcal{U}M| \xrightarrow{g} M$ are homotopy equivalences. particular, $H^*(M) \xrightarrow{\sim} H^*(|\mathcal{U}M|)$ is an isomorphism.*

Note. The construction of $\mathcal{U}M$ also works if M is any paracompact sp rather than a manifold. Theorem 7.3 still holds in this case.

Combining Theorem 7.3 with Theorem 4.5 implies that

$$H^*(M) = H^*(|\mathcal{U}M|) = H^*_D(A(\mathcal{U}M)).$$

Using the isomorphism $H^*_{(sing.)}(M) = H^*_{DeRham}(M)$, we have reproved Th rem 7.1.

Trivial Exercise. Verify that the double complex $(A\mathcal{U}M, d, \delta)$ of Theor 7.1 is indeed the same as $A(\mathcal{U}M)$ defined as the double complex of differen forms on the semi-simplicial manifold $\mathcal{U}M$.

Returning to characteristic classes, we see now that pulling back differen forms on NG to $A(U \cap V) \oplus A(U \cap V \cap W) \oplus \cdots$ as we did earlier correspo to a m double complexes $\rho : ANG \to AUM$.

To treat the map ρ semi-simplicially we prove:

Theorem 7.4. *Let G be a Lie group. Let $\eta = (E \to M)$ be a G-bun on a manifold M with cocycle $\{g_{ab}\}$ relative to the cover $\mathcal{U} = \{\mathcal{U}_a\}$ of M. $F : \mathcal{U}M \to NG$ be the map*

$$\begin{array}{ccccccc}
\mathcal{U}M = (\mathcal{U}_0 M & \leftrightarrows & \mathcal{U}_1 M & \leftleftarrows & \mathcal{U}_2 M & \cdots) \\
F \downarrow \quad F \downarrow & & F \downarrow & & F \downarrow & \\
NG = (\; * & \leftrightarrows & G & \leftleftarrows & G \times G & \cdots)
\end{array}$$

defined by

$$U_a \to * \quad \textit{for all} \quad a$$
$$g_{ab} : U_a \cap U_b \to G \quad \textit{for all} \quad a \le b$$

$(g_{ab}, g_{bc}) : U_a \cap U_b \cap U_c \to G \times G$, *etc. Then*

1) *F is a morphism of double complexes (i.e., it commutes with ∂_i and ϵ_i).*
2) *The induced map $|F| : M \cong |\mathcal{U}M| \to |NG| = BG$ is a classifying map for the bundle η.*
3) *$F^* : ANG \to A\mathcal{U}M$ is just the map ρ previously defined.*

Proof. Assertion 3 is obvious. We prove assertion 1 in the special case

$$\begin{array}{ccc} \mathcal{U}_2M & \overleftarrow{\overleftarrow{\overleftarrow{\longleftarrow}}} & \mathcal{U}_3M \\ F\big\downarrow & & F\big\downarrow \\ G\times G & \overleftarrow{\overleftarrow{\overleftarrow{\longleftarrow}}} & G\times G\times G \end{array}$$

(the general case is almost identical).

We restrict to $U_{abcd} \subset \mathcal{U}_3M$, and prove that $\partial_i \circ F = F \circ \partial_i, i = 0, 1, 2, 3$.

$\underline{i = 0}$

$$\begin{array}{ccc} U_{bcd} & \xleftarrow[= \text{ inclusion}]{\partial_0} & U_{abcd} \\ F = \big\downarrow (g_{bc}, g_{cd}) & & F = \big\downarrow (g_{ab}, g_{bc}, g_{cd}) \\ G \times G & \xleftarrow{\partial_0} & G \times G \times G \\ (y, z) & \xleftarrow{\partial_0} & (x, y, z) \end{array}$$

The diagram obviously commutes.

$\underline{i = 1}$

$$\begin{array}{ccc} U_{acd} & \xleftarrow[= \text{ inclusion}]{\partial_1} & U_{abcd} \\ F = \big\downarrow (g_{ac}, g_{cd}) & & F = \big\downarrow (g_{ab}, g_{bc}, g_{cd}) \\ G \times G & \xleftarrow{\partial_1} & G \times G \times G \\ (xy, z) & \xleftarrow{\partial_1} & (x, y, z) \end{array}$$

the diagram commutes because $g_{ac} = g_{ab}g_{bc}$ by the cocycle condition. The case $i = 2$ is like $i = 1$, and $i = 3$ is like $i = 0$.

Exercise. Prove that F commutes with ϵ_i, the degeneracy maps.

To prove assertion 2, we construct a semi-simplicial bundle map. We sume without loss of generality (see Section 1) that $\eta = (E \to M)$ is a *prin* G-bundle. Cover E by the opens $\pi^{-1}U_a = E|U_a$, and let $\mathcal{U}E = \mathcal{U}_0E \leftleftarrows \mathcal{U}_1$ be the corresponding semi-simplicial manifold. The maps $E|U_a \to U_a$ in a map $\mathcal{U}E \to \mathcal{U}M$ with fiber G. The map $|\mathcal{U}E| \to |\mathcal{U}M|$ also has fibe Let $\{\lambda_a\}$ be a partition of unity on M subordinate to $\{U_a\}$. Then $\{\lambda_a$ is a partition of unity on E subordinate to $\{E|U_a\}$. We use these two p tions of unity to construct maps $M \to |\mathcal{U}M|$ and $E \to |\mathcal{U}E|$ as in the pro Theorem 7.3.

Exercise. The diagram

$$\begin{array}{ccc} E & \longrightarrow & |\mathcal{U}E| \\ \downarrow & & \downarrow \\ M & \longrightarrow & |\mathcal{U}M| \end{array}$$

commutes, and the map $E \to |\mathcal{U}E|$ commutes with the action of G.

Now recall [cf. Theorem 6.1] the semi-simplicial manifold $PG =$

$$G \;\overleftarrow{\overrightarrow{\longleftarrow}}\; G \times G \;\overleftarrow{\overleftarrow{\longleftarrow}}\; G \times G \times G \cdots$$

defined by $\partial_i(x_0, \ldots, x_n) = (x_0, \ldots, \hat{x}_i, \ldots, x_n)$. We define a map $\mathcal{U}E \to$ as follows. Let $\{f_a\}$ be a set of maps which are horizontal projections for bundle $E \xrightarrow{\pi} M$ [cf. Section 1]. Recall that this means that $f_a : (E|U_a)$ - (G regarded as the fiber of the bundle here), and that for all $y \in E|U_a \cap U$ $\pi(y) = x$, then

$$f_b(y) = g_{ba}(x) \cdot f_a(y).$$

Now define a map $\mathcal{U}E \to PG$

$$\begin{array}{ccccc} \mathcal{U}_0E & \overleftarrow{\overrightarrow{\longleftarrow}} & \mathcal{U}_1E & \overleftarrow{\overleftarrow{\longleftarrow}} & \mathcal{U}_2E \quad \cdots \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G & \overleftarrow{\overrightarrow{\longleftarrow}} & G \times G & \overleftarrow{\overleftarrow{\longleftarrow}} & G \times G \times G \end{array}$$

by

$$E|U_{a_0} \cap \cdots \cap U_{a_p} \longrightarrow G^{p+1} \quad \text{for each } a_0 \le \ldots \le a_p$$
$$y \longrightarrow (f_{a_0}(y), \ldots, f_{a_p}(y)).$$

Exercise. Check that $\mathcal{U}E \to PG$ commutes with ∂_i and ϵ_i.

Exercise. Check that the diagram

$$\begin{array}{ccc} \mathcal{U}E & \longrightarrow & PG \\ \downarrow & & \downarrow \\ \mathcal{U}M & \longrightarrow & NG \end{array}$$

commutes, where $PG \to NG$ is defined (as in Theorem 6.1) by

$$(x_0, x_1, \ldots, x_n) \longrightarrow (x_0 x_1^{-1}, x_1 x_2^{-1}, \ldots, x_{n-1} x_n^{-1}),$$

and other maps are as defined so far in this proof. Check also that $\mathcal{U}E \to PG$ commutes with the action of G.

This exercise proves that

$$\begin{array}{ccc} |\mathcal{U}E| & \longrightarrow & |PG| \\ \downarrow & & \downarrow \\ |\mathcal{U}M| & \longrightarrow & |NG| \end{array}$$

commutes. Composing this with the previous diagram

$$\begin{array}{ccc} E & \longrightarrow & |\mathcal{U}E| \\ \downarrow & & \downarrow \\ M & \longrightarrow & |\mathcal{U}M| \end{array}$$

and using Theorem 6.1, we get a G-bundle map

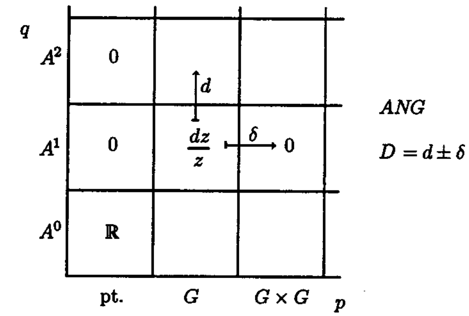
$$\begin{array}{ccccc} E & \longrightarrow & |PG| & = & EG \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & |NG| & = & BG. \end{array}$$

Hence the map $M \to |NG|$ must classify the bundle $\eta = (E \to M)$. Since the map factors $M \to |\mathcal{U}M| \to |NG| = BG$ and $M \to |\mathcal{U}M|$ is a homotopy equivalence, assertion 2 is proved. Q.E.D. (Theorem 7.4)

Corollary 7.4.1. *The map $H^*(BG) \xrightarrow{\sim} H^*_D ANG \to H^*_D A\mathcal{U}M \xrightarrow{\sim} H^*M$ gives the characteristic classes of the bundle η.*

Comment. Theorem 7.4 is also true if M is a paracompact space but not a manifold, provided that in assertion 3 we replace $ANG \to A\mathcal{U}M$ by $S^*NG \to S^*\mathcal{U}M$ (singular cohomology). However G must in general have a decent topology (for example G = Lie group) so that $|PG| \to |NG|$ will be a (locally trivial) fibration. If G has a bad topology then $|NG_Z|$ classifies if M is any paracompact space (see Sections 4,6 and [6], Stasheff's appendix).

Example. Let $G = GL(1, \mathbb{C}) = \mathbb{C}^*$. Then as we have seen in Sectio $\frac{dz}{z} \in A^1G$ is a primitive form on G, so in the double complex ANG (where now mean *complex*-valued smooth forms on NG)



$D\left(\frac{dz}{z}\right) = 0$. Therefore dz/z is a 2-cocycle of ANG. We know from our se simplicial theory that $H^*_D(ANG) = H^*(BG)$, and we know from other meth [53] that $H^*BG = \mathbb{C}[c_1]$, a polynomial algebra generated by $c_1 \in H^2BG$, wh c_1 is called the first Chern class. (In Section 9, we shall calculate $H^*_D Ai$ directly). Thus we can write $dz/z = xc_1$, where x is some complex consta We shall find x by comparing the pullbacks of dz/z and c_1 on the same comp line bundle.

Now let $M = \mathbb{C}P^1$, and let $\eta = (\pi : E \to M)$ be the canonical comp line bundle over M defined as follows: Let $M = (\mathbb{C} \times \mathbb{C} - 0)/\mathbb{C}^*$

$$= \{(z_1, z_2)\}/((z_1, z_2) \sim (zz_1, zz_2))(z \in \mathbb{C}^*)$$

Let $E = ((\mathbb{C} \times \mathbb{C} - 0) \times \mathbb{C})/\mathbb{C}^* =$

$$\{(z_1, z_2; z_3)\}/((z_1, z_2; z_3) \sim (zz_1, zz_2; zz_3)).$$

Define $\pi : E \to M$ by $\pi(z_1, z_2; z_3) = (z_1, z_2)$. Cover M by U and V (orde: $U \leq V$) where

$$U = \{(1, z)\} \quad \text{and} \quad V = \{(z, 1)\}.$$

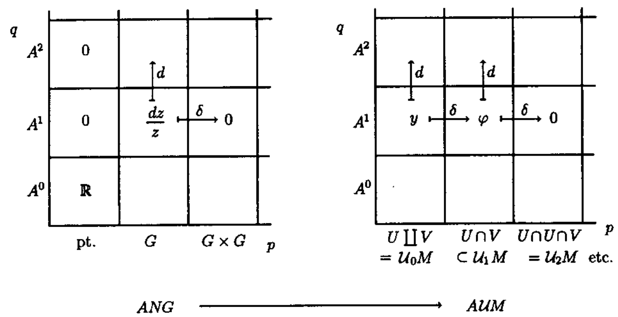
Identify U and V with $\mathbb{C}$ in the obvious way. In $E|U \cap V$, we have $(z, 1; w)$ $(1, z^{-1}; z^{-1}w)$ so the transition function $g_{uv} : U \cap V \to \mathbb{C}^*$ is given $g_{uv}((z, 1)) = z^{-1}$. Coordinatizing $U \cap V$ by the map

$$U \cap V \subset V \longrightarrow \mathbb{C}, \quad (z, 1) \longmapsto z$$

we have

$$\begin{aligned} g^*_{uv}(dz/z) &= dg_{uv}/g_{uv} = d(z^{-1})/z^{-1} \\ &= -z^{-2}dz/z^{-1} = -dz/z. \end{aligned}$$

Thus the map $ANG \to A\mathcal{U}M$ defined by the cocycle of the bundle takes dz/z in A^1G to $-dz/z$ in $A^1\mathcal{U}_1M$. Let $\varphi = -dz/z \in A^1\mathcal{U}_1M$.



Now φ is a D 2-cocycle on $A\mathcal{U}M$, hence generates an element of $H^2(|\mathcal{U}M|) = H^2(M)$. To find out which element, we imitate the proof of Lemma 7.2 to find $y \in A^1(U \coprod V)$ such that $\delta y = \varphi$ (see diagram). As in that proof, we let $\{\lambda_u, \lambda_v\}$ be a partition of unity subordinate to $\{U, V\}$ and define $y = h(\varphi)$. We get

$$\begin{aligned} y(U) &= (-1)^1(\lambda_u)(U \cap U) + \lambda_v\varphi(U \cap V)) \\ &= -\lambda_v\varphi(U \cap V) \\ y(V) &= (-1)^1(\lambda_u\varphi(V \cap U) + \lambda_v\varphi(V \cap V)) \\ &= +\lambda_u\varphi(U \cap V) \quad (\text{since} \quad \varphi(V \cap U) = -\varphi(U \cap V)). \end{aligned}$$

$dy \in A^2(U \coprod V)$ is given by

$$\begin{aligned} dy(U) &= -(d\lambda_v)\varphi(U \cap V) - \lambda_v d\varphi(U \cap V) \\ &= -(d\lambda_v)\varphi(U \cap V) \quad (\text{since} \quad d\varphi = 0) \\ dy(V) &= (d\lambda_u)\varphi(U \cap V) \end{aligned}$$

Observe that since $\lambda_v = 1 - \lambda_u, dy(U)$ and $dy(V)$ agree on $U \cap V$. Thus dy is a form in A^2M. Now $Dy = (d + (-1)^1\delta)y = dy - \delta y = dy - \varphi$, so dy and φ are D-cohomologous in $A\mathcal{U}M$. Thus the class of φ in $H^2_D(A\mathcal{U}M)$ goes to the class of dy in H^2M under the isomorphism $H^*_D(A\mathcal{U}M) \to H^*M$.

To evaluate the class of dy in H^2M, we integrate dy over M. Observe that

$$\text{support} \quad dy \subset \quad \text{supp } d\lambda_u = \text{supp } d\lambda_v \subset U \cap V \subset V,$$

so

$$\int_M dy = \int_V dy(V) = \int_V (d\lambda_u)\varphi(U \cap V).$$

Now identifying V with $\mathbb{C}$, we have $U \cap V = \mathbb{C} - \{0\}$, and we can choose λ_u, λ_v so that supp $d\lambda_u \subset \{z | \, |z| \leq 1\} = D$. Then $\lambda_u \equiv 1$ on $(\overline{\mathbb{C} - D})$ and

$\int_V (d\lambda_u)\varphi(U \cap V) = \int_D (d\lambda_u)\varphi(U \cap V) = \int_D d(\lambda_u\varphi) =$ (by Stoke's Theo
$\int_{\partial D} \lambda_u\varphi = \int_{|z|=1} 1 \cdot (-dz/z) = -\int_{\theta=0}^{2\pi}((d(e^{i\theta}))/(e^{i\theta})) = -2\pi i.$

Now $c_1 \in H^2BG$ is defined so that its pullback to H^2M is 1 for the bu η [53], so we have shown that

$$\frac{dz}{z} = -2\pi i c_1 \in H^2ANG = H^2BG.$$

8. More applications of semi-simplicial techniques

In Section 7 we showed that one can replace a manifold M with the s simplicial manifold $\mathcal{U}M$ formed from an open cover of M. We showed $|\mathcal{U}M|$ and M have the same homotopy type, and that $H^*M = H^*_D A\mathcal{U}M$.

$\mathcal{U}M$ consists of some open sets, together with the information of how are glued together to form M. There are also other ways to represent a s M or X semi-simplicially.

1) Making no use of the topology of X, define the semi-simplicial space X

$$X \leftleftarrows X \leftleftarrows X \quad \cdots$$

where all face and degeneracy maps are identity$_X$. Then the degene maps collapse $X \times \Delta^i$, $i > 0$, onto $X \times \Delta^0$ in $|X'|$, so the geometric r ization is $|X'| = X \times \Delta^0 = X$.

2) Making the most extreme use of the topology of X, define a semi-simpli set (discrete space)

$$\Sigma X = \Sigma_0 X \leftleftarrows \Sigma_1 X \leftleftarrows \Sigma_2 X \quad \cdots$$

where

$$\Sigma_p = \{\text{singular } p\text{-simplices of } X\},$$

and ∂_i and ϵ_i are the usual face and degeneracy maps of singular homol Since ΣX is discrete, $|\Sigma X|$ is a CW complex.

In computing the D-cohomology of the double complex $S^*\Sigma_* X$ $\oplus_{p,q} S^q\Sigma_p X$ with coefficients in the ring R, the discreteness of $\Sigma_p X$ plies that

$$\begin{aligned} E_1^{p,q} &= H^q_d(S^*\Sigma_p X) \quad (d = \text{singular coboundary map}) \\ &= H^q(\Sigma_p X) \quad (\text{singular cohomology}) \\ &= \begin{cases} 0 & \text{if} \quad q \neq 0 \\ F(\Sigma_p X) & \text{if} \quad q = 0, \end{cases} \end{aligned}$$

where

$$F(\Sigma_p X) = \text{the module of functions from } \Sigma_p X \text{ to } R$$
$$= \text{module of singular } p\text{-cochains of } X$$
$$= S^p(X).$$

Since the E_1 term has zeroes except when $q = 0$, we get $E_2 = E_\infty = H_D(S^*\Sigma X)$, so that $H^p_D(S^*\Sigma X) = E_2^{p,0} = H^p_\delta(E_1) = H^p_\delta(S^*X) = H^*(X)$. Applying Theorem 4.1, we see that

$$H^*(|\Sigma X|) = H^*_D(S^*\Sigma X) = H^*X,$$

so that the evaluation map on simplices

$$\text{eval} : |\Sigma X| \longrightarrow X$$

induces an isomorphism in cohomology (and similarly in homology). The map also induces isomorphism of all homotopy groups [52], [50].

3) Let us reconsider the semi-simplicial manifold $\mathcal{U}M$. If M is a differentiable n-manifold, we can choose a covering $\mathcal{U}$ of M such that U_a and every nonempty finite intersection $U_{a_0\cdots a_p}$ are diffeomorphic to $\mathbb{R}^n$; this will be taken as the definition of a *good* covering. For example, one can put a Riemannian metric on M and choose each U_a to be geodesically convex, or one can triangulate M and take the U_a to be the star neighborhoods in some subdivision. Then the semi-simplicial manifold $\mathcal{U}M$ has the form

$$\begin{array}{ccccc} \mathcal{U}_0M & \leftleftarrows & \mathcal{U}_1M & \leftleftarrows & \mathcal{U}_2M \;\cdots \\ \| & & \| & & \| \\ \coprod_a \mathbb{R}^n & & \coprod_{a\le b} \mathbb{R}^n & & \coprod_{a\le b\le c} \mathbb{R}^n \\ & & \text{with } U_{ab} \neq 0 & & \text{with } U_{abc} \neq 0 \end{array}$$

where the map ∂_i is the inclusion

$$U_{a_0\cdots a_p} \longrightarrow U_{a_0\cdots \hat{a}_i \cdots a_p}.$$

The manifold $\mathcal{U}M$ is a "free resolution" of M if one regards

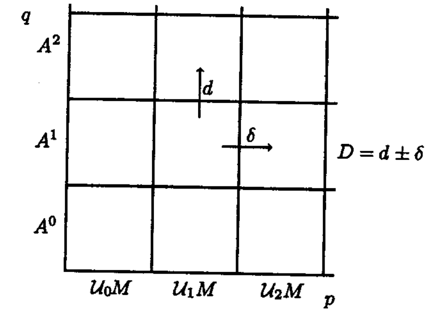
$$\mathcal{U}_i M = \coprod \mathbb{R}^n$$

as the analog for spaces of a free module. In fact, the cochain complex kA^qUM *is* a free resolution of the module A^qM by modules of the form $\oplus_a A^q\mathbb{R}^n$; the exactness of

$$0 \longrightarrow A^qM \longrightarrow A^q\mathcal{U}_0M \xrightarrow{\delta} A^q\mathcal{U}_1M \xrightarrow{\delta} \cdots$$

was proved in Lemma 7.2.

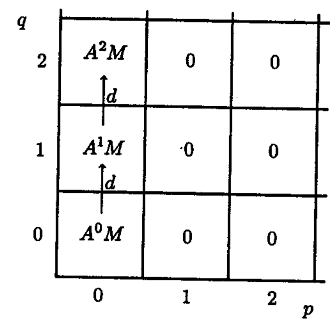
Regarding $A\mathcal{U}M$ as a double complex



we get a simple proof, due to Weil [71], of the isomorphism between $H^*_{Cech}M$ and $H^*_{DeRham}(M)$: The rows of $A\mathcal{U}M$ are exact by Lemma 7.2 (i.e., $0 \to A^qM \to A^q\mathcal{U}_0M \xrightarrow{\delta} A^q\mathcal{U}_1M \to \cdots$ is exact) while the columns are now exact by the Poincaré Lemma, i.e.,

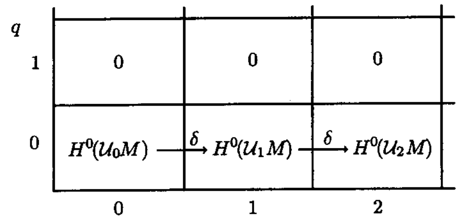
$$0 \longrightarrow \mathbb{R} \longrightarrow A^0\mathbb{R}^n \xrightarrow{d} A^1\mathbb{R}^n \xrightarrow{d} \cdots$$

is exact. Computing $H^*_D A\mathcal{U}M$ by spectral sequences, we can filter by q to get $E_1 = H_\delta(A\mathcal{U}M) =$



yielding $H^*_D A\mathcal{U}M = H(A^*M) = H^*_{DeRham}(M)$, or filter by p to get $E_1 =$

$H_d(A\mathcal{U}M) =$



The latter complex is the Čech cochain complex

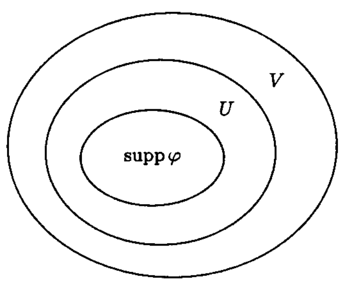
$$\check{C}^0(M,\mathcal{U}) \xrightarrow{\delta} \check{C}^1(M,\mathcal{U}) \xrightarrow{\delta} \cdots$$

of functions which assign a constant to each nonempty intersection $U_{a_0} \cap \cdots \cap U_{a_p}$, so

$$E_2^{p,0} = H_\delta^p(\check{C}^*(M,\mathcal{U})) = \check{H}^p_{Cech}(M,\mathcal{U}) = E_\infty^{p,0} = H_D^p(A\mathcal{U}M).$$

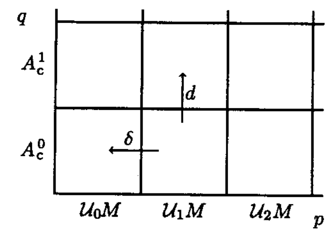
Thus $H^*_{DeRham}(M) = H^*_D(A\mathcal{U}M) = \check{H}^*_{Cech}(M,\mathcal{U})$. Since this isomorphism is independent of which good cover U is used, it follows that $\check{H}^*(M,\mathcal{U}) = \check{H}^*(M)$; that is, we have also proved that one can compute Čech cohomology by using a good cover, without passing to the direct limit.

By a similar approach we prove Poincaré duality for an n-manifold M. Let $\mathcal{U}$ be a *finite* good cover of M. Let the functor A_c^q denote q-forms with compact support. This functor is *covariant* with respect to inclusions of open submanifolds, since a form φ with compact support can be extended to be 0 outside its support.



The exterior derivative $d : A_c^q \to A_c^{q+1}$ is the usual one. We define $H_c^q(M)$, the *DeRham cohomology of M with compact support*, to be $H_d^q(A_c^*M)$. Thus the double complex $A_c\mathcal{U}M$ is a chain complex with respect to δ and a cochain

complex with respect to d.

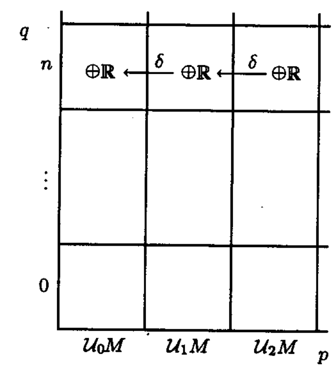


We set $D = d \pm \delta$ and grade $A_cU\mathcal{U}M$ by $q-p$ to make it a D-cochain comp
Since M is an n-manifold the complex is 0 above the n-th row, so the spec
sequence for $H^*_D A_c\mathcal{U}M$ converges.

The complex

$$0 \longrightarrow (A_c^0\mathbb{R}^n \xrightarrow{d} A_c^1\mathbb{R}^n \longrightarrow \cdots \longrightarrow A_c^n\mathbb{R}^n) \xrightarrow{I} \mathbb{R} \longrightarrow 0$$

is exact, where the map I is integration of an n-form with respect to a ch
orientation of $\mathbb{R}^n$. Each column of the double complex $A_c\mathcal{U}M$ is a direct
of copies of the cochain complez in parenthesis. Thus filtering $A_c\mathcal{U}M$ b
$E_1^{p,q} = H_d(A_c\mathcal{U}M) = H_c(\mathcal{U}M)$ is the complex



where $\oplus\mathbb{R} = H_c^n(\mathcal{U}_pM)$ means one copy of $\mathbb{R}$ for each copy of $\mathbb{R}^n$ in the sp
$\mathcal{U}_pM = \coprod \mathbb{R}^n$.

To define the connecting maps $\oplus\mathbb{R} \to \oplus\mathbb{R}$, we must take into account lo
orientations and use local coefficients, just as in Čech homology theory w
local coefficients [63]. Specifically, we regard $H_c^n(U_{a_0} \cap \cdots \cap U_{a_p})$ as hav
coefficients in $H_c^n(U_{a_0}) \cong \mathbb{R}$. The set $\{H_c^n(U_a)\}$ is a local system on M

copies of $\mathbb{R}$ (it is essentially the orientation sheaf of M [32]), and our complex

$$\begin{array}{ccccc} H^n_c(\mathcal{U}_0 M) & \xleftarrow{\delta} & H^n_c(\mathcal{U}_1 M) & \xleftarrow{\delta} & \cdots \\ \| & & \| & & \\ \oplus\mathbb{R} & \longleftarrow & \oplus\mathbb{R} & \longleftarrow & \cdots \end{array}$$

is exactly the complex of Čech chains with coefficients in the local system. Therefore, the δ-homology of the complex is the Čech homology $\check{\mathrm{H}}_p(M;\text{ local})$ of M with coefficients in the local system. Thus $E_2^{p,q}$ of the double complex is

q		
n	$\check{H}_0(M;\text{local})$	$\check{H}_1(M;\text{local})$
$\vdots$		
0	0	0
	0	1 $\quad p$

Therefore $E_2 = E_\infty$, and

$$\check{\mathrm{H}}_p(M;\text{local}) \cong H_D^{n-p}(A_c\mathcal{U}M).$$

The rows of A_cUM,

$$0 \longleftarrow A^q_c M \longleftarrow (A^q_c\mathcal{U}_0 M \xleftarrow{\delta} A^q_c\mathcal{U}_1 M \xleftarrow{\delta} \cdots)$$

are exact by a modification of Lemma 7.2.

Thus filtering $A_c\mathcal{U}M$ by q, we get $E_1 = H_\delta(A_c\mathcal{U}M) =$

q		
2	$A^2_c M$	0
	$\uparrow d$	
1	$A^1_c M$	0
	$\uparrow d$	
0	$A^0_c M$	0
	0	1 $\quad p$

and

$$E_2^{0,q} = H^q_d E_1 = H^q_d(A^*_c M) = H^q_c(M).$$

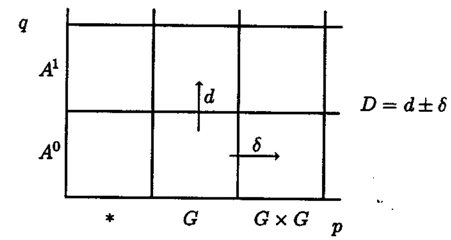
Therefore

$$H^q_c(M) = H^q_D(A_cUM) = \check{\mathrm{H}}_{n-q}(M;\text{local}).$$

This is Poincaré duality.

9. Computation of HANG

Recall that ANG is the double complex



of differential forms on the semi-simplicial manifold NG (cf. Section 6). To fin $H_D(ANG) = H^*(BG)$, we could filter ANG by p, getting a spectral sequen converging to $H_D(ANG)$, with $E_1 = H_d(ANG)$ and $E_1^{p,q} = H^q(G^p)$ (DeRha cohomology).

If, instead, we filter ANG by q we get another spectral sequence co verging to $HANG(\underset{def.}{=} H_D(ANG))$, but with E_1 term $H_\delta(ANG)$, i.e., $E_1^{p,q}$ $H^p_\delta(A^q NG)$. This has been computed, and the answer has a rather nice form

Theorem 9.1. (Bott, Hochschild).

$$H^p_\delta(A^q NG) = H^{p-q}_c(G; S^q g^*)$$

where G is any Lie group, g is its Lie algebra, $g^* = Hom_{\mathbb{R}}(g,\mathbb{R})$ is the du of g, S^q is the q-th symmetric power (hence $S^q g^*$ is the module of degree polynomials on g), and H_c means continuous cohomology in the sense of th cohomology of groups, in particular of Lie groups [38]. [$H^{p-q}_c(G; S^q g^*)$ is th $(p-q)$-th continuous cohomology group of G acting on the module $S^q g^*$ by th adjoint action.]

Proof. See [8].

We recall [19] that for discrete groups G, $H^*_{EM}(G;\mathbb{R})$, the Eilenberg MacLane cohomology of the group G, is defined as the cohomology of

$$\text{Functions } (G,\mathbb{R}) \rightrightarrows \text{Functions } (G\times G,\mathbb{R}) \underset{\longrightarrow}{\rightrightarrows} \cdots$$

where the maps are the duals of those of NG. $H_{continuous}(G) = H_c(G)$ and $H_{smooth}(G)$ are defined analoagously for Lie groups G by looking only at smooth or continuous functions, respectively.

We state now two well-known facts about continuous cohomology of groups. (See [38].)

Fact 1. Let G be a Lie group and N a module on which G acts. Then $H^0_c(G;N) = \mathrm{Inv}_G N$, the G-invariants of N(i.e., $\{n \in N | gn = n \;\forall g \in G\}$). (This is the same result as in the discrete group case).

Fact 2. If G is a compact Lie group and M is any module on which G acts, then $H^i_c(G;M) = 0$ if $i > 0$. (The proof makes use left-invariant or Haar integration on G).

Corollary 9.1.1.

$$H^p_\delta(A^q NG) = \begin{cases} 0 & \text{if} \quad p < q \;\; (\text{above diagonal}) \\ Inv_G S^p g^* & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \text{ and } G \text{ compact.} \end{cases}$$

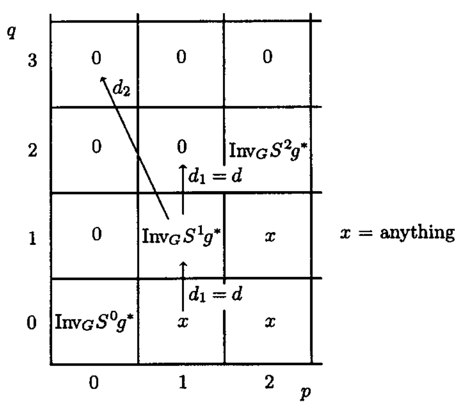
Example. Let $G = GL(n,\mathbb{R})$, $g = gl(n,\mathbb{R})$. Then

$$\mathrm{Inv}_G S^* g^* = \mathbb{R}[c_1,\ldots,c_n] = \mathbb{R}[s_1,\ldots,s_n],$$

a polynomial algebra with generators c_i or s_i, where for $A \in g$, (A an $n \times n$ matrix),

$$c_i(A) = \text{coefficient of} \quad t^i \quad \text{in} \det(I + tA),$$
$$s_i(A) = tr(A^i). \quad (\text{cf. appendix of [6]})$$

Thus for all G, the E_1 term of ANG, filtering by q, is



All diagonal entries are cocycles for $d_1, d_2, d_3, \ldots$ in the spectral sequence, so

$$E_1^{p,p} \xrightarrow{onto} E_\infty^{p,p} \xrightarrow{1-1} H_D^{2p}(ANG).$$

The composition is called the *Chern–Weil homomorphism*

$$\varphi : \mathrm{Inv}_G S^p g^* \longrightarrow H^{2p}(ANG) \xrightarrow{=} H^{2p}(BG).$$

Corollary 9.1.2. *If G is compact, $\varphi : Inv_G S^p g^* \to H^{2p}(BG)$ is an isc morphism.*

Example. Let $G = O(n)$, $g = o(n) = \{n \times n$ real matrices A such tha $A^t = -A\} = \{$skew-symmetric matrices$\}$. Then $tr(A^p) = 0$ for p odd, and i follows that $\mathrm{Inv}_G S^p g^* = 0$ for p odd. Thus

$$H^*(BO(n)) = \mathrm{Inv}_G S^* g^* = \mathbb{R}[c_2, c_4, \ldots, c_{[n/2]}].$$

Thus $H^*(B\check{G}L(n,\mathbb{R})) = \mathbb{R}[c_2, c_4, \ldots, c_{[n/2]}]$. $c_{2i} \in H^{4i}(BG\check{L}(n)) = i$-th Pon tryagin class (cf. Section 6).

Example. Since $\mathrm{Inv}_{U(n)} u(n) = \mathbb{R}[c_1,\ldots,c_n]$, it follows tha $H^*(BGL(n,\mathbb{C})) = H^*(BU(n)) = \mathbb{R}[c_1,\ldots,c_n]$, where $c_i \in H^{2i} = i$-th Cher class.

Special case. $H^*(BGL(1,\mathbb{C})) = \mathbb{R}[c_1]$, where c_1 = class of $-\frac{1}{2\pi i}\frac{dz}{z}$ i $A^1\mathbb{C}^* = A^1 GL(1,\mathbb{C})$ in ANG [cf. example at end of Section 7]. Note that $\frac{d}{z}$ is a left invariant form on $\mathbb{C}^*$, hence is an element of the dual, g^*, of the Li algebra $gl(1,\mathbb{C}) = g$.

Exercise. Try to find universal formulas for c_1 and c_2 for $GL(2,\mathbb{C})$ bundle in terms of the transition functions g_{ab} (of the same type as $c_1 = -\frac{1}{2\pi i}\frac{dg_{ab}}{g_{ab}}$ fo $GL(1,\mathbb{C})$ bundles.)

Comment. For oriented n-plane bundles with n even (structure grou $G = GL^+(n,\mathbb{R})$ or $SO(n)$), there is an Euler class $\chi \in H^n(BG)$, with $\chi^2 = c_n$ For example, for $SO(2)$ bundles, $\chi \in \mathrm{Inv}^I_{SO(2)} so(2)$ is given by $\chi\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$ where $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in so(2)$. Observe that $\chi^2 = \det = c_2$. χ does not aris for $O(2)$ (or any $O(n)$) bundles, because conjugating $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in o(2)$ b $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O(2)$ yields $\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$, hence takes χ to $-\chi$.

10. Discrete subgroups of Lie groups, and bundles with discrete structure group

If D is a discrete subgroup of a Lie group G, it is a difficult problem to find $H^*_{EM}(D)$, the Eilenberg–MacLane cohomology [19] of the group D. $H^*_{EM}(D)$ is isomorphic to $H^*(BD)$, where BD is, as usual, the classifying space for bundles with structure group D (see Section 5). Now the inclusion $D \to G$ induces a map $BD \to BG$, hence a map $H^*BG \to H^*BD$. Thus one can find a few elements of H^*BD, which is hard to compute, as images of elements of H^*BG, which is relatively easy to compute. For example, one can study $D = GL(n, \mathbb{Z})$ as a discrete subgroup of $G = GL(n, \mathbb{R})$, and study the images of the Pontryagin classes (H^*BG) in H^*BD. However, this procedure is not very useful because it constructs only a small fraction of the elements of H^*BD.

To get a better hold on H^*BD, one studies discrete subgroups of Lie groups in a universal way by defining for each Lie group G the discrete group G^δ (also called G_δ) which is the same group as G but with the discrete topology. The identity map $G^\delta \to G$ is continuous, and any discrete subgroup D of G maps $D \to G^\delta \to G$. This induces maps $BD \to BG^\delta$ and $H^*BG^\delta \to H^*BD$. Thus to find elements of H^*BD in a universal way, we compute H^*BG^δ and study its image in H^*BD.

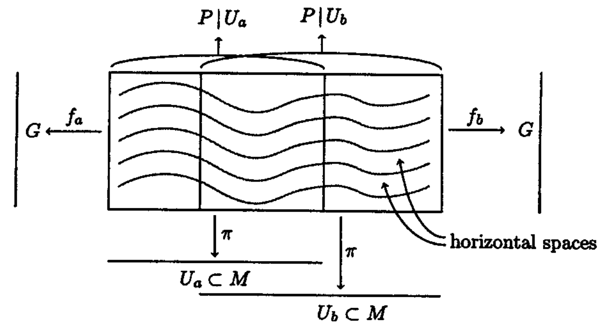
What does it mean for a bundle over a space X to have structure group G^δ? It means that the transition functions $\{g_{uv} : U \cap V \to G^\delta\}$ are continuous with respect to the discrete topology on G, that is, g_{uv} is constant on each connected component of $U \cap V$. If we assume the cover is good, then g_{uv} is constant on $U \cap V$. Every G^δ bundle can be considered also as a G bundle via the map $G^\delta \to G$.

Now let

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

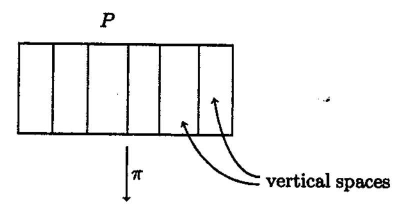
be a principal G^δ bundle, i.e., a principal G-bundle whose structure group is G^δ. Let $\{U_a\}$ be a good cover of M which trivializes P. Let $\{f_a : P|U_a \to G\}$

be the horizontal projections (see Section 1).



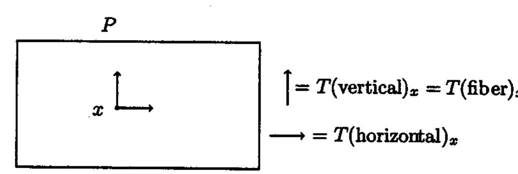
On $P|U_a$, the fibers of f_a are submanifolds diffeomorphic to U_a, and similarl; on $P|U_b$. On $P|U_a \cap U_b$, $f_a = g_{ab} f_b$, where g_{ab} is a *constant* in G since th structure group is G^δ. Therefore {fibers of f_a} = {fibers of f_b} in $P|U_a \cap U_b$.

Definition. The *horizontal spaces* of P are the connected subspaces of *I* which are locally fibers of horizontal projections f_a. (See diagram above.) Th *vertical spaces* or *fibers* of P are the ordinary fibers of the bundle $\pi : P \to M$.



Since f_a and π are submersions and $P|U_a \underset{(\pi, f_a)}{\longrightarrow} U_a \times G$ is a diffeomorphism, th horizontal spaces and fibers are manifolds, the sum of their dimensions equal dimension (P), and they are *transversal*, i.e., at any $x \in P$,

$$TP_x = T(\text{horizontal space})_x + T(\text{fiber})_x.$$

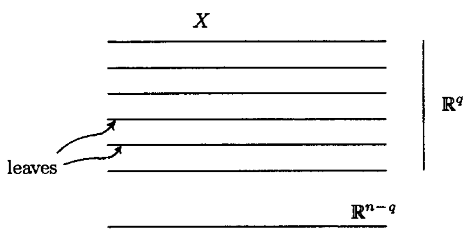


Definition. A codimension q *foliation* $\mathcal{F}$ on an n-manifold X is a decomposition of X into $(n-q)$-dimensional submanifolds, called *leaves*, such that every point of X has a neighborhood U and a diffeomorphism

$$g : \mathbb{R}^{n-q} \times \mathbb{R}^q \longrightarrow U$$

which carries each submanifold $\mathbb{R}^{n-q} \times \{y\}$, $y \in \mathbb{R}^q$, diffeomorphically into a leaf. (See [6].)

Thus locally, X looks like



Of course, globally the foliation may be twisted.

The *tangent bundle of the foliation*, $T\mathcal{F}$, is the subbundle of TX consisting of vectors tangent to the leaves. The normal *bundle of the foliation*, $\nu\mathcal{F}$, is $TX/T\mathcal{F}$. If one puts a Riemannian metric on TX, one can embed $\nu\mathcal{F}$ in TX so that $TX \cong T\mathcal{F} \oplus \nu\mathcal{F}$.

Returning to the bundle P, we see now that the horizontal spaces are the leaves of a foliation transversal to the fibers of P. On $P|U_a$ this foliation is trivial, i.e., a product, but globally it may be twisted.

In the language of connections [6] the bundle $\pi : P \to M$ has a flat connection whose flat subspaces are the horizontal spaces or leaves. To summarize:

Theorem 10.1. *Let $\pi : P \to M$ be a bundle. Then the following three statements are equivalent:*

1. *The bundle has a discrete structure group (or its structure group can be reduced to a discrete group).*
2. *The bundle has a foliation transversal to the fibers (of π), whose leaves have the same dimension as M.*
3. *The bundle has a flat connection.*

How does one find characteristic classes of G_δ bundles? We can put a differential form ψ on G and pull it back to $P|U_a$ via the map $f_a : P|U_a \to G$. Since f_a and f_b differ by a left translation in G (by g_{ab}), $f_a^*\psi$ and $f_b^*\psi$ will agree on $P|U_a \cap U_b$ if $\psi \in \{\text{left invariant forms on } G\} \underset{def.}{=} \text{Inv}_G AG$. (The action of $x \in G$ on AG is L_x^*, induced by left multiplication $L_x : G \to G$.) Thus there is a natural map $\text{Inv}_G AG \to AP = \{\text{forms defined globally on } P\}$.

The module of left invariant forms $\text{Inv}_G AG$ is a finite dimensional vector space, since each such form is uniquely determined by its value on the tangent space at the unit element $e \in G$. Writing g for the Lie alg of G (which is defined either as T_eG or as the vector space of vector field G invariant under left multiplication by elements of G, with the commut (bracket) operation $[X,Y]$), we see that $\text{Inv}_G A^1G = g^*$, the dual of g, and $\text{Inv}_G A^qG$ =left invariant q-forms $= \Lambda^q g^*$, where Λ^q denotes the q-th ext power. That is, $\text{Inv}_G A^qG$ can be regarded as the module of alternating q-li real-valued forms on the vector space g.

We can reduce our calculations with differential forms completely tc algebra calculations. A standard fact from geometry is the DeRham ext derivative d, when restricted to $\text{Inv}_G A^qG = \Lambda^q g^*$, has the formula

$$d\varphi(X_1,\ldots,X_{q+1}) = \Sigma_{i<j}(-1)^{i+j}\varphi([X_i,X_j],X_1,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_{q+1}$$

where $\varphi \in \Lambda^q g^*$ is a q-linear alternating form on g and $X_i \in g$. Motivate this formula, one defines the cohomology $H^*(g;\mathbb{R})$ of *any* Lie algebra g t the cohomology of the complex

$$\mathbb{R} = \Lambda^0 g^* \xrightarrow{d} \Lambda^1 g^* \xrightarrow{d} \Lambda^2 g^* \longrightarrow \cdots$$

where d is defined by the formula above (cf. infra, see also [14]).

We have defined a natural map $\text{Inv}_G AG \to AP$. This map commutes the DeRham d, so it induces a map $H^*(g;\mathbb{R}) = H^*(\text{Inv}_G AG) \to H^*(P)$. our goal was to construct characteristic classes for G^δ bundles, and these sh lie in $H^*(M)$. How can we pull down cohomology classes from the total s P to the base space M of the bundle? We proceed in two different ways.

First, suppose the G^δ bundle $\pi : P \to M$ is trivialized as a G-bundle, is, is G-bundle isomorphic to $M \times G$. Then π has a section

$$s : M \to M \times \{e\} \subset M \times G \cong P$$

and we can pull down forms and classes from P to M by the maps

$$s^* : AP \longrightarrow AM \quad \text{and} \quad s^* : H^*P \longrightarrow H^*M.$$

In general, both maps depend on the trivialization chosen.

Bundles with structure group G^δ which are trivialized as G-bundles been classified by Thurston by the following scheme (see [54], p. 84, for ge properties of this construction). Let G^I = path space of G = space of n $I = [0,1] \to G$ (with compact-open or evaluation topology). G^I maps to $p \to p(0)$. Let G_1 be the fiber product of G^δ and G^I:

$$\begin{array}{ccccc} G_1 & \longrightarrow & G^I & & p \\ \downarrow & & \downarrow & & \downarrow \\ G_\delta & \longrightarrow & G & & p(0) \end{array}$$

That is, $G_1 = \{(x,p) \in G_\delta \times G^I | x = p(0)\}$. $G_1 \to G_\delta$ is a homotopy equivalence. Now map $G_1 \to G$ by $(x,p) \to p(1)$. This is a Hurewicz fibration, and we call its fiber (inverse image of $e \in G$) $\overline{G}$. $\overline{G}$ is a group since we can multiply paths in G pointwise. We have $0 \to \overline{G} \to G_1 \to G \to e$, or (by a common abuse of notation) $0 \to \overline{G} \to G_\delta \to G \to e$, where we identify G_1 with G_δ since they have the same homotopy type.

Exercise. Check that a $\overline{G}$-bundle over M is given by a G-bundle $Q \to M \times I$ and by G-bundle maps

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longleftarrow & M \times G \\ \downarrow & & \downarrow & & \downarrow \\ M \times \{0\} & \subset & M \times I & \longleftarrow & M \times \{1\} \end{array}$$

where P is a G^δ bundle.

Definition. A *trivialization* of a G-bundle $E \to M$ is a bundle map

$$\begin{array}{ccc} E & \longrightarrow & M \times G \\ \downarrow & & \downarrow \\ M & \overrightarrow{=} & M \end{array}$$

(equivalently, a map $E \to G$). Two trivializations f, g are *homotopic* if there is a trivialization

$$\begin{array}{ccc} E \times I & \xrightarrow{f_t} & M \times I \times G \\ \downarrow & & \downarrow \\ M \times I & = & M \times I \end{array} \qquad \begin{array}{c} \text{such that} \quad f_0 = f \\ \\ \text{and} \quad f_1 = g; \end{array}$$

an equivalent condition is that the maps $E \to G$ induced by f and g be homotopic maps.

Refer to the last exercise. By standard bundle theory [41], there exists a bundle map (not unique)

$$\begin{array}{ccc} Q & \xrightarrow{\approx} & (Q|M \times 1) \times I \\ \downarrow & \swarrow & \\ M \times I & & \end{array}$$

Since $Q|M \times 1 = M \times G$, we get a trivialization

$$\begin{array}{ccc} Q & \xrightarrow{\approx} & M \times I \times G \\ \downarrow & \swarrow & \\ M \times I & & \end{array}$$

Restricting to $P = Q|M \times \{0\}$, we get a trivialization

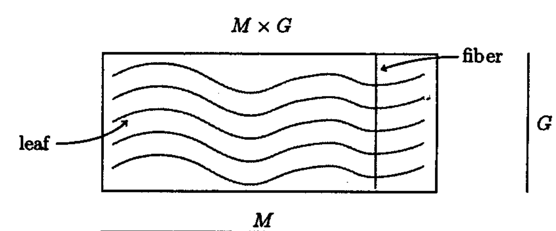
$$\begin{array}{ccc} P & \xrightarrow{\approx} & M \times G \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Exercise. All trivializations of P obtained in this way are homotopic.

Thus we get a second characterization of $\overline{G}$-bundles. A $\overline{G}$-bundle over is a G^δ bundle $P \to M$ together with a trivialization

$$\begin{array}{ccc} P & \longrightarrow & M \times G \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

of P as a G-bundle. That is, a $\overline{G}$-bundle on M is a foliation of $M \times G$ by lea of dimension $= \dim M$, which are transversal to the fibers of $M \times G \to M$.



Definition. Two G-bundles $E \to M$ and $E' \to M$ are *homotopic* if there i G-bundle $F \to M \times I$ and bundle maps

$$\begin{array}{ccc} E & \xrightarrow{\approx} & F|M \times \{0\} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \times \{0\} \end{array} \quad \text{and} \quad \begin{array}{ccc} E' & \xrightarrow{\approx} & F|M \times \{1\} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \times \{1\}. \end{array}$$

Exercise. Verify that there is indeed a canonical isomorphism of sets fr {homotopy classes of $\overline{G}$ bundles for the first characterization of $\overline{G}$ bundles} {homotopy classes of $\overline{G}$ bundles for the second characterization of $\overline{G}$ bundl

Using our semi-simplicial construction on the group $\overline{G}$, we construct $B\overline{G} = |N\overline{G}_Z|$ (unwinding; see Sections 4,6). $B\overline{G}$ classifies $\overline{G}$-bundles.

Our work so far has been to construct maps $H^*(g) \to H^*(M)$ which are natural with respect to maps of $\overline{G}$-bundles. The naturality implies the existence of a map $H^*(g) \to H^*(B\overline{G})$ (see [6], pp. 70–72), so that

$$\begin{array}{ccc} H^*(g) & \longrightarrow & H^*(B\overline{G}) \\ & \searrow & \downarrow \\ & & H^*(M) \end{array}$$

commutes, where $H^*(B\overline{G}) \to H^*(M)$ is induced by a classifying map of the bundle, $M \to B\overline{G}$.

Suppose now we try to construct characteristic classes for G^δ-bundles which are not necessarily $\overline{G}$-bundles. That is, how can we find classes in $H^*(BG^\delta)$? Suppose instead of the principal G^δ bundle $\pi : P \to M$ we consider the associated bundle $P \times_G F \to M$ with the same cocycle $\{g_{uv}\}$ but with fiber F, where F is a manifold on which G acts smoothly on the left (cf. Section 1). The same argument as before shows that $P \times_G F$ has a horizontal foliation, and that there is a natural map $\mathrm{Inv}_G AF \to A(P \times_G F)$ of G-invariant forms on F into forms on $P \times_G F$. In particular, for each subgroup $H \subset G$ one can choose $F = G/H$. In this case the module $\mathrm{Inv}_G A(G/H)$ maps into $A(P \times_G (G/H))$. Specializing further, if we choose $H = K$ so that $G/K \approx \mathbb{R}^q$ for some q (which we can do by taking K to be a maximal compact subgroup of G, using Cartan's theory), then $P \times_G (G/K) \to M$ has a section s, and all sections are homotopic, so there is a well-defined map $s^* : H^*(P \times_G (G/K)) \to H^*(M)$. Now the relative Lie algebra cohomology $H^*(g, K; \mathbb{R}) = H^*(g, K)$ is defined to be equal to $H^*(\mathrm{Inv}_G A(G/K))$ (cf. infra). Hence we have defined a map $H^*(g, K) \to H^*(M)$ which is natural with respect to G^δ-bundle maps. This implies that there is a map $H^*(g, K) \to H^*(BG^\delta)$ such that

$$\begin{array}{ccc} H^*(g,K) & \longrightarrow & H^*(BG^\delta) \\ & \searrow \quad \swarrow & \\ & H^*(M) & \end{array}$$

commutes for any G^δ bundle on M. In the specific case $G = GL(n, \mathbb{R})$ or $GL^+(n, \mathbb{R})$(GL^+ classifies *oriented n-plane bundles*) we have $g = gl(n, \mathbb{R})$ and $K = O(n)$ or $SO(n)$, respectively, is the maximal compact subgroup of G. We have constructed maps

$$\varphi_1 : H^*(gl(n)) \longrightarrow H^*(B\overline{G})$$
$$\varphi_2 : H^*(gl(n), K) \longrightarrow H^*(BG^\delta).$$

If $D \subset G$ is a discrete subgroup of G, for example $D = GL(n, \mathbb{Z})$, then inclusion $D \subset G^\delta$ induces a map $H^*(g, K) \to H^*(BG^\delta) \to H^*(BD)$.

A priori, φ_1 and φ_2 need not be injective or even non-zero maps. But deep theory of Borel and Harish-Chandra implies

Theorem 10.2. *If G is a semi-simple Lie group (e.g., $SL(n, \mathbb{R})$ with algebra g and maximal compact subgroup K, then the map*

$$\varphi : H^*(g, K) \longrightarrow H^*(BG^\delta)$$

defined previously is injective.

Proof. By the theory of Borel and Harish-Chandra, there is subgr $D \subset G$ which acts discretely on G/K by left multiplication, and such t $D\backslash G/K$ is compact. But G/K is contractible and D acts freely so

$$\begin{array}{ccc} G/K & = & ED \\ \downarrow & & \downarrow \\ D\backslash G/K & = & BD \end{array}$$

is a universal D bundle. Thus there are maps

$$H^*(g, K) \longrightarrow H^*(BG^\delta) \longrightarrow H^*(BD) = H^*(D\backslash G/K)$$

Hence it suffices to show that $f : H^*(g, K) \to H^*(D\backslash G/K)$ is injective. V f is induced by the map $\mathrm{Inv}_G(A(G/K)) \subset \mathrm{Inv}_D(A(G/K)) = A(D\backslash G/.$ and this map takes left-invariant volume form on G/K to a volume form $D\backslash G/K$. Now we can assume $D\backslash G/K$ is orientable (by changing D if ne sary), and it is compact, so a volume form on it represents a non-zero elem of $H^*(D\backslash G/K)$ in the top dimension, $p = \dim G - \dim K$. Now p is the top dimension of $H^*(g, K)$, and $H^p(g, K)$ and $H^p(D\backslash G/K)$ are both so $H^p(g, K) \to H^p(D\backslash G/K)$ is an isomorphism. But Poincaré duality hc on $H^*(g, K)$ and on $H^*(D\backslash G/K)$, i.e., multiplication $H^i \times H^{p-i} \to H^p =$ is a nondegenerate bilinear pairing. Furthermore, $H^*(g, K) \to H^*(D\backslash G/$ commutes with multiplication.

Exercise. This implies that $H^*(g, K) \to H^*(D\backslash G/K)$ is injective in ev dimension. Q.E

Consider now the complex of singular cochains on BG^δ which are conti ous in the topology of BG (since $G^\delta \to G$ can be taken to be an isomorphi of sets, BG^δ can be taken to be BG with a finer topology). We define cohomology of this complex to be the *continuous cohomology of* BG^δ.

Theorem 10.3. (Van Est, Bott, Haefliger). *$H^*(g, k)$ is isomorphic to continuous cohomology of BG^δ.*

Proof. [9].

11. Lie algebra cohomology

Let L be a Lie algebra. That is L is a (for now, finite dimensional real) vector space with a skew symmetric bilinear bracket product:

$$L \times L \xrightarrow{[,]} L.$$

Skew symmetric means (for $X, Y \in L$) $[X,Y] = -[Y,X]$. We also require that $[,]$ satisfy the Jacobi identity:

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]] \quad (X,Y,Z \in L).$$

Now for $X \in L$ we have a linear map $ad_X : L \to L$ defined by

$$ad_X(Y) = [X,Y].$$

Then the Jacobi identity says that ad_X is a derivation of $L : ad_X([Y,Z]) = [ad_X(Y),Z] + [Y, ad_X(Z)]$ for $X,Y,Z \in L$.

Exercise. $ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X$ for $X, Y \in L$.

Let $A^q(L)$ be the set of q-linear alternating forms on L. (i.e., $f \in A^q(L)$ is a multilinear map $\underbrace{L \times \cdots \times L}_{q-factors} \to \mathbb{R}$ such that for $X_1, \ldots, X_q \in L$, we have $f(X_1, \ldots, X_q) = -f(X_{\partial(1)}, \ldots, X_{\partial(q)})$ whenever ∂ is a permutation of q elements which interchanges two of them.)

We can define a map $d: A^q(L) \to A^{q+1}(L)$ by $(d\varphi)(X_1, \ldots, X_{q+1}) = \Sigma_{i<j}(-1)^{i+j}\varphi([X_i, x_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{q+1})$ for $\varphi \in A^q(L)$ and $X_1, \ldots, X_{q+1} \in L$. If $\varphi \in A^\circ(L) \simeq \mathbb{R}$ then $d\varphi = 0$. Then d is an antiderivation since $d^2 = 0$ and

$$d(\eta \wedge \omega) = (d\eta) \wedge \omega + (-1)^{deg\,\eta} \wedge d\omega$$

for $\eta, \omega \in A^*(L) = \oplus_{q=0}^{\infty} A^q(L)$. Therefore, d is determined by its value on $A^1(L)$:

$$(d\varphi)(X,Y) = -\varphi([X,Y])$$

for $\varphi \in A^1(L)$, and $X, Y \in L$.

Example. If L is the Lie algebra of left invariant vector fields of a Lie group, G, then $A^*(L)$ is identified with the left invariant differential forms on the manifold of G. Our d on $A^*(L)$ is then induced by the usual DeRham d on the differential forms.

Definition. The cohomology of a Lie algebra L, $H^*(L)$, is defined as the cohomology of the sequence:

$$0 \xrightarrow{d} A^\circ(L) \xrightarrow{d} A^1(L) \xrightarrow{d} A^2(L) \longrightarrow \cdots$$

If L is the Lie algebra of a Lie group G then there is a natural map $H^*(L)$ $H^*_{DR}(G)$ = DeRham cohomology of the manifold of G. This map need not an isomorphism.

Example. Let $G = \mathbb{R}$ under addition, with Lie algebra $g \simeq \mathbb{R}$ genera by $\partial/\partial x$. Then

$$H^q(g) = \begin{cases} \mathbb{R} & \text{for} \quad q = 0 \quad \text{(generated by 1)} \\ \mathbb{R} & \text{for} \quad q = 1 \quad \text{(generated by dx)} \\ 0 & \text{for} \quad q \geq 2. \end{cases}$$

This is not isomorphic to the DeRham cohomology, $H^*_{DR}(G) = \mathbb{R}$ only dimension 0. However, g is also the Lie algebra of $S^1 = \mathbb{R}/\mathbb{Z}$ and $H^q_{DR}(S^1$ $H^q(g)$.

Theorem 11.1. *If G is compact and connected, then $H^*(g) \to H^*_{DR}$ is an isomorphism.*

The idea of the proof is to get an inverse by averaging (i.e., integrating) form on G to obtain a left-invariant q-form (see Chevalley–Eilenberg [14] the same cohomology class in $H^q_{DR}(G)$.

But we still have the problem for an arbitrary Lie algebra, L, to m (functorially) a space Y such that $H^*(L) \simeq H^*(Y)$.

Homework. Compute $H^*(gl(2,\mathbb{R}))$ or $H^*(sl(2,\mathbb{R}))$. After spending sc time trying to compute directly from the definition, the reader will gain sc appreciation for general techniques as computation aids. (See Hochschild–Se spectral sequence, for instance [39].)

12. Gel'fand–Fuks

Let M be a C^∞ manifold of dimension n, and let $L = L(M)$ be the algebra of all C^∞ vector fields on M with the C^∞ topology. Let us recall w the C^∞ topology means for the set of maps $\mathbb{R}^n \to \mathbb{R}$. A basic neighborhoo $f : \mathbb{R}^n \to \mathbb{R}$ is $N(f, K, r, \epsilon)$, where $K \subseteq \mathbb{R}^n$ is compact, $r \geq 0$ is an integer, $\epsilon > 0$. Then $g \in N(f, K, r, \epsilon)$ if and only if restricted to K, the values of g its partial derivatives $\partial^s g/\partial x^{\alpha_1} \cdots \partial x^{\alpha_s}$ of order $s \leq r$ are within ϵ of thos f. Since, locally, vector fields on M are given by n-tuples of functions $\mathbb{R}^n \to$ we obtain a topology on $L(M)$ independent of charts. We can define $A^q(L)$ be the q-linear alternating forms on L as before and consider the cohomol of the complex $A^\circ(L) \to A^1(L) \to$.

However, this object is too large to hope to compute. So instead of q-forms we will consider only the continuous q-forms. (For finite dimensio Lie algebras linearity of the q-forms implies continuity.)

Definition. $A_c^q(L)$ = the *continuous* q-alternating forms on $L = L(M)$. $A_c(L) = \oplus_{q=0}^{\infty} A_c^q(L)$.

Definition. The Gel'fand–Fuks cohomology of M, $H_{GF}^*(M) = H^*(A_c(L(M)))$ = cohomology of the complex:

$$0 \longrightarrow A_c^\circ(L) \xrightarrow{d} A_c^1(L) \xrightarrow{d} A_c^2(L) \longrightarrow \cdots$$

Think of $L(M)$ as the Lie algebra of the group of diffeomorphisms of M, Diff (M). A curve $t \to g_t$ through the identity of Diff (M) gives rise to a family of curves $t \to g_t(m)$ through each point $m \in M$. So a "tangent vector" at $id \in$ Diff (M) corresponds to a vector field on M, which is an element of $L(M)$.

Gel'fand–Fuks stated the next theorem for compact manifolds [26]. However, the only property of compact manifolds which we will require is that M possess a covering by a *finite* collection of open sets such that these open sets and all finite intersections of them are all copies of $\mathbb{R}^n$ (or empty). Such a cover will be called *"good."* Note that if M is compact then it can be covered by a finite number of geodesically convex open sets in some Riemannian metric. But there are other obvious examples, for instance $M = \mathbb{R}^n$.

Theorem 12.1. (Gel'fand–Fuks) *If M has a good covering then $H_{GF}^r(M)$ is finite dimensional for each r.*

Theorem 12.2. (Gel'fand–Fuks) $H_{GF}^*(S^1) = E(\omega) \otimes P(y)$, *the exterior algebra generated by a class ω of dimension 3 tensored with the polynomial algebra generated by a class y of dimension 2. Further, ω and y have the following representatives. Let $X, Y, Z \in L(M)$. Then $X = f(x)\partial/\partial x$, $Y = g(x)\partial/\partial(x)$, and $Z = h(x)\partial/\partial x$, where f, g, and h are smooth periodic functions $\mathbb{R} \to \mathbb{R}$ (periodic so we get a map $S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}$). Then:*

$$\omega(W, Y, Z) = \det \begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix}_0 .$$

Gel'fand–Fuks had said to integrate this around the circle, but we can instead evaluate at (an arbitrary point) $0 \in S^1$ and get a cohomologous representative form. See [24], [26] 9.2.

$$y(X, Y) = \int_0^{2\pi} \begin{vmatrix} f' & f'' \\ g' & g'' \end{vmatrix}_x dx$$

We can not just evaluate this one at a point since we would not get a cocycle. ω and y are clearly continuous forms since the values $\omega(X, Y, Z)$ and $y(X, Y)$ do not change much whenever $f, f', f'', g, g', g'', h, h', h''$ do not change much.

By way of contrast, an ordinary differential form on M is also a function on vector fields, but with the values in $C^\infty(M)$ = smooth functions $M \to \mathbb{R}$. Our forms, $A_c^q(L)$, have values in $\mathbb{R}$.

The plan of this proof is to compute $H_{GF}^*(\mathbb{R}^n)$ first and then apply (semi-simplicial machinery to get at $H_{GF}^*(M)$.

13. Special Case: Gel'fand–Fuks of $\mathbb{R}^1$

A vector field $X \in L(\mathbb{R}^I)$ is of the form $f(x)\partial/\partial x$, $f : \mathbb{R} \to \mathbb{R}$ smoo Then a 1-cochain is a distribution, that is, a continuous linear functional $C^\infty(\mathbb{R}^1)$ = smooth maps $\mathbb{R}^1 \to \mathbb{R}$.

Consider now in $A_c^1 L\mathbb{R}^1$ the subcomplex $A_{pt}^1 L\mathbb{R}^1$, of distributions wh support is contained at the origin 0 of $\mathbb{R}^1$. In general a q-form $\varphi \in A_{pt}^q L\mathbb{R}^1$ support at 0 in the sense that $\varphi(X_1, \ldots, X_q) = 0$ whenever the vector fie $X_1, \ldots, X_q$ vanish in a neighborhood of 0.

Example. The Dirac δ-function: If $X = f(x)\partial/\partial x$ then $\delta(X) = f$ We also have the derivatives of δ. $\delta'(X) = -f'(0)$, $\delta''(X) = f''(0)$, $\delta^{[k]}(X) = (-1)^k f^{[k]}(0)$. The conventional $(-1)^k$ sign is used to make notation consistent with the following:

Any smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support defines a distri tion (for $f : \mathbb{R}^1 \to \mathbb{R}$)

$$f \longrightarrow \varphi(f) = \int_{-\infty}^{\infty} \varphi(x) f(x)\, dx.$$

Since φ is smooth, φ' is defined and is the distribution:

$$f \longrightarrow \varphi'(f) = \int_{-\infty}^{\infty} \varphi'(x) f(x)\, dx.$$

Then by integrating by parts:

$$\varphi'(f) = \varphi(x) f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \varphi(x) f'(x)\, dx = (0 - 0) - \varphi(f').$$

This is consistent with our notation for the derivative of the δ-function: $\delta'(f$ $-\delta(f')$. Now I claim that $A_{pt}^1 L(\mathbb{R}^1)$ is the vector space generated by the 1-fo $\{\delta, \delta', \delta'', \ldots\}$. And $A_{pt}^* L\mathbb{R}^1$ is the exterior algebra generated by $\{\delta, \delta', \delta'', .$ To see this we must look at what "continuous in the C^∞ topology" me Let $\varphi \in A_{pt}^1 L\mathbb{R}^1$ and $0 : \mathbb{R} \to \mathbb{R}$ be identically 0. Then by linearity $\varphi(0)$ = By continuity, for all $\epsilon > 0$, there exists a basic neighborhood $N(0, K,$ of $0 \in C^\infty(\mathbb{R})$, such that $\varphi(N(0, K, r, \eta)) \subseteq (-\epsilon, \epsilon)$. Since φ has suppor the origin, we can take K to be the origin of $\mathbb{R}$. r is a positive integer $\eta > 0$. Suppose f and its first r derivatives vanish at the origin $=K$. T $f \in N(0, K, r, \eta)$, so that $\varphi(f) \in (-\epsilon, \epsilon)$. In fact $\varphi(f) = 0$, for otherwise, $\lambda \in \mathbb{R}$, λf would also vanish to r-th order at the origin. So $\lambda f \in N(0, K,$ and $\varphi(\lambda f) = \lambda\varphi(f)$ can be $> \epsilon$ for large λ. So *φ can only depend on a fi number of derivatives* of its arguments. As a corollary, φ cannot disting between "flat" functions. That is, if $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 0$, $f^{[k]}(0) = 0$ for all k, then for every continuous linear functional $\varphi \in A^1_{pt}L\mathbb{R}^1$, $\varphi(f(x)\partial/\partial x) = 0$.

Computation. Let $X = f(x)\partial/\partial x$, $Y = g(x)\partial/\partial x$ in $L(\mathbb{R}^1)$.

$$\begin{aligned}(\delta' \wedge \delta'')(X, Y) &= \delta'(X)\delta''(Y) - \delta''(X)\delta'(Y) \\ &= -f'(0)g''(0) - f''(0)(-g'(0)) \\ &= -\det \begin{vmatrix} f' & f'' \\ g' & g'' \end{vmatrix}_0 .\end{aligned}$$

What is d on $A^*_{pt}(\mathbb{R}^1)$? Let $X, Y \in L(\mathbb{R}^1)$ as above.

$$\begin{aligned}(d\delta)(X, Y) &= -\delta([X, Y]) = -\delta([f(x)\partial/\partial x,\ g(x)\partial/\partial x]) \\ &= -\delta(f(x)g'(x)(\partial/\partial x) - g(x)f'(x)(\partial/\partial x)) \\ &= -f(0)g'(0) + g(0)f'(0) \\ &= \delta \wedge \delta'(X, Y).\end{aligned}$$

Thus $d\delta = \delta \wedge \delta'$. If we extend the action of $'$ to be a derivation of $A^*_{pt}L\mathbb{R}^1$ then $'$ will commute with d (Verify! Cf. the next section in which $'$ is just the Lie derivative denoted by $\theta_{\partial/\partial x}$). Then we can compute:

$$\begin{aligned}d(\delta') &= (d\delta)' = (\delta \wedge \delta')' = \delta' \wedge \delta' + \delta \wedge \delta'' = \delta \wedge \delta'' \\ d(\delta'') &= (d\delta')' = (\delta \wedge \delta'')' = \delta' \wedge \delta'' + \delta \wedge \delta''' \quad \text{etc.}\end{aligned}$$

Consider the 3-form $\omega = \delta \wedge \delta' \wedge \delta''$. Then ω is a cocycle since $d\omega = -d(\delta' \wedge d\delta') = -d\delta' \wedge d\delta' + \delta' \wedge d^2\delta' = 0$. For $X = f(x)\partial/\partial x$, $Y = g(x)\partial/\partial x$, and $Z = h(x)\partial/\partial x$ we have

$$\begin{aligned}\omega(X, Y, Z) &= \delta \wedge \delta' \wedge \delta''(f(x)\partial/\partial x, g(x)\partial/\partial x, h(x)\partial/\partial x) \\ &= \det \begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix}_0\end{aligned}$$

Proposition 13.1. *$H^*(A^*_{pt}L\mathbb{R}^1)$ is generated by the trivial class 1 in dimension 0 and by ω in dimension 3.*

Proof. We will defer the proof until Section 14. See [28].
Our plan will then be:

1) to compute $H^*(A_{pt}L\mathbb{R}^n)$, and
2) to show that the inclusion $A^*_{pt}L\mathbb{R}^n \to A^*_c L\mathbb{R}^n$ induces an isomorphism in cohomology. This corresponds to the Poincaré Lemma in DeRham theory: we will study the contraction of $\mathbb{R}^n$ given by $f_t : \mathbb{R}^n \to \mathbb{R}^n$, $x \to tx$, for $t \in \mathbb{R}$, $x \in \mathbb{R}^n$.

14. Techniques and basic formulae

Let g be a Lie algebra, $A^q(g) = q$-linear alternating forms on g. Re that, if g is the Lie algebra of a Lie group G, then $A^q(g) \simeq \mathrm{Inv}_G A^q(G) =$ invariant q-forms on the underlying manifold of G. For $g = L(M) = C^\infty$ vec fields on a manifold M, we can only take continuous forms.

Definition. For each $X \in g$ we define the *interior product*:

$$\begin{aligned}\iota_X : A^q(g) &\to A^{q-1}(g) \quad \text{by} \\ (\iota_X\varphi)(X_1, \ldots, X_{q-1}) &= \varphi(X, X_1, \ldots, X_{q-1})\end{aligned}$$

for $\varphi \in A^q(g); X_1, \ldots, X_{q-1} \in g$. We have $\iota_X : A^\circ(g) \to 0$ and $\iota_X : A^1(g)$ $\mathbb{R} = A^\circ(g)$ is a duality.

Exercise. ι_X is an anti-derivation. That is, $\iota_X^2 = 0$ and

$$\iota_X(\omega \wedge \eta) = (\iota_X\omega) \wedge \eta + (-1)^{\deg \omega}\omega \wedge (\iota_X\eta).$$

Definition. For $X \in g$, we define the *Lie derivative* $\theta_X : A^q(g) \to A^q$ by the relation

$$\theta_X = \iota_X d + d\iota_X.$$

Remark. θ_X is chain homotopic to 0 via ι_X so that θ_X is 0 in $H^*(g)$.

Exercise. Prove the following useful formulae: (for $X, Y \in g$)

$$\begin{aligned}d\theta_X &= \theta_X d \\ \theta_X \quad &\text{and} \quad \iota_X \quad \text{are linear in} \quad X \\ \theta_{[X,Y]} &= \theta_X\theta_Y - \theta_Y\theta_X \equiv [\theta_X, \theta_Y] \\ \iota_{[X,Y]} &= \theta_X\iota_Y - \iota_Y\theta_X \equiv [\theta_X, \iota_Y] \\ (\theta_X, \varphi)(Y) &= -\varphi([X, Y]), \quad \text{for} \quad \varphi \in A^1(g) \\ \theta_X \quad &\text{is a derivation of} \quad A^*(g).\end{aligned}$$

Remark. Suppose θ_X splits up $A^*(g)$ into eigenspaces:

$$A^*(g) = \bigoplus_\lambda A_\lambda$$

where θ_X acts on A_λ by scalar multiplication by $\lambda \in \mathbb{R}$. Then $d(A_\lambda) \subseteq$ and $H^*(A(g)) = \oplus_\lambda H^*(A_\lambda)$. Furthermore, if $\lambda \neq 0$, $H^*(A_\lambda) = 0$ beca $id = (1/\lambda)\theta_X = (1/\lambda)(\iota_X d + d\iota_X)$ is homotopic to 0 in cohomology. Theref $H^*(A_\circ) \simeq H^*(A(g))$, where the isomorphism is induced by the inclusion of zero eigenspace $A_\circ$ into $A(g)$.

Using this principle, we can compute $H^*(A_{pt}(L\mathbb{R}^1))$, where $A_{pt}(L\mathbb{R}^1) =$ continuous forms on $L\mathbb{R}^1$ with support at the origin. Let $R = x(\partial/\partial x)$ be the radial vector field on $\mathbb{R}^1$. Then for $X = f(x)\partial/\partial x$ we have:

$$\begin{aligned}(\theta_R\delta)(X) &= -\delta([R, X]) = -\delta(xf'(x)(\partial/\partial x) - f(x)(\partial/\partial x)\\ &= f(0) = \delta(X).\end{aligned}$$

Therefore, $\theta_R\delta = \delta$.

Since δ' is just $\theta_{\partial/\partial x}(\delta)$, we have:

$$\begin{aligned}\theta_R\delta' = \theta_R\theta_{\partial/\partial x}\delta &= \theta_{\partial/\partial x}(\theta_R\delta) + \theta_{[x(\partial/\partial x),\partial/\partial x]}(\delta)\\ &= \theta_{\partial/\partial x}(\delta) + \theta_{-\partial/\partial x}(\delta) = 0.\end{aligned}$$

Inductively, we obtain:

$$\theta_R\delta^{[k]} = (1-k)\delta^{[k]},$$

which gives an eigenspace decomposition of $A^1_{pt}(L\mathbb{R}^1)$. The eigenvalue $(1-k)$ is called the *weight* of $\delta^{[k]}$. Since θ_R is a derivation, the weight of a product is the sum of the weights: $A_\lambda \cdot A_\mu \subseteq A_{\lambda+\mu}$. Here are some examples:

	weight						
	1	0	-1	-2	-3	-4	$\cdots$
1-forms	$\{\delta$	δ'	δ''	δ'''	δ''''	$\cdots$	
2-forms	$\{\delta\wedge\delta'$	$\delta\wedge\delta''$	$\delta\wedge\delta'''$	$\delta\wedge\delta''''$	$\cdots$		
			$\delta'\wedge\delta'''$	$\delta\wedge\delta''''$	$\cdots$		
3-forms	$\{$		$\cdots$				
		$\delta\wedge\delta'\wedge\delta''$	$\cdots$				

We note that the 0-eigenspace of $A^*_{pt}L\mathbb{R}^1$ is generated by $\{1, \delta', \delta\wedge\delta'', \delta\wedge\delta'\wedge\delta''\}$ as a vector space. Since d acts by $1 \to 0, \delta' \to \delta\wedge\delta''$, $\delta\wedge\delta'' \to 0$, $\delta\wedge\delta'\wedge\delta'' \to 0$, the cohomology, $H^*(A_\circ)$, of the 0-eigenspace (and, therefore, $= H^*(A_{pt}L\mathbb{R}')$) is generated by 1 in dimension 0 and $\omega = \delta\wedge\delta'\wedge\delta''$ in dimension 3. This proves

15. Distributions again

Let us return to the larger problem of $A^*_cL\mathbb{R}^1$, the continuous forms (with aribitrary support) on $L\mathbb{R}^1$.

Exercise. $\varphi \in A^q_cL\mathbb{R}^1$ is continuous $\Rightarrow$ φ has compact support. In fact the values of φ only depend on a finite number of derivatives of its arguments on a compact set K of $\mathbb{R}^1$ (The *support* of φ is the intersection of all closed sets K such that for all $g : \mathbb{R} \to \mathbb{R}$ smooth, vanishing on a neighborhood of K we have $\varphi(g) = 0$. Use the K in the definition of the C^∞-topology.)

Theorem 15.1. (Schwartz [55]. Cf. Gel'fand [23].) *Every continu linear functional $\varphi : C^\infty(\mathbb{R}) \to \mathbb{R}$ (where $C^\infty(\mathbb{R})$ = smooth maps $\mathbb{R} \to \mathbb{R}$ in C^∞ topology) is a finite linear combination of terms of the following form:*

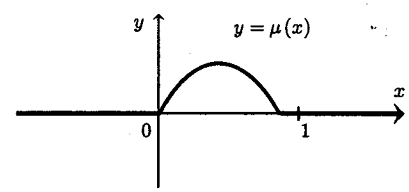
$$f \longrightarrow \int_{-\infty}^{\infty} \mu(x) f^{[k]}(x)dx$$

where $f^{[k]}(x)$ is the k-th derivative of f and $\mu(x)$ is a continuous funci $\mathbb{R} \to \mathbb{R}$ with compact support. In Schwartz's language, every distribution ı compact support is the sum of derivatives of continuous distributions.

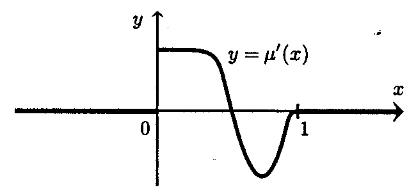
Example. What about the Dirac δ? How can δ be put in the above foı Let $\mu : \mathbb{R} \to \mathbb{R}$ be a C^∞ except at the origin such that

$$\mu(x) = \begin{cases} 0 & \text{for} \quad x \le 0 \quad \text{or} \quad x \ge 3/4 \\ x & \text{for} \quad 0 \le x \le 1/4. \end{cases}$$

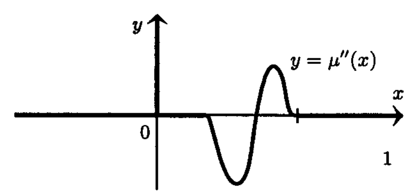
Then μ looks like:



And μ' looks like:



μ' is discontinuous at 0. And intuitively μ'' looks like the sum of a smo function, ν, and a Dirac-δ spike at 0.



Although μ'' is not rigorously defined as a function, it is well-defined as a distribution.

$$\begin{aligned}\mu''(f) = +\mu(f'') &= \int_{-\infty}^{\infty} \mu(x) f''(x)dx = \int_0^1 \mu(x)f''(x)dx \\ &= \mu(x)f'(x)|_0^1 - \left\{\int_0^1 \mu'(x)f'(x)dx\right\} \\ &= (0-0) - \left\{\mu'(x)\ f(x)|_0^1 - \int_0^1 \mu''(x)f(x)dx\right\} \\ &= -0 + f(0) + \int_{-\infty}^{\infty} \nu(x)f(x)dx\end{aligned}$$

so that as distributions it makes sense to say $\mu'' = \nu + \delta$. We have shown:

$$\delta(f) = \int_{-\infty}^{\infty} \mu(x)f''(x)dx - \int_{-\infty}^{\infty} \nu(x)f(x)dx.$$

We can write this as

$$\delta = \int_{-\infty}^{\infty} \mu(x)\delta_x'' dx - \int_{-\infty}^{\infty} \nu(x)\delta_x dx,$$

where δ_x and δ_x'' are just the Dirac-δ function and its second derivative, but at x instead of the origin $\delta_x(f) = f(x)$, $\delta_x''(f) = f''(x)$, etc.

We repeat some important facts we have learned:

Proposition 15.2. *The distributions (continuous linear functionals $C^\infty(\mathbb{R}) \to \mathbb{R}$) which have support at the origin are all given by finite linear combinations of the δ-function and its derivatives.*

Proposition 15.3. *Furthermore, the finite linear combinations of the δ_x for $x \in \mathbb{R}$ and their derivatives are dense in the space of distributions with compact support (with the dual topology on the space of distributions).*

This is just the statement that for $f : \mathbb{R} \to \mathbb{R}$, $\int \mu(x)\delta_x^{[k]}(f)dx = \int \mu(x) f^{[k]}(x)dx$ can be arbitrarily closely approximated by a finite sum $\sum_{i=1}^N \mu(x_i)f^{[k]}(x_i)(x_i - x_{i-1})$, since $\mu(x)$ is continuous with compact support. The dual topology is the one which makes all the evaluations: {distributions with compact support} $\to \mathbb{R}$ by $\varphi \to \varphi(f)$ continuous for each $f \in C^\infty(\mathbb{R} \to \mathbb{R})$.

16. Generalization to $\mathbb{R}^n$

Again we denote by $L\mathbb{R}^n$ the smooth vector fields on $\mathbb{R}^n$ with the C^∞ topology, and by $A_{pt}L\mathbb{R}^n = \oplus_{q=0}^{\infty} LA_{pt}^q\mathbb{R}^n$ the continuous alternating q-forms on $L\mathbb{R}^n$ with support at pt= origin of $\mathbb{R}^n$. Any $X \in L\mathbb{R}^n$ looks like $X = \sum_{i=1}^n f_i(x_i, \ldots, x_n)\partial/\partial x_i$. Instead of one δ function, we now have n of them:

Definition. $\delta^i(X) = f_i(0, \cdots, 0)$, for X as above, and for $i = 1, \ldots,$ Instead of ordinary derivatives we now have n partial derivatives:

Definition.

$$\begin{aligned}\delta_j^i &= \theta_{\partial/\partial x_j}(\delta^i) \\ \delta_{jk}^i &= \theta_{\partial/\partial x_k}(\theta_{\partial/\partial x_j}\delta^i) = \theta_{\partial/\partial x_k}\delta_j^i \\ \delta_{jkl}^i &= \theta_{\partial/\partial x_l}(\delta_{jk}^i)\end{aligned}$$

and so forth inductively, for $i, j, k, l, = 1, \ldots, n$.

Exercise. For $X \in L\mathbb{R}^1$ as above, $\delta_j^i(X) = -\partial f_i/\partial x_j$, $\delta_{jk}^i(X)$ $\partial^2 f_i/\partial x_j \partial x_k$, etc. Thus, $\delta_{j_1 \cdots j_r}^i$ is symmetric in its lower indices.

In fact, $A_{pt}L\mathbb{R}^n$ is the exterior algebra generated by the 1-forms $\{\delta^i, \delta$ $\delta_{jk}^i, \cdots\}$ in $A_{pt}^1 L\mathbb{R}^n$. The derivative d of this complex is computed as follow Let $X = \sum_{\alpha=1}^n (f_\alpha)\partial/\partial x_\alpha$ and $Y = \sum_{\beta=1}^n (g_\beta)\partial/\partial x_\beta$ be in $L\mathbb{R}^n$. Then

$$\begin{aligned}(d\delta^i)(X,Y) &= -\delta^i([X,Y]) \\ &= -\delta^i\left(\sum_{\alpha,\beta} f_\alpha(\partial g_\beta/\partial x_\alpha)\partial/\partial x_\beta - g_\beta(\partial f_\alpha/\partial x_\beta)\partial/\partial x_\alpha\right) \\ &= -\left(\sum_\alpha f_\alpha(\partial g_i/\partial x_\alpha) - \sum_\beta g_\beta(\partial f_i/\partial x_\beta)\right) \\ &= \sum_j (\delta^j \wedge \delta_j^i)(X,Y).\end{aligned}$$

Therefore, $d\delta^i = \sum_j (\delta^j \wedge \delta_j^i)$. We will often omit the summation sign wh summing over repeated indices.

$$(d\delta_k^i) = d\theta_{\partial/\partial x_k}\delta^i = \theta_{\partial/\partial x_k}(d\delta^i) = \theta_{\partial/\partial x_k}(\delta^j \wedge \delta_j^i) = \delta_k^j \wedge \delta_j^i + \delta^j \wedge \delta_{jk}^i,$$

since $\theta_{\partial/\partial x_k}$ is a derivation. d is defined inductively on $A_{pt}^1 L\mathbb{R}^n$, and exten to all of $A_{pt}^* L\mathbb{R}^n = \Lambda^* A_{pt}^1 L\mathbb{R}^n$ since d is an anti derivation. For the geometer notice that $d\delta_k^i - \delta_k^i \wedge \delta_j^i$ looks like a curvature formula.

Question. Is $H^*(A_{pt}L\mathbb{R}^n)$ finite dimensional? Consider the radial vect field $R = \sum_i x_i(\partial/\partial x_i)$. Then just as in the $n = 1$ case we have $\theta_R \delta^i$ $\delta^i, \theta_R\delta_j = 0$, $\theta_R \delta_{jk}^i = -\delta_{jk}^i, \theta_R \delta_{j_1 \cdots j_r}^i = (1-r)\delta_{j_1 \cdots j_r}^i$. Then on $A_{pt}^* L\mathbb{R}^n$, acts with eigenvalues $n, n-1, \ldots, 0, -1, \ldots$. For example, $\delta^1 \wedge \delta^2 \wedge \cdots \wedge$ has weight n. Each eigenspace A_λ of $A_{pt}L\mathbb{R}^n$ is finite dimensional and stat under d. $H^*(A_{pt}L\mathbb{R}^n) = H^*(\oplus_{\lambda=-\infty}^n A_\lambda) = \oplus_\lambda H^*(A_\lambda) = H^*(A_\circ)$ is fini dimensional since θ_R is multiplication by λ on A_λ and $\theta_R = \iota_R d + d\iota_R$ homotopic to 0. For $\lambda \neq 0$, therefore, $H^*(A_\lambda) = 0$. Therefore,

Proposition 16.1. *$H^*(A_{pt}L\mathbb{R}^n)$ is finite dimensional.*

17. Digression: Tautological forms

Consider the frame bundle $F(M) \xrightarrow{\pi} M$ with fiber $GL(n,\mathbb{R})$ and structure group also $GL(n,\mathbb{R})$. This is the principal bundle associated to the tangent bundle of a manifold M. A point of $F(M)$ looks like $(x; v_1, \ldots, v_n)$ where $x \in M$ and $v_1, \ldots, v_n$ form a basis for the tangent space T_xM at x. π is projection to $x \in M$.

There are some canonical 1-forms $\omega^1, \ldots, \omega^n$ on $F(M)$ defined as follows: Let X be a tangent vector at $(x; v_1, \ldots, v_n) \in F(M)$. Then $\pi_* X$ is a tangent vector at $x \in M$ and can be written in the basis $v_1, \ldots, v_n$. Then $\omega^i(X)$ is defined by:

$$\pi_*(X) = \omega^1(X)v^1 + \cdots + \omega^n(X)v^n.$$

Second approach. Consider the charts $f : \mathbb{R}^n \to M$. For $k = 0, 1, 2, \cdots$ we can identify two such maps f and g whenever

$$f(0) = g(0), \frac{\partial f}{\partial x_i}\Big|_0 = \frac{\partial g}{\partial x_i}\Big|_0, \ldots, \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big|_0 = \frac{\partial^k g}{\partial x_{i_1} \cdots \partial x_{i_k}}\Big|_0$$

where the requirements are well-defined via any chart on M. The equivalence class of f is called the *k-jet* of f, $j^k f$. We denote by $F_k(M)$ the collection of k-jects of charts $f : \mathbb{R}^n \to M$.

The set, $F_0(M)$, of 0-jets is just M via the target $= f(0)$. The set, $F_1(M)$, of 1-jets is just the frame bundle $F(M)$. We get a tower of fiber bundles:

$$\begin{array}{c} \vdots \\ \downarrow \\ F_3(M) \\ \downarrow \\ F_2(M) \\ \downarrow \\ F_1(M) = F(M) \\ \downarrow \pi \\ M = F_0(M) \end{array}$$

On $F_2(M)$ we have the tautologous forms ω^i which pull back from $F_1(M)$. But we also have others: $\omega^i_j = j$-th partial of ω^i. This construction may be a little clearer in coordinates. Locally a map $g : \mathbb{R}^n \to M$ looks like $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(x_1, \ldots, x_n) \to (g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n))$$

Then

$$j^0 g = (g_1(0), \ldots, g_n(0)) = \text{target}$$

$$j^1 g = \left(g_i(0); \frac{\partial g_i}{\partial x_j}\Big|_0\right)_{i,j}$$

$$j^2 g = \left(g_i(0); \frac{\partial g_i}{\partial x_j}\Big|_0; \frac{\partial^2 gi}{\partial x_j \partial x_k}\Big|_0\right)_{i,j,k} \quad \text{etc.}$$

Let $f : \mathbb{R}^n \to M$ be a chart. Let X be a tangent vector to $j^1 f \in F_1(M$
X is represented by a curve $t \to j^1 f_t$ with $f_0 = f$, and f_t a family of chart
Differentiating formally,

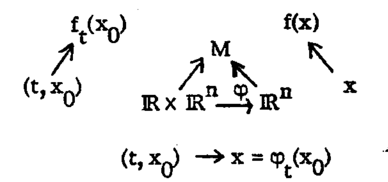
$$\frac{\partial f_t}{\partial t}\Big|_{t=0} = \frac{\partial x_i}{\partial t} \cdot \frac{\partial f_0}{\partial x_i}.$$

Then $\omega^i(X) \equiv \partial x^i/\partial t = i$-th component of $\pi_* X$ (which is represent ed by the curve $t \to f_t(0)$) when we are at the frame $(f_1(0), \ldots$ $f_n(0), \partial f(0)/\partial x_1, \ldots, \partial f(0)/\partial x_n)$.

On $F_2(M)$, f_t has well-defined derivatives of order 2, so that $\partial x_i/\partial t$ ha 1st order partials. We define ω^i_j on $F_2(M)$:

$$\omega^i_j(X) = -\frac{\partial}{\partial x_j}\left(\frac{\partial x^i}{\partial t}\right).$$

On $F_3(M)$, $\omega^i_{jk}(X) = \partial^2(\partial x^i/\partial t)/(\partial x_j \partial x_k)$ is defined, and so forth. The reade may have noticed that we have differentiated $\partial f_t/\partial t = \sum(\partial x_i/\partial t)(\partial f/\partial x$ formally, and $(\partial x_i/\partial t)$ does not really make sense until we interpret x as function of t. However, $f_t(x_0) = f(x(t, x_0))$ will hold if we set $x(t, x_0)$: $f^{-1} \circ f_t(x_0)$, a curve through each $x_0 \in \mathbb{R}^n$ in a neighborhood of the origi $x(0, x_0) = x_0$ since $f_0 = f$. Then, the expressions $\partial x_i/\partial t$, $(\partial/\partial(x_0)_j)(\partial x_i/\partial t$ etc. make sense. We have a commutative diagram:



where φ is given by $\varphi_t = f^{-1} \circ f_t : \mathbb{R}^n \to \mathbb{R}^n$. $t \to \varphi_t$ is a curve through th identity of Diff $(\mathbb{R}^n)$ whose derivative (at $t = 0$) is a vector field

$$Y = (\partial x_1/\partial t)_{t=0}\, \partial/\partial x_1 + \cdots + (\partial x_n/\partial t)_{t=0}\, \partial/\partial x_n$$

on $\mathbb{R}^n$ written in our coordinates $x_1, \ldots, x_n$ of $\mathbb{R}^n$; $x(t, x_0) = \varphi_t(x_0) = f^{-1}$ $f_t(x_0)$.

If we scrap the abusive (but common) notations for partial derivatives, w can define our tautological forms more rigorously. For our tangent vector X c $F_k(M)$ represented by $t \to j^k f_t$, $\varphi_t = f^{-1} \circ f_t$ is defined up to its k-jet at th origin. After differentiating, Y is defined only up to its $k-1$ jet at $0 \in \mathbb{R}$ Thus, $\delta^i(Y), \delta^i_j(Y), \ldots, \delta_{j_1 \cdots j_{k-1}}(Y)$ are defined (where $\delta^i, \delta^i_j, \ldots \in A^1_{pt} L\mathbb{R}^n$ before). For example, $\delta^i(Y) = \delta^i(\sum_j (\partial x_j/\partial t)\partial/\partial x_j) = \partial x_i/\partial t$.

Definition. $\omega^i(X) \equiv \delta^i(Y), \omega^i_j(X) \equiv \delta^i_j(Y), \ldots$, so that $\omega^i, \omega^i_j, \omega^i_{j_1, \cdots, j_k-}$ are well-defined one-forms on $F_k(M)$.

Thus we have a natural map $\delta^i \to \omega^i, \delta^i_j \to \omega^i_j, \delta^i_{jk} \to \omega^i_{jk}, \ldots$ of $A^*_{pt}L\mathbb{R}^n$-tautological forms on $F_\infty(M)$, where $F_\infty(M) = \varprojlim_k F_k(M)$, and $A^*F_\infty(M) = \varinjlim_k A^*F_k(M)$ (here A^* means "differential forms on").

18. Sections of the frame tower and characteristic classes

The fibers of $F_{k+1}(M) \to F_k(M)$ are all affine spaces for $k \geq 1$ (since invertibility is only a condition on the 1-jet or first derivatives). So a partition of unity gives us sections $F_k(M) \to F_{k+1}(M)$ unique up to homotopy. Since the fiber of $F_1(M) \to M$ is $GL_{(n,\mathbb{R})}$, which is not contractible, we do not in general get a section $\partial : M \to F_1(M)$. When we do have one, M is called parallelizable, for a section of the frame bundle is just a trivialization of the tangent bundle of M. Note that the section need not be unique, even up to homotopy.

We obtain characteristic classes in $H^*(F_1(M))$ by the composition:

$$H^*(A_{pt}) \to H^*(F_\infty(M)) \xrightarrow{\partial^*} H^*(F_1(M))$$

However, only when M has a trivial tangent bundle can these classes be pulled down to $H^*(M)$ where we want them. Unless we modify our approach we will not get very interesting classes on M.

Remark. $H^*(A^*_{pt}L\mathbb{R}^n)$ can be described slightly differently. Let $\mathfrak{a}_n$ be the Lie algebra of formal-power-series vector fields on $\mathbb{R}^n$. Any $X \in \mathfrak{a}_n$ is of the form $X = \sum f_i \frac{\partial}{\partial x_i}$ where $f_i \in \mathbb{R}[[x_1, \ldots, x_n]]$ is a formal power series in $x_1, \ldots, x_n$. (No convergence is required.) Under $[\,,\,]$ $\mathfrak{a}_n$ is a topological Lie algebra. The topology is defined by stipulating, for instance, that $x_1^{200}\frac{\partial}{\partial x_1}$ be small. More formally, $\mathbb{R}[[x_1, \ldots, x_n]]$ has a topology relative to the ideal $m = (x_1, \ldots, x_n)$. The basic neighborhoods of 0 are the powers of m.

Exercise. The algebra of continuous forms, $A^*_c(\mathfrak{a}_n)$, on $\mathfrak{a}_n$ is just our friend $A^*_{pt}L\mathbb{R}^n$.

An $\mathbb{R}$-vector space basis for $\mathfrak{a}_n$ is $\{\partial/\partial x_i, (x_i)\partial/\partial x_j, (x_i, x_j)\partial/\partial x_k, \ldots\}_{i,j,\ldots}$ Note that $gl(n, \mathbb{R})$ sits in $\mathfrak{a}_n$ generated by the $\{(x_i)\partial/\partial x_j\}_{i,j}$ part.

Exercise. Verify that the map of a matrix $(\alpha^j_i) \in gl_{n,\mathbb{R}}$ to $\sum \alpha^j_i(x_i\partial/\partial x_j) \in \mathfrak{a}_n$ is a Lie algebra isomorphism onto the $\{(x_i)\partial/\partial x_j\}_{i,j}$ subspace. Furthermore, the skew symmetric matrices $so(n, \mathbb{R}) \subseteq gl(n, \mathbb{R})$ are mapped onto the subspace generated by $\{(x_i)\partial/\partial x_j - (x_j)\partial/\partial x_i\}_{i,j}$.

We can, therefore, speak of the relative cohomology $H^*(\mathfrak{a}_n, so(n))$ as the cohomology of $A^0_c(\mathfrak{a}_n, so_n) \xrightarrow{d} A^1_i(\mathfrak{a}_n, so_n) \xrightarrow{d} \cdots$ where

$$A^q_c(\mathfrak{a}_n, so(n)) = \{\varphi \in A^q_c(\mathfrak{a}_n) | \iota_x\varphi = 0 = \theta_X\varphi \quad \text{for all} \quad X \in so_n\}.$$

Then we have a natural map:

$$H^*(\mathfrak{a}_n, so(n)) \longrightarrow H^*(F_\infty(M)/SO(n))$$

where $F_\infty(M)/SO(n)$ is obtained from $F_\infty(M)$ by identifying points which a on the same orbit of the right action of $SO(n)$. (If $f : \mathbb{R}^n \to M$ represen a point in $F_\infty(M)$ and if $\alpha \in SO(n)$ then $f \circ \alpha : \mathbb{R}^n \to M$ represents a other point of $F_\infty(M)$.) Note that the forms on $F_\infty(M)/SO(n)$ correspon to the differential forms on $F_\infty(M)$ which are zero when applied to vecto tangent to the fiber $\simeq SO(n)$ of $F_\infty(M) \to SO(n)$, and are also invaria under the right action of $SO(n)$. Since $SO(n)$ is connected, invariance (b ing fixed) by $SO(n)$ is equivalent to invariance (being killed) by $so(n)$. A least when M is oriented, $F_\infty(M)/S(n) \to M$ has a section, and up to h motopy, there are exactly two sections, one for each orientation. This is b cause we can reduce the structure group of $F_1(M) \to M$ to be $GL(n, \mathbb{R})$ = matrices with positive determinant. Then the fiber of $F_1(M) \to M$ $GL(n, \mathbb{R})^+/SO(n)$ is contractible because $SO(n)$ is a maximal compact su group of $GL(n, \mathbb{R})^+$. We get characteristic classes on M by the compositi $H^*(\mathfrak{a}_n, so_n) \to H^*(F_\infty(M)/SO(n)) \xrightarrow{\partial^*} H^*(M)$.

Remark. If M is not orientable, then we must replace $SO(n)$ with $O(n$ in the above discussion. $F_1(M)/O(n) \to M$ always has a section since fiber i $GL(n, \mathbb{R})/O(n)$. This is contractible since $O(n)$ is a maximal compact subgrou of $GL(n, \mathbb{R})$. $H^*(\mathfrak{a}_n, O(n))$ is defined as the cohomology of

$$A^0_c(\mathfrak{a}_n, O(n)) \xrightarrow{d} A^1_c(\mathfrak{a}_n, O(n)) \xrightarrow{d} \cdots.$$

Think of $\mathfrak{a}_n$ as the Lie algebra of the pseudogroup G of infinite jets of map $\mathbb{R}^n \to \mathbb{R}^n$. Then $A^q_c(\mathfrak{a}_n)$ corresponds to left invariant (continuous) differenti q-forms on G. The subalgebra, $so(n)$ of $\mathfrak{a}_n$ corresponds to the subgroup $SO(n$ or $O(n)$ of G. (See Haefliger [36] for details.) Then $A^q_c(\mathfrak{a}_n, O(n)) = \{$lef invariant q-forms on $G/O(n)\} \simeq$ left invariant q-forms on G which are zero i the $O(n)$ directions and invariant under the right actions of $O(n)$ on G. W define $A^q_c(\mathfrak{a}_n, SO(n))$ similarly. Since $SO(n)$ is connected, $A^q_c(\mathfrak{a}_n, SO(n)) =$ $A^q_c(\mathfrak{a}_n, so(n))$ as defined earlier. In fact, we will see that $H^*(\mathfrak{a}_n, so_n)$ contain the Pontryagin classes up to dimension n and the Euler class (if n is even). Bu there are also more classes in $H^*(\mathfrak{a}_n, so_n)$ above dimension $n = \dim M$, whic must, therefore, go to 0 in $H^*(M)$. However, these will show up non-triviall for foliations. The same holds for $H^*(\mathfrak{a}_n, O_{(n)})$ except that there is, of cours no Euler class. These characteristic classes come to $H^*(M)$ entirely by the wa M is glued together by the charts $\mathbb{R}^n \to M$. We will return to this point late when we consider foliations of M.

19. Return to $H^*(A_{pt}L\mathbb{R}^n)$

Recall that $A_{pt}L\mathbb{R}^n = \oplus_{q=0}^{\infty} A_{pt}^q L\mathbb{R}^n$ = continuous alternating q-forms with support at the origin of $\mathbb{R}^n$, on the vector fields on $\mathbb{R}^n$. This is the same as the continuous q-forms on the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$. So we identify $A_{pt}(L\mathbb{R}^n)$ with $A_c^*(\mathfrak{a}_n)$ and $H^*(A_{pt}L\mathbb{R}^n)$ with $H^*(\mathfrak{a}_n)$. Let $V = \mathbb{R}^n$. A basis for the dual, V^*, is the coordinate functions $\{x_1, \dots, x_n\}$. Identifying V with its tangent space at the origin, $\{\partial/\partial x_1, \dots, \partial/\partial x_n\}$ is a basis for V. Then $\mathfrak{a}_n$ is the direct product

$$\mathfrak{a}_n = \prod_{p=-1}^{\infty} S^{p+1}(V^*) \otimes V,$$

where $S^{p+1}(V^*)$ = polynomials of degree $p+1$ in $x_1, \dots, x_n$, and the direct product (instead of sum) reflects the formal power series in $\mathfrak{a}_n$. $\mathfrak{a}_n$ is the infinite jets (inverse limit of k-jets) at the origin of vector fields on $\mathbb{R}^n$. Note that $gl(n,\mathbb{R})$ is a subalgebra of $\mathfrak{a}_n$ (as the $p = 0$ factor, $V^* \otimes V$). Thus, we regard $\mathfrak{a}_n$ as the $gl(n,\mathbb{R})$-module.

Let $R = \sum (x_i)\partial/\partial x_i$, the radial vector field, which generates the center, Z, of $gl(n,\mathbb{R})$. $\mathfrak{a}_n$ breaks up under the action of R into (finite dimensional) eigenspaces $S^{p+1}(V^*) \otimes V$ with eigenvalue, or *weight*, p. For example, for $\partial/\partial x_1 \in S^{-1+1}(V^*) \otimes V = V, [R, \partial/\partial x_1] = [\sum(x_i)\partial/\partial x_i, \partial/\partial x_1] = -\partial/\partial x_1$ with eigenvalue $p = -1$. Since $R \in Z$, each eigenspace is preserved by $gl(n,\mathbb{R})$. Since $gl(n,\mathbb{R}) = sl(n,\mathbb{R}) \oplus Z$, where $sl(n,\mathbb{R})$ is semi-simple, we see that $\mathfrak{a}_n$ is completely reducible as $gl(n,\mathbb{R})$-module. Therefore, $gl(n,\mathbb{R})$ is called a *reductive subalgebra* of $\mathfrak{a}_n$. (See Appendix. See also [39].) We use the Hochschild–Serre spectral sequence relative to a reductive subalgebra. The spectral sequence converges to $H^*(\mathfrak{a}_n)$.

Consider the 1-forms δ_j^i the 2-forms Ω_j^i defined by

$$\Omega_j^i = d\delta_j^i - \sum_k \delta_j^k \wedge \delta_k^i = \delta^k \wedge \delta_{jk}^i.$$

(Recall that $\delta^i, \delta_j^i, \delta_{jk}^i, \dots \in A^1(\mathfrak{a}_n)$ are dual to $\partial/\partial x_i, -(x_j)\partial/\partial x_i, (x_j x_k)\partial/\partial_{x_i, \dots} \in \mathfrak{a}_n$.) Let $\underline{W} \subseteq A_{pt}^* L\mathbb{R}^n$ be the subalgebra they generate. Then $d(\underline{W}) \subseteq \underline{W}$ and the inclusion $\underline{W} \to A_{pt}^* L\mathbb{R}^n$ induces the isomorphism in cohomology:

Theorem 19.1.

$$H^*(A_{pt}L\mathbb{R}^n) \simeq H^*(W).$$

Proof. Denote the continuous $\mathbb{R}$-valued q-forms on $\mathfrak{a}_n$ by $C^q(\mathfrak{a}_n;\mathbb{R}) = C^q(\mathfrak{a}_n)$. $C^*(\mathfrak{a}_n;\mathbb{R}) = C^*(\mathfrak{a}_n) = \oplus C^q(\mathfrak{a}_n)$. We filter $C^*(\mathfrak{a}_n)$ by the subspaces:

$$A^p = \oplus_{q=-\infty}^{\infty} A^{p,q}$$

where

$$A^{p,q} = \begin{cases} 0 \quad if \quad q<0 \\ C^{p+q}(\mathfrak{a}_n) \text{ if } p \le 0 \\ \{\omega \in C^{p+q}(\mathfrak{a}_n) | \omega(X_1, \dots, X_{p+q}) = 0 \\ \text{whenever at least } \ q+1 \text{ of the} \\ X_i \ \text{ are in } \ gl(n,\mathbb{R}) \end{cases}$$

The corresponding spectral sequence begins with

$$E_0^{p,q} = C^q(gl(n,\mathbb{R}); A^{p,0})$$
$$E_1^{p,q} = H^q(gl(n,\mathbb{R}); A^{p,0})$$

where $A^{p,0}$ is the coefficient module. (See appendix for definition of cohomolog of a Lie algebra with coefficients in a module.) Since $A^{p,0}$ are the p-cochair which vanish when any one of their arguments is in the subalgebra $gl(n,\mathbb{R}$ $A^{p,0} = C^p(\mathfrak{a}_n, gl(n,\mathbb{R}); \mathbb{R}) \equiv$ cochains "relative to $gl(n,\mathbb{R})$."

For convenience, we define:

$$g^{(p)} = S^{p+1}V^* \otimes V \subseteq \mathfrak{a}_n$$
$$g_{(p)} = g^{(p)*} = S^{p+1}V \otimes V^* \subseteq C^1(\mathfrak{a}_n)$$

for $p = -1, 0, 1, \dots$. The radial vector field $R \in gl(n,\mathbb{R})$ acts on $g^{(p)}$ and $g_($ with eigenvalue p and $-p$ respectively. $g^{(0)} = gl(n,\mathbb{R})$.

$$C^1(\mathfrak{a}_n) = \oplus_{p=-1}^{\infty} g_{(p)}$$
$$C^p(\mathfrak{a}_n) = \Lambda^p C^1(\mathfrak{a}_n) \quad \text{is a direct sum of subspaces}$$
$$= \oplus_{p_{-1}+p_0+\cdots=p} \Lambda^{p-1} g_{(-1)} \otimes \Lambda^{p_0} g_{(0)} \otimes \Lambda^{p_1} g_{(1)} \otimes \cdots.$$

The *weight* (eigenvalue for R) of each of these subspaces is $p_{-1} - p_1 - 2p_2 - 3p_3 \cdots$. $A^{p,0}$ is the subspace of $C^p(\mathfrak{a}_n)$ for which $p_0 = 0$, i.e., no factor of $g_{(0)}$ $gl(n,\mathbb{R})^*$. Since the coefficient module, $A^{p,0}$, of $E_1^{p,q} = H^q(gl(n,\mathbb{R}); A^{p,}$ breaks up into eigenspaces for R, we can replace the coefficient module in tl E_1 term by the submodule of $gl(n,\mathbb{R})$-invariants. (See Appendix. Also [39].)

$$E_1^{p,q} \simeq H^q(gl(n,\mathbb{R}),\ C^p(\mathfrak{a}_n, gl(n,\mathbb{R})))$$
$$= H^q(gl(n,\mathbb{R}),\ C^p(\mathfrak{a}_n, gl(n,\mathbb{R}))^{gl(n,\mathbb{R})}).$$

$C^p(\mathfrak{a}_n, gl(n,\mathbb{R}))^{gl(n,\mathbb{R})} \equiv \{\omega \in C^p(\mathfrak{a}_n, gl(n,\mathbb{R})) | X \cdot \omega = 0$ for all $x + gl(n,\mathbb{R}$ is computed from scratch in Godbillon [30]. However, we will apply a theore: on the invariant theory of $gl(n,\mathbb{R})$. (See Atiyah et al. [2]; Weyl [72]). What a: the invariants of the $gl(n,\mathbb{R})$-module,

$$\underbrace{V \otimes \cdots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ times}}?$$

(See Appendix for definition of the $gl(n,\mathbb{R})$ action on V^* and tensor products There are none (except 0) if $r \neq s$ (apply R = identity matrix). So suppo:

$r = s$. Let $\{v_i\}$ be a basis for V and $\{v^i\}$, the dual basis for V^*. Then the invariants are spanned as a vector space by

$$\sum_{i_1,\cdots,i_r}^{n} v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{i_{\theta(1)}} \otimes \cdots \otimes v^{i_{\theta(1)}} \otimes \cdots \otimes v^{i_{\theta(r)}},$$

for each permutation, θ, of $1, \ldots, r$.

Exercise. Show that the elements $\sum_{i,j} v_i \otimes v_j \otimes v^i \otimes v^j$ and $\sum_{i,j} v_i \otimes v_j \otimes v^j \otimes v^i$ of $V \otimes V \otimes V^* \otimes V^*$ are invariant (i.e., killed) under the action of $gl(n, \mathbb{R})$.

Recall that

$$A^{p,0} = c^p(a_n, gl(n,\mathbb{R})) = \oplus_{p_{-1}+p_1+p_2+\cdots=p} \Lambda^{p-1} V^* \otimes \Lambda^{p_1}(S^2 V \otimes V^*) \otimes \Lambda^{p_2}(S^3 V \otimes V^*) \otimes \cdots$$

where no factor appears from $\Lambda^* S^1 V \otimes V^* = C^*(gl(n,\mathbb{R})) \subseteq C^*(a_n)$. Each summand of $A^{p,0}$ is a sub $gl(n,\mathbb{R})$-module of

$$\underbrace{V \otimes \cdots \otimes V}_{(2p_1+3p_2+\cdots \text{ factors})} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{(p_{-1}+p_1+p_2+\cdots \text{ factors})}.$$

(Verify that the action via θ of $gl(n,\mathbb{R}) \subseteq a_n$ on $A^{P,0} \subseteq C^p(a_n)$ is the same as the usual action of $gl(n,\mathbb{R})$ on $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$.) In particular, for there to be any invariants under the action of $gl(n,\mathbb{R})$ we must have

$$2p_1 + 3p_2 + \cdots = p_{-1} + p_1 + p_2 + \cdots .$$

This is just a restatement of the requirement that the weight $= p_{-1} - p_1 - 2p_2 - \cdots$ be zero. We get only invariants of the following form (where $\{\delta^i\}$ is a basis of V^*, $\{\delta^i_{jk}\}$ is a basis of $S^2V \otimes V^* \subseteq V \otimes V \otimes V^*$, etc. and all indices must be in pairs, one upper and one lower for summation):

$$\delta^a \wedge \delta^i_{ai} = \Omega^i_i = \text{trace}\,(\Omega)$$
$$\delta^a \wedge \delta^i_{ai} \wedge \delta^b \wedge \delta^j_{bj} = \Omega^i_i \Omega^j_j = (tr(\Omega))^2$$
$$\delta^a \wedge \delta^i_{aj} \wedge \delta^b \wedge \delta^j_{bi} = \Omega^i_j \Omega^j_i = tr(\Omega^2)$$

etc. (Summation over repeated indices)

We get only the (invariant) polynomials in the $\Omega^i_j = \delta^a \wedge \delta^i_{aj}$. In fact, we recognize the traces of the powers of the matrix Ω, whose entries are Ω^i_j. To prove that these are the *only* invariants, we play off the symmetry in the lower indices of $\delta^i_{jk}, \delta^i_{jkl}, \ldots$ against the anti-symmetry of the wedge product, $\wedge$. For example interchanging the dummy variables a and b in the following possible invariant:

$$\begin{aligned}\delta^a \wedge \delta^b \wedge \delta^i_{ab} \wedge \delta^j_{ij} &= (\delta^b \wedge \delta^a) \wedge \delta^i_{ba} \wedge \delta^j_{ij} \\ &= -(\delta^a \wedge \delta^b) \wedge \delta^i_{ba} \wedge \delta^j_{ij} \\ &= -\delta^a \wedge \delta^b \wedge \delta^i_{ab} \wedge \delta^j_{ij}\end{aligned}$$

So both sides are 0. Similarly, (plus the pigeon-hole principle) we find th can be no factors of δ^i_{jkl}, etc. We are only left with polynomials in the Ω^i_j. ' speak of polynomials since $\{\Omega^i_j\}$ have degree 2, and, therefore, commute.

$\underline{W}$ is generated by the δ^i_j's in $C^*(gl(n,\mathbb{R}))$ and the Ω^i_j's in $C^*(a_n, gl(n, \mathbb{R}$ The filtration on $C^*(a_n)$ induces one on W with E_0 term:

$$E_0^{p,q} = C^q(gl(n,\mathbb{R}), P^{p/2}(\Omega^i_j)),$$

where $P^{p/2}(\Omega^i_j) = 0$ if p is odd. If p is even, $P^{p/2}(\Omega^i_j)$ is the homogenec polynomials in the Ω^i_j's of degree $p/2$ as a polynomial (degree p as a form $C^*(a_n)$). $P^{p/2}(\Omega^i_j) \subseteq C^p(a_n, gl(n,\mathbb{R}))$. The E_1 term is

$$\begin{aligned}E_1^{p,q} &= H^q(gl(n,\mathbb{R}),\ P^{p/2}(\Omega^i_j)) \\ &= H^q(gl(n,\mathbb{R}),\ P^{p/2}(\Omega^i_j)^{gl(n,\mathbb{R})}).\end{aligned}$$

Since $C^p(a_{n,gl(n,\mathbb{R})})^{gl(n,\mathbb{R})} = P^{p/2}(\Omega^i_j)^{gl(n,\mathbb{R})}$, the inclusion $W \to C^*(a)$ induc isomorphism on the E_1 terms (commuting with the succeeding differenti since the filtrations are the same) and, therefore, on the E_∞ terms. Therefo

$$H^*(W) \simeq H^*(a_n).$$

Q.E.

Corollary 19.1.1. *$H^q(a_n)$ is finite dimensional and is 0 for $q > 2n + r$*

Proof. There are only n^2 of the δ^i_j and at most n of the $\Omega^i_j = \delta^a \wedge \delta^i_{aj}$ c be multiplied together without getting 0. Each Ω^i_j has dimension 2.

Corollary 19.1.2. *Every class in $H^*(a_n)$ contains a representative cocy which depends only on the 2-jets of its arguments. That is, we never need u representative forms like $\delta^i_{jkl}, \delta^i_{jklm}, \cdots$ in cohomology.*

Exercise. Let δ and Ω be the matrices whose (i,j)-th entries are the forms δ^i_j and the 2-forms Ω^i_j, respectively. Then $d\delta = \Omega - \delta \wedge \delta$, and $d\Omega$ $\sum_{i,j} X^i_j(x^j)\partial/\partial x^i \in gl(n,\mathbb{R}) \subseteq a_n$, let X be the corresponding matrix, $(X^i_j$ of scalars. Then $\iota_{\tilde{X}}\delta = -X$, $\iota_{\tilde{X}}\Omega = 0$, $\theta_{\tilde{X}}\delta = -[X,\delta]$, and $\theta_{\tilde{X}}\Omega = -[X,\Omega]$.

Let W be the free (anti-) commutative algebra generated by the symbc $\{\delta^i_j\}$, $\{\Omega^i_j\}$ of dimensions 1 and 2 respectively, $i,j = 1,2,\ldots,n$. Define d $W \to W$ by

$$d\delta^i_j = \Omega^i_j - \delta^i_k \wedge \delta^k_j.$$

Then in order that $d^2 = 0$ we must define:

$$\begin{aligned}d\Omega^i_j = d(\delta^i_k \wedge \delta^k_j) &= (d\delta^i_k) \wedge \delta^k_j - \delta^i_k \wedge (d\delta^k_j) \\ &= (\Omega^i_k - \delta^i_1 \wedge \delta^1_k) \wedge \delta^k_j - \delta^i_k \wedge (\Omega^k_j - \delta^k \wedge \delta^1_j) \\ &= \Omega^i_k \wedge \delta^k_j - \delta^i_k \wedge \Omega^k_j.\end{aligned}$$

W is called the *Weil algebra* of $gl_{(n,\mathbb{R})}$. In general, a Lie algebra g has Weil algebra:

$$W(g) = \Lambda^* g^* \otimes S^* g^*$$

where $\Lambda^* g^*$ (resp. $S^* g^*$) is the alternating (resp. symmetric) tensor algebra of the dual g^* to g. The degree of an element in $\Lambda^p g^* \otimes S^q g^*$ is taken to be $p+2q$. In our case, $\{\Omega^i_j\}$ generate $S^1 g^*$ (degree $=2 \Rightarrow$ they commute), and $\{\delta^i_j\}$ generate $\Lambda^1 g^*$ (degree $=1 \Rightarrow$ they anti-commute).

Proposition 19.2.

$$H^*(W) = \begin{cases} \mathbb{R} & \text{in} \quad \dim = 0 \\ & \text{is acyclic.} \\ 0 & \text{in} \quad \dim \geq 1. \end{cases}$$

Proof. Consider the antiderivation k of degree -1 defined by $k(\delta^i_j) = 0$, $k(\Omega^i_j) = \delta^i_j$. Then

$$\begin{aligned} (dk+kd)(\delta^i_j) &= k(\Omega^i_j - \delta^i_k \wedge \delta^k_j) = \delta^i_j \\ (dk+kd)(\Omega^i_j) &= d(\delta^i_j) + k(\Omega^i_k \wedge \delta^k_j - \delta^i_k \wedge \Omega^k_j) \\ &= (\Omega^i_j - \delta^i_k \wedge \delta^k_j) + (\delta^i_k \wedge \delta^k_j - (-\delta^i_k \wedge \delta^k_j)) \\ &= \Omega^i_j + \delta^i_k \wedge \delta^k_j. \end{aligned}$$

We say for now that an element $\varphi \in W$ has weight q if $\varphi \in \oplus^q_{r=0} \Lambda^*(gl_n)^* \otimes S^r(gl^*_n)$. That is, φ is a polynomial in the $\{\Omega^i_j\}$ (with coefficients in the exterior algebra generated by the $\{\delta^i_j\}$) of degree $\leq q$. The weight of the element $0 \in W$ is considered to be -1.

Now suppose φ is homogeneous of degree $m > 0$ (recall that δ^i_j has degree 1, and Ω^i_j has degree 2). Then $(kd+dk)\varphi$ is still homogeneous of degree m. $(kd+dk)$ is a derivation of degree 0. Check that if φ has weight $q \geq 0$ then:

$$\varphi_1 = \varphi - \frac{1}{m-q}(\delta k + k\delta)\varphi \quad \text{has weight} \quad \leq q-1.$$

If φ were a cocycle then so is φ_1, and φ and φ_1 differ by a coboundary. We continue inductively

$$\varphi_2 = \varphi_1 - \frac{1}{m-(q-1)}(\delta k - k\delta)\varphi_1 \quad \text{has weight} \quad \leq q-2.$$

Finally, φ_q has weight 0 and $\varphi_{q+1} = 0$. If φ is a cocycle, then φ differs from $\varphi_{q+1} = 0$ by a coboundary. (Cf. H. Cartan [12] p. 58 for the generalization to $W(g)$ for any Lie algebra g.)

We have a homomorphism (commuting with d) $W \to \underline{W}$ (onto). What is the kernel? Since $\Omega^i_j \to d\delta^i_j - \delta^k_j \wedge \delta^i_k = \delta^k \wedge \delta^i_{jk}$, if we multiply more than n of the $\{\Omega^i_j\}$ together we get 0. This is the only added relation. The kernel of $W \to \underline{W}$ is the ideal in W generated by all polynomials in the $\{\Omega^i_j\}$ of degree

$> 2n$, where we count each Ω^i_j as degree 2. Therefore, we truncate W t $\underline{W}$ in this way.

$$\begin{aligned} \underline{W} &= W/(\text{polynomials of degree} > 2n \quad \text{in} \, \{\Omega^i_j\}) \\ &= \oplus^n_{q=0}(\Lambda^* gl(n,\mathbb{R})^* \otimes s^q gl(n,\mathbb{R})^*). \end{aligned}$$

In the spectral sequence converging to $H^*(\underline{W})$ we had $E^{p,q}_1 = 0$ if p is or if $p > 2n$. Otherwise:

$$\begin{aligned} E^{2r,q}_1 &\simeq H^q(gl(n,\mathbb{R}), (S^r gl(n,\mathbb{R})^*)^{gl(n,\mathbb{R})}) \\ &\simeq H^q(gl(n,\mathbb{R})) \otimes (S^r gl(n,\mathbb{R})^*)^{gl(n,\mathbb{R})}. \end{aligned}$$

Since $E^{p,q}_1 = 0$ if p is odd, all the d_1 maps are 0 and $E^{p,q}_2 = E^{p,q}_1$. Now invariant polynomials on $gl(n,\mathbb{R})$,

$$(S^* gl(n,\mathbb{R})^*)^{gl(n,\mathbb{R})} = P(c_1,\ldots,c_n)$$

is a polynomial algebra generated by $c_1, c_2, \ldots, c_n$ where $c_i \in S^i gl(n,\mathbb{R}$ $\pm$ the coefficient of λ^{n-i} in the characteristic polynomial $\det|\lambda I - \Omega|$ o $n \times n$ matrix $\Omega = (\Omega^k_j)$ of 2-forms. (See Bott [6].) For instance $c_1(\Omega) =$ $tr(\Omega)$, $c_n(\Omega) = \pm\det(\Omega)$. After normalizing c_i by some constant factor, it be the usual i-th Chern class. This will become clearer when we interpret as a connection matrix and (Ω^k_j) as the corresponding curvature matrix. Bott [6] for definition of Chern classes via a curvature matrix.) The E_2 = term is, thus:

$$\begin{aligned} E_2 &= H^*(gl(n,\mathbb{R})) \otimes \underline{P}(c_1,\ldots,c_n) \\ E_2 &= E(h_1,h_2,h_3,\ldots,h_n) \otimes \underline{P}(c_1,\ldots,c_n) \end{aligned}$$

where $E(h_1,h_2,\ldots,h_n)$ is the exterior algebra in the h_i of degree $2i-1$, $P(c_1,\ldots,c_n)$ is the polynomial algebra in the c_i, degree $> 2n$ (counting c_i as degree $2i$).

Homework. Use the Hochschild–Serre spectral sequence [39] relativ the reductive subalgebra $gl(n-1,\mathbb{R})$ of $gl(n,\mathbb{R})$ to show that $H^*(gl_{n,\mathbb{R}}$ $E(h_1,\ldots,h_n)$. Use induction on n and the theorem on invariants of $V \otimes \cdot$ $V \otimes V^* \otimes \cdots \otimes V^*$.

20. The unitary trick to compute $H^*(gl(n,\mathbb{R}))$.

We can also use the unitary trick by extending the base field to ([23] p. 337). Then $gl(n,\mathbb{C}) = gl(n,\mathbb{R}) \otimes \mathbb{C}$ has cohomology $H^*(gl(n,\mathbb{C})$ $H^*(gl(n,\mathbb{R})) \otimes \mathbb{C}$. But $gl(n,\mathbb{C}) \simeq u(n) \otimes_{\mathbb{R}} \mathbb{C}$, where $u(n) = \{X \in gl(n,\mathbb{C}) | X$ $-\overline{X}\}$ is the Lie algebra of the unitary group $U(n)$. Since $U(n)$ is compact connected, we have

$$H^*(u(n)) = H^*(U(n))$$

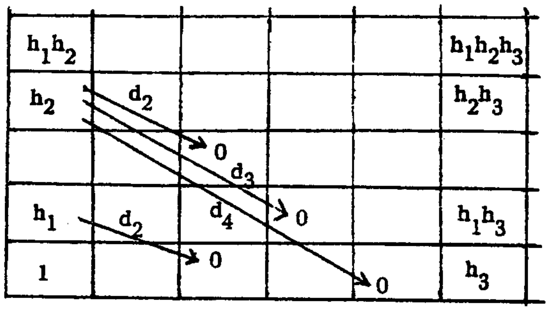
where $H^*(U(n))$ is the real cohomology of the topological space $U(n)$, which we now compute. $U(n)$ maps onto S^{2n-1} = unit ball in $\mathbb{C}^n$ with fiber $U(n-1)$:

$$\begin{array}{ccc} U(n-1) & \longrightarrow & U(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

Now the Serre spectral sequence for a fibration [58] (converging to $H^*(U(n))$) has E_2 term:

$$\begin{aligned} E_2^{p,q} &= H^p(S^{2n-1}, H^q(U(n-1))) \\ &= H^p(S^{2n-1}) \otimes H^q(U(n-1)) \end{aligned}$$

where, for $n \geq 2$, we have $\pi_1(S^{2n-1}) = 0$, so we do not need local coefficients. By an induction hypothesis $H^q(U(n-1)) = E(h_1, \ldots, h_{n-1})$. We sketch for $n = 3$. $H^*(U(2)) = E(h_1, h_2)$ and $H^*(S^{2n-1}) = E(h_3)$.



Since all the d_r maps are $0, E_2 = E_\infty$, and $H^*(gl(n,\mathbb{R})) = H^*(U(n)) = E(h_1, h_2, \ldots, h_n)$, where $\dim h_i = 2i - 1$.

Let $E \to M$ be a principal $U(n)$-bundle. A connection is a projection $\pi : TE \to TU(n)$ (from the tangent space of E to the tangent space of the fiber) which is equivariant under the action of $U(n)$ (on the right). Thus, we regard $\pi : TE \to u(n)$ as a matrix of 1-forms. The kernel of π is the "horizontal space." The corresponding curvature is a matrix of 2-forms $\Omega = d\pi - \pi \wedge \pi$ (Cf. Bott [6]), which takes pairs of vectors in $\Lambda^2 TE$ into $u(n)$. π dualizes to a map

$$\Lambda^1(u(n))^* = u(n)^* \to A^1(E) = \text{1-forms on } E.$$

Also the curvature dualizes to a map

$$S^1(u(n))^* = u(n)^* \to A^2(E).$$

We extend these to a map

$$W(u(n)) = \Lambda^* u(n)^* \otimes S^* u(n)^* \to A^*(E),$$

which will commute with d. Now, we wish to find a space Y_n with the sa cohomology as $H^*(\mathfrak{a}_n) = H^*(\underline{W}(u(n)))$. Since $W(u(n))$ is acyclic, it provi a good model for cochains on the total space of the universal principal $U($ bundle, $EU(n) \to BU(n)$, where $E(U(n))$ is contractible. (Note that $BU(n$ $BGL(n,\mathbb{C})$ since any $GL(n,\mathbb{C})$-bundle can be reduced to a $U(n)$-bundle.) N for $E \to M$ with M an n-manifold, polynomials in the components of matrix Ω of degree $> n$ vanish (see Section 21; cf. Bott [6] section 6) so

$$\underline{W}(u(n)) \to A^*(E),$$

where $\underline{W}(u(n)) = W(u(n))/(S^{n+1}(u(n))^*)$ is the truncated Weil algebra $u(n)$. Since this map is functorial, there exists a map

$$\underline{W}(u(n)) \to A^*(Y_n),$$

where $Y_n \to BU(n)^{2n}$ is the restriction of $EU(n)$ to the $2n$-skeleton, $BU(n)$ of $BU(n)$. $Y_n \to BU(n)^{2n}$ classifies $U(n)$-principal bundles over n-manifolc

$$\begin{array}{ccc} Y_n & \longrightarrow & EU(n) \\ \downarrow & & \downarrow \\ BU(n)^{2n} & \longrightarrow & BU(n) \simeq BGL(n,\mathbb{C}) \end{array}$$

In fact, we have the following

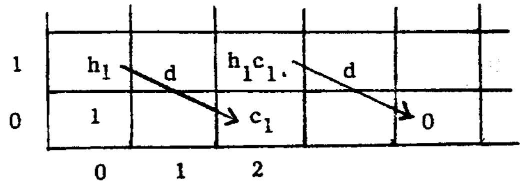
Theorem 20.1. (Gel'fand–Fuks) $H^*(\mathfrak{a}_n) = H^*(\underline{W}(u(n))) = H^*(Y_n)$ *the same cohomology as the complex:*

$$E(h_1, \ldots, h_n) \otimes P(c_1, \ldots, c_n),$$

where $E(h_1, \ldots, h_n)$ is the exterior algebra generated by $\{h_i\}$ of degree 2i 1, $P(c_1, \ldots, c_n)$ is the polynomial algebra with generators $\{c_i\}$ of dimens $2i$ divided by the relations $c_1^{a_1} \cdots c_n^{a_n} = 0$ whenever $\sum_{k=1}^n (ka_k) > n$, and differential is given by $d(h_i) = c_i$.

We recognize $E(h_1, \ldots, h_n) \otimes \underline{P}(c_p, \ldots, c_n)$ as the E_2 term of the spect sequence converging to $H^*(\mathfrak{a}_n)$. The d_r maps are 0 for r odd, and $d_{2i}(h_i) =$ (Gel'fand–Fuks [28] p. 337–340. Cf. H. Cartan [12] p. 27, Godbillon [30] p. 10, 14). Since $H^*(W(u(n)))$ is acyclic, we can choose $h_i \in W(u(n))$ such tl $dh_i = c_i$ and the same will hold for their images under $W(u(n)) \to \underline{W}(u(n)$

Example. $n = 1$. $H^*(\mathfrak{a}_1) = H^*(E(h_1) \otimes P(c_1)/c_1^2 = 0)$ with $\dim h_1$ 1, $\dim c_1 = 2$, $d(h_1) = c_1$, $d(c_1) = 0$, $d(h_1 c_1) = c_1^2 = 0$.



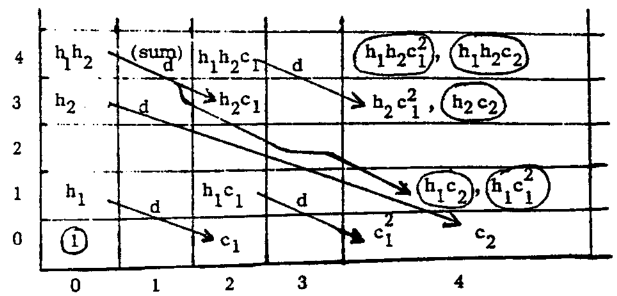
There is cohomology generated by 1 in dim= 0, and by $\omega = h_1c_1$ in dim= 3.

Example. $n = 1$, $U(1) = S^1$. $BU(1) = \mathbb{C}P^\infty$, the infinite Grassmannian has 2-skeleton: $BU(1)^2 = \mathbb{C}P^1 = S^2$. Then the pullback $Y_1 = S^3$ and the map $S^3 \to S^2$ is the Hopf fibration:

$$\begin{array}{ccc} Y_1 = S^3 & \longrightarrow & EU(1) \\ \downarrow & & \downarrow \\ BU(1)^2 = S^2 & \longrightarrow & BU(1) = \mathbb{C}P^\infty \end{array}$$

$$H^*(Y_1) = H^*(S^3) = \left\{ \begin{array}{ll} \mathbb{R} & \text{in dim } 0,3 \\ 0 & \text{otherwise} \end{array} \right\} = H^*(\mathfrak{a}_1).$$

Example. $n = 2$. $H^*(\mathfrak{a}_2) = H^*(E(h_1, h_2) \otimes P(c_1, c_2)/(c_1^3, c_1c_2, c_2^2))$.



The generators of cohomology are circled. There are 1 in dim= 0, 2 in dim= 5, 1 in dim= 7, and 2 in dim= 8. Vey computed an explicit basis for $H^*(\mathfrak{a}_n)$ ([30] p. 13, [37] p. 383; cf. [28] p. 340.)

Proposition 20.2. *The* $\{h_{i_1} \cdots h_{i_r} c_{j_1} \cdots c_{j_s}\}$ *such that*

$$1 \le i_1 < i_2 < \cdots < i_r \le n \quad \textit{for} \quad r = 1, \ldots, n,$$
$$1 \le j_1 \le j_2 \le \cdots \le j_s \le n \quad \textit{for} \quad s = 1, \ldots, n,$$
$$j_1 + \cdots + j_s \le n < i_1 + j_1 + \cdots + j_s, \textit{ and } i_1 < j_1$$

form a basis of $H^*(\mathfrak{a}_n)$.

Then the following two propositions are immediate:

Proposition 20.3. (Gel'fand–Fuks) *Except for the trivial class in dime sion 0,* $H^*(\mathfrak{a}_n)$ *starts in dimension* $2n + 1$ *and goes no higher than* $n^2 + 2n$.

Proposition 20.4. (Gel'fand–Fuks) *Multiplication is trivial in* $H^*(\mathfrak{a}_n)$.

Proof. Note that for $h_{i_1} \cdots h_{i_r} c_{j_1} \cdots c_{j_s}$ in the Vey basis, $i_1 < j_1$ impli $j_1 + \cdots + j_s > (n/2)$. Thus, all products are truncated.

21. Connection with connections

Recall our tower of frame bundles $\cdots \to F_2(M) \to F_1(M) \to M$ ov a manifold M. Choosing a connection on $\pi : F_1(M) \to M$ is the same choosing a section $\partial : F_1(M) \to F_2(M)$ which is equivariant under the actic of $gl(n, \mathbb{R})$. The forms ω^i_j on $F_2(M)$ pull back to a connection form, the matri $(\partial^*\omega^i_j)$, on $F_1(M)$. A tangent vector X on $F_1(M)$ is mapped to the matr $(\partial^*\omega^i_j(X)) \in gl(n, \mathbb{R}) \simeq$ tangent space of the fiber $\simeq GL(n, \mathbb{R})$. This projectic $T(F_1(M)) \to T$ (fiber) is a connection on the bundle $F_1(M) \to M$. (Verify Then the invariant polynomials in the curvature forms,

$$\Omega^i_j = d\omega^i_j - \omega^k_j \wedge \omega^i_k = \omega^k \wedge \omega^i_{kj}$$

(where we have pulled ω^k_j and ω^i_{kj} down to $F_1(M)$ by suitable sections), w be basic forms (i.e., come from the base, M, via π^*) and give us characterist classes in $H^*(M)$.

22. Reduction of $H^*_{GF}(\mathbb{R}^n)$ to $H^*(W)$

The object of this section is to show that the inclusion

$$A^*_{pt}L\mathbb{R}^n \longrightarrow A^*_c L\mathbb{R}^n$$

induces isomorphism in cohomology. (See Bott [7].) To simplify the notation v first take $n = 1$. For $x \in \mathbb{R}^1$, we denote by δ_x the evaluation at x : $\delta_x(f) = f(x$ for $f : \mathbb{R} \to \mathbb{R}$. $\delta'_x, \delta''_x, \ldots, \delta^{[k]}_x, \ldots$ will be the evaluation of the 1st, 2nd, .. kth, ... derivatives at $x \in \mathbb{R}^1$. $\delta'_x(f) = f'(x)$, etc. (We will forget about the sign in this section for convenience.) In this context, Schwartz's theorem sa that any continuous linear functional on $C^\infty(\mathbb{R}^1)$ can be written as a finite su of ones of the form:

$$\varphi = \int_{-\infty}^{\infty} \mu(x)\, \delta^{[k]}_x dx$$

where $\mu(x)$ is continuous $\mathbb{R} \to \mathbb{R}$ with compact support. This means for $f : \mathbb{R} \to \mathbb{R}$ smooth, $\varphi(f) = \int \mu(x)\delta_x^{[k]} f = \int \mu(x) f^{[k]}(x)dx$. Then since μ is continuous with compact support,

$$d\varphi = \int_{-\infty}^{\infty} \mu(x)\,[d(\delta_x^{[k]})]dx$$

where we already have seen how to compute $d(\delta_x^{[k]})$ in the $A_{pt}^* L\mathbb{R}^1$ case.

What are the 2-forms? For $x, y \in \mathbb{R}^1$ we can of course have $\delta_x^{[k_1]} \wedge \delta_y^{[k_2]}$ as a 2-form. We can also take finite linear combinations of these with x, y, k_1, and k_2 varying. Limits of these are integrals:

$$\varphi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\,\delta_x^{[k_1]} \wedge \delta_y^{[k_2]} dxdy.$$

At first $\mu(x,y)$ may have some Dirac-δ spikes, but as in the 1-form case, we find that the most general continuous 2-form on $L\mathbb{R}^1$ can be written as a finite sum of φ's as above with $\mu(x,y)$ *continuous*. This is the Schwartz kernel theorem. To generalize to the $\mathbb{R}^n$ case we replace $\delta_x^{[k]}$ with some $(\partial^k/\partial x_{i_1} \cdots \partial x_{i_k})\delta_x^j$ where j indicates which component of $\mathbb{R}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

The general continuous q-form on $L\mathbb{R}^1$ will be finite sums of forms of the form:

$$\int \mu(x_1, \ldots, x_q)\,\delta_{x_1}^{[k_1]} \wedge \cdots \wedge \delta_{x_q}^{[k_q]} dx_1 \cdots dx_q.$$

When evaluating such an expression $\delta_{x_1}^{[k_1]} \wedge \cdots \wedge \delta_{x_q}^{[k_q]}$ on q vector fields $f(x)\frac{\partial}{\partial x}, \ldots, f_q(x)\frac{\partial}{\partial x}$ we must not forget the alternation.

Consider the function (for $t \in \mathbb{R}$) $f_t : \mathbb{R}^1 \to \mathbb{R}^1$ by $x \to tx$. By the chain rule, we compute f_t^* for $t \neq 0$. I claim:

$$f_t^*\delta_x = t\delta_{(x/t)} \quad \text{for} \quad t \neq 0$$

This is because $f_t^* : \frac{\partial}{\partial x}|_x \longrightarrow t\frac{\delta}{\delta x}|_{tx}$. As $t \to \infty$, the support (i.e., x/t) goes to 0, but the form blows up because of the factor of t in front. We also have by the chain rule:

$$f_t^*\delta_x' = t(\delta_{x/t}')(1/t) = \delta_{x/t}'.$$

Namely, $(f_t^*\delta_x')(f(y)\partial/\partial y) = \delta_x'(tf(y/t)\partial/\partial y) = tf'(x/t)(1/t)$. We also have

$$f_t^*\delta_x'' = (1/t)\delta_{x/t}'',$$

and in general:

$$f_t^*\delta_x^{[k]} = t^{(1-k)}\delta_{x/t}^{[k]}.$$

As $t \to \infty$ there is no blowing up in $f_t^*\delta_x^{[k]}$ for $k \geq 1$. Only $f_t^*\delta_x$ blows up for $k = 0$. We are reminded of our earlier decomposition into eigenspaces of θ_R where $R = (x)\partial/\partial x$, the infinitesimal generator of f_t. Consider the q-form $\varphi^q = \int \mu(x_1, \ldots, x_q)\delta_{x_1}^{[k_1]} \wedge \cdots \wedge \delta_{x_q}^{[k_q]} dx_1 \cdots dx_q$. A priori $f_t^*\varphi^q$ can grow no faster than t^q as $t \to \infty$. But we can say more. It never grows faster than t^1.

Lemma 22.1. *Let $f_t^* : A^q(L\mathbb{R}^1) \to A^q(L\mathbb{R}^1)$ as above. Then $(1/t)f_t^*$ have a limit as $t \to \infty$. Furthermore, this limit is in $A_{pt}^* L\mathbb{R}^1$ and will be eigenvector for $\theta_{(x)\partial/\partial x}$ with eigenvalue (or weight) 1.*

Recall that $A_{pt}^*(L\mathbb{R}^1)$ breaks up a direct sum of eigenspaces $A_{pt}(\lambda)$ $\lambda = 1, 0, -1, -2, \ldots$ of the operator $\theta_{(x)\partial/\partial x}$. To repeat the lemma, $\lim_{t\to\infty}(1/t$ exists and takes values in $A_{pt}(1)$.

Exercise. $\lim_{t\to\infty}(1/t)f_t^*$ is identity on $A_{pt}(1)$, and 0 on $A_{pt}(k)$ for $k \leq$ Suppose we have shown that the limit exists and is finite. Since the suppo shrink to zero, the limit has support at the origin. Further, $\gamma = \lim_{t\to\infty}(1/t)f_t^*$ a projection, $\gamma \circ \gamma = \gamma$, since for $s, t \in \mathbb{R}$, $f_s^* \circ f_t^* = f_{st}^*$, and (assuming that t limits as $t \to \infty$ exist) the composition of limits = limit of compositions.

We obtain a projection which commutes with d:

$$A^* L\mathbb{R}^1 \xrightarrow{\ \gamma\ } A_{pt}(1) \qquad (\iota : A_{pt}(1) \to A^* L\mathbb{R}^1)$$

which is split by the inclusion ι. $\gamma \circ \iota =$ identity. Since $H^*(A_{pt}(1)) = 0$ (Rec that $\theta_{(x)\partial/\partial x} = \iota_{(x)\partial/\partial x}d + d\iota_{(x)\partial/\partial x}$ is homotopic to 0 but is 1 on $A_{pt}(1)$.) u is a cocycle in $A^* L\mathbb{R}^1$ then γu is a coboundary, dv, for $v \in A_{pt}(1)$. A $u - \iota dv \in \ker\gamma$.

Idea of proof of Lemma. Since $\delta_x', \delta_x'', \ldots$ give us no problem, the wo case is illustrated by the form: $\varphi = \int \mu(x,y)\delta_x \wedge \delta_y dxdy$. Since μ is continuo with compact support, $f_t^*\varphi = t^2 \int \mu(x,y)\delta_{x/t} \wedge \delta_{y/t} dxdy$. Now as $t \to \infty$, have the t^2 factor blowing up to order 2. However, we also have $\delta_{x/t} \wedge \delta_{y/t}$ $\delta_0 \wedge \delta_0 = 0$ by the alternation. Explicitly, let $x = f(x)\partial/\partial x$, $y = g(y)\partial/\partial y$ $L\mathbb{R}^1$. Then $f(x) = a_0 + a_1 x + x^2(h_1(x))$ and $g(y) = b_0 + b_1 y + y^2 h_2(y)$

$$\begin{aligned}&(\delta_{x/t} \wedge \delta_{y/t}(X,Y) = f(x/t)g(y/t) - f(y/t)g(x/t)\\ &= (a_0 + a_1(x/t) + (x/t)^2 h_1(x/t))(b_0 + b_1(y/t) + (y/t)^2 h_2(y/t))\\ &- (a_0 + a_1(y/t) + (y^2/t)h_1(y/t))(b_0 + b_1(x/t) + (x/t)^2 h_2(x/t)) = \frac{1}{t}\{\cdots\}\end{aligned}$$

Because of this $1/t$ factor $f_t^*\varphi$ can blow up only like $t^2 \cdot (1/t) = t^1$.

Remark. In the $\mathbb{R}^n$ case the worst that $f_t^*\varphi$ can blow up is to order n as illustrated by the term $\delta_{x_1}^1 \wedge \cdots \wedge \delta_{x_n}^n$ where superscript means compone in $\mathbb{R}^n$, not derivative. In this case the image of γ is the n-eigenspace of θ where $R = x_1\partial/\partial x_1 + \cdots + x_n\partial/\partial x_n$ is the infinitesimal generator of f_t $(x_1, \ldots, x_n) \to (tx_1, \ldots, tx_n)$.

Granting this lemma, one extends it as follows:

Lemma 22.2. *Let $F^n \supseteq F^{n-1} \supseteq \cdots \supseteq F^0 \supseteq F^{-1} \supseteq \cdots$ be a filtration of $A_c^* L\mathbb{R}^1$ defined by:*

$$F^k = \{\omega \in A_c^* L\mathbb{R}^1 | \gamma_k \omega = \lim_{t\to\infty} \frac{1}{t^k}\omega \quad \textit{is well defined}\}.$$

Then one has a split exact sequence:

$$0 \longrightarrow F^{k-1} \longrightarrow F^k \xrightarrow{\gamma_k} A_{pt}(k) \longrightarrow 0.$$

Remark. By 22.1, we have $A_c^* L\mathbb{R}^1 = F^1$. In the $\mathbb{R}^n$ case, we would have $A_c^* L\mathbb{R}^n = F^n$.

Proposition 22.3. *If $A_{pt}^* L\mathbb{R}^1$ is graded by weight then the associated graded group of $\{F^k\}_{k=-\infty}^1$ is $\oplus_{k=-\infty}^1 A_{pt}(k) = A_{pt}^* L\mathbb{R}^1$.*

The situation is described by a diagram:

$$\begin{array}{ccccccc} \cdots & F^{-1} & \hookrightarrow & F^0 & \hookrightarrow & F^1 & = A_c^* L\mathbb{R}^1 \\ & \iota\uparrow\downarrow\gamma_{-1} & & \iota\uparrow\downarrow\gamma_0 & & \iota\uparrow\downarrow\gamma_1 & \\ \cdots & A_{pt}(-1) & & A_{pt}(0) & & A_{pt}(1) & \end{array}$$

where $\gamma_1 = \lim_{t\to\infty}(1/t)f_t^*$ defined on all of $A_c^* L\mathbb{R}^1 = F^1$, $\gamma_k = \lim_{t\to\infty} t^{-k} f_t^*$ defined on $F^k = \ker(\gamma_{k+1})$ for $k = 0, -1, \ldots$, and ι is inclusion. We have already shown that any cocycle, u, in F^1 is cohomologous to a cocycle, $u_2 = u - i\gamma_1 u$, in F^0, since $H^*(A_{pt}(1)) = 0 \Rightarrow \gamma_1 u$ is a coboundary. Repeating the process, $u_3 = u_2 - \iota\gamma_0 u_2$ is a cocycle in F^{-1}. After we show that $H^*(F^{-1}) = 0$, u_3 will be a coboundary in F^{-1}. Then the map $H^*(AL\mathbb{R}^1) \to H^*(A_{pt}(0))$ induced by $u \to \gamma_0(u - \iota\gamma_1 u)$ is the isomorphism inverse to ι^*. $H^*(A_{pt}(0))$ has already been computed in 20.1 and 20.2.

To complete the proof, we need a homotopy formula to show $H^*(F^{-1}) = 0$. Again let $R = (x)\partial/\partial x$ be the infinitesimal generator of f_t.

Lemma 22.4. $\frac{d}{dt} f_t^* = (1/t)\{\iota_R d + d\iota_R\} f_t^*$.

Proof.

$$t\frac{d}{dt} f_t^* = \lim_{\Delta t\to 0} t \frac{f_{t+(t\Delta t)}^* - f_t^*}{(t\Delta t)} = \lim_{\Delta t \to 0}\left(\frac{f_{1+\Delta t}^* - 1}{\Delta t}\right) f_t^*.$$

Now

$$\lim_{\Delta t\to 0} \frac{f_{t+\Delta t}^* - 1}{\Delta t} = \frac{d}{dt}\Big|_{t=1} f_t^* \quad \text{and } \theta_R = \iota_R d + d\iota_R.$$

So it suffices to show $d/dt|_{t=1}(f_t^*) = \theta_R$. But this is just the statement that θ_R is the Lie derivative in the R direction. For example, let us practice what

these symbols mean on a vector field $f(x)\partial/\partial x$ and 1-form $\delta_a^{[k]}$ = evaluation k-th derivative at $x = a$ in $\mathbb{R}$.

$$\begin{aligned} \frac{d}{dt}\Big|_{t=1}[(f_t^* \delta_a^{[k]})\left(f(x)\frac{\partial}{\partial x}\right) &= \frac{d}{dt}\Big|_{t=1}(t^{1-k} f^{[k]}(a/t)) \\ &= [(1-k)t^{-k} f^{[k]}(a/t) + t^{1-k} f_{(a/t)}^{k+1}(-at^{-2})]_{t=1} \\ &= (1-k) f^{[k]}(a) - a f^{[k+1]}(a) \end{aligned}$$

$(1)_k$

On the other hand,

$$\begin{aligned} (\theta_{x(\partial/\partial x)}\delta_a^{[k]})(f(x)\frac{\partial}{\partial x}) &= -\delta_a^k[(x(\partial/\partial x), f(x)\partial/\partial x]) \\ &= -\delta_a^k(xf'(x)\frac{\partial}{\partial x} - f(x)\frac{\partial}{\partial x}) \\ &= \frac{\partial^k}{\partial x^k}\Big|_a (f(x) = xf'(x)) \end{aligned}$$

$(2)_k$

For $k = 0$, $(2)_0 = f(a) - af'(a) = (1)_0$. The inductive step is given by $(2)_{k+1}$ $d/dx|_a\{(2)_k\} = d/dx|_a\{(1-k)f^{[k]}(x) - xf^{[k+1]}(x)\}$ (by inductive hypothesi $= (1-k)f^{[k+1]}(a) - (f^{[k+1]}(a) + af^{[k+2]}(a)) = (1)_{k+1}$. Therefore, $(1)_k = (2$ for all $k \geq 0$. We integrate from 1 to T to obtain:

$$\begin{aligned} \int_1^T \frac{d}{dt} f_t^* &= \int_1^T \{\iota_R d + d\iota_R\} f_t^*(dt/t) \\ f_T^* - 1 &= \int_1^T \iota_R f_t^* d + d\iota_R f_t^*(dt/t) \\ &= K_T d + dK_T \\ \text{where} \quad K_T(\omega) &= \int_1^T \iota_R f_t^*(\omega)(dt/t) \end{aligned}$$

In general, $K_\infty = \lim_{T\to\infty} K_T$ need not exist, but if $\omega \in F^{-1}$, $K_\infty(\omega)$ will exist ar also be in F^{-1}. Since $\lim_{T\to\infty} f_T^* = 0$ on F^{-1}, we will have shown $H^*(F^{-1}) =$ For $\omega \in F^{-1}$:

$$K_T(\omega) = \int_1^T \iota_R t f_t^*(\omega)(dt/t^2).$$

Changing variables, let $\lambda = (1/\lambda)$, $d\lambda = -(dt/t^2)$.

$$K_T(\omega) = \int_{1/T}^1 \iota_R((1/\lambda) f_{1/\lambda}^*(\omega)) d\lambda$$

which converges to

$$K_\infty(\omega) = \int_0^1 \iota_R((1/\lambda) f_{1/\lambda}^*(\omega)) d\lambda$$

Since $\lim_{\lambda\to\infty}(1/\lambda)f^*_{1/\lambda}(\omega)$ is defined for $\omega \in F^{-1}$.

But we still have to show that $K_\infty(F^{-1}) \subset F^{-1}$. First note that ι_R commutes with f_t^*, and $f_s^* f_t^* = f_{st}^*$.

$$\begin{aligned} f_t^* \circ K_\infty &= \int_0^1 f_t^* \iota_R (1/\lambda) f^*_{1/\lambda} d\lambda \\ &= \int_0^1 (1/\lambda)\iota_R f^*_{(t/\lambda)} d\lambda \\ &= \int_0^1 (t/\lambda)\iota_R f^*_{(t/\lambda)} (d\lambda/t) \\ &= \int_0^{1/t} (1/\nu)\iota_R f^*_{(1/\nu)} d\nu \to 0 \quad \text{as } t\to\infty \quad \text{on } F^{-1} \end{aligned}$$

where $\nu = \lambda/t$, $d\nu = d\lambda/t$.

23. Cohomology of subalgebras of $L\mathbb{R}^n$

Generalizing the argument for $n = 1$, we have

Proposition 23.1. *The inclusion*

$$A^*_{pt} L\mathbb{R}^n \hookrightarrow A^*_c L\mathbb{R}^n$$

induces isomorphism in cohomology:

$$H^*_{GF}(\mathbb{R}^n) = H^*(\mathfrak{a}_n) = H^*(Y_n).$$

Y_n is the restriction of the total space of universal principal $U(n)$-bundle to the $2n$-skeleton of $BU(n)$, as in Section 20.

Corollary 23.1.1. *$H^*_{GF}(\mathbb{R}^n)$ is finite dimensional.*

We can refine the proof to certain subalgebras of $L\mathbb{R}^n$ which correspond to certain subgroups of Diff $\mathbb{R}^n$ = diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$. Recall that $L\mathbb{R}^n$ is the analogue of the Lie algebra for the infinite dimensional group Diff $\mathbb{R}^n$. However, there are some unsolved cases.

Example. The Hamiltonian case. Let $x_1,\ldots,x_n,y_1,\ldots,y_n$ be coordinates on $\mathbb{R}^{2n}$. Let $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. The diffeomorphisms $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which preserve ω form a subgroup of Diff $\mathbb{R}^{2n}$ whose Lie algebra is $L_H\mathbb{R}^{2n} = \{X \in L\mathbb{R}^n | \theta_X\omega = 0\}$. Since θ_X is the Lie derivative, this says ω is constant along a flow which is tangent to the vector field X. We can ask what the inclusion

$$A^*_{pt} L_H\mathbb{R}^n \to A^*_c L_H\mathbb{R}^n$$

induces in cohomology. But here the argument involving the contraction $f_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ breaks down because f_t does not preserve ω. By way of contrast, for

the subalgebra $\{X \in L\mathbb{R}^n | \theta_X\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}\}$ of $L\mathbb{R}^n$, the contract argument remains valid. Note that $H^*(A_{pt}L_H\mathbb{R}^n)$ is just $H^*_c(\mathfrak{a}_{2n;H})$ wh $\mathfrak{a}_{2n;H}$ is the subalgebra of $\mathfrak{a}_{2n}$ containing those $X = \sum_i(f_i\partial/\partial x_i + g_i\partial/\partial$ such that $\theta_X\omega = 0$. f_i and g_i are formal power series in $x_1,\ldots,x_n,y_1,\ldots,$

The first question we would like to know about $H^q(A_cL_H\mathbb{R}^n)$ is whethe not it is finite dimensional. Gel'fand used a computer to find some very ex classes in there. The requirement that $\theta_X\omega = 0$ imposes certain symme conditions on the $\delta^i_{j_1\ldots j_r}$ of $A_cL_H\mathbb{R}^n$. (See [29].) Another problem is to t ω = volume form on $\mathbb{R}^n$ and consider the volume-preserving diffeomorphis This corresponds to the Lie algebra of vector fields X on $\mathbb{R}^n$ such that $\theta_X\omega =$ It is hard to adapt the Gel'fand–Fuks original proof to other groups with contraction property. But we will use the abstract nonsense approach to ve fields on manifolds.

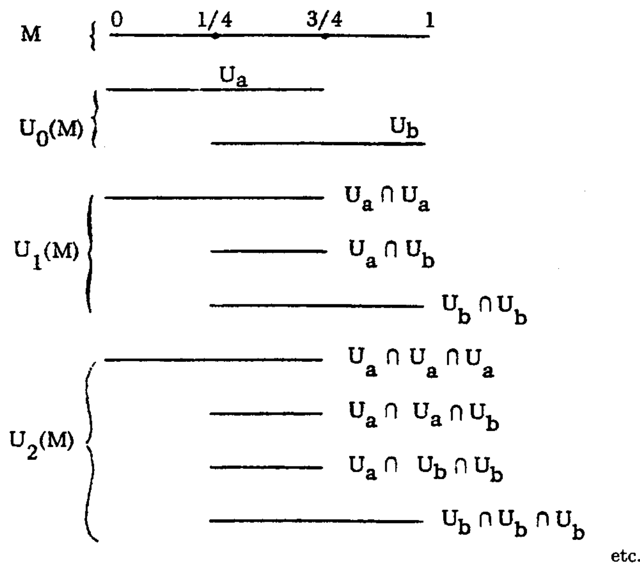
24. Gel'fand–Fuks on manifolds. Semi-simplicial methods

Let M be a manifold, $U(M) = \{U_a\}_a$ be a good covering of M. Re that this means that $U_a, U_a \cap U_b, U_a \cap U_b \cap U_c, \ldots$ are all copies of $\mathbb{R}^n$ (or empty). We also assume $a, b, \ldots$ taken over some finite, linearly ordered in set. Then we have our semi-simplicial manifold:

$$\begin{aligned} U_0(M) &= \coprod_a U_a \quad \text{where } \coprod \text{ means disjoint union.} \\ U_1(M) &= \coprod_{a\le b}(U_a \cap U_b) \\ U_2(M) &= \coprod_{a\le b\le c}(U_a \cap U_b \cap U_c), \text{ etc.} \end{aligned}$$

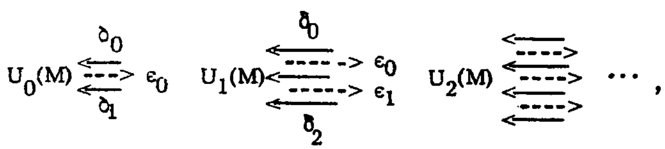
The maps $\partial_0,\ldots,\partial_k : U_k(M) \to U_{k-1}(M)$ are defined by the inclusi $\partial_i : U_{a_0} \cap\cdots\cap U_{a_k} \to U_{a_0}\cap\cdots\cap \widehat{U}_{a_i} \cap\cdots\cap U_{a_k}$ where $\widehat{}$ means omit as us We also have the degeneracy maps $\epsilon_0,\ldots,\epsilon_k : U_k(M) \to U_{k+1}(M)$ given by identity maps $\epsilon_i : U_{a_0}\cap\cdots\cap U_{a_k} \to U_{a_0}\cap\cdots\cap U_{a_i}\cap\cdots\cap U_{a_k}$.

Example. $M = (0,1)$, the open interval, can be covered by $U_a = (0,3$ and $U_b = (1/4, 1)$. We have

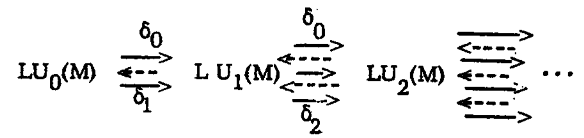


etc.

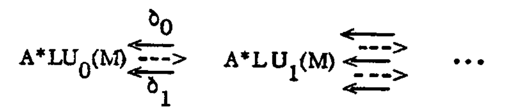
Notice that for $U_2(M)$ and higher we get more and more copies of $\mathbb{R}^n$, but these will all be degenerate. Because of the repeated subscripts, we will not see anything new besides $(0, 3/4), (1/4, 3/4)$, and $(1/4, 1)$. Now to the semi-simplicial manifold,



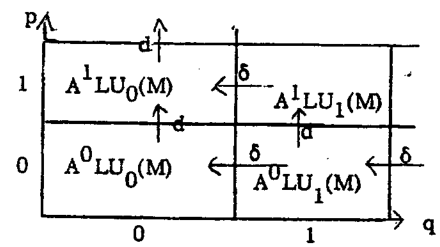
we apply the functor L = "vector fields on." L is a contravariant functor with respect to the maps ∂_i because vector fields pull back, or restrict, to open subsets. Contrast this situation with the fact that vectors at a point (but not vector fields) are pushed foward by differentiable maps. We obtain a co-semi-simplicial Lie algebra:



We now apply the contravariant functor $A^* = \oplus_p A^p$ where A^p = "continuo
cochains on" to obtain the semi-simplicial module:



Remark. Although we are doing cohomology, the map $\delta = \sum_i (-1)^i \partial_i$ w act like homology, $\delta : A^p LU_q(M) \to A^p LU_{q-1}(M)$. We will also have the of the Gel'fand–Fuks theory, $d : A^p LU_q(M) \to A^{p+1} LU_q(M)$. On the doub complex $\{A^p LU_q(M)\}_{p,q}$ we have the total derivative $D = d \pm \delta$ as usual. V have two spectral sequences convergeing to H^*_D (double complex), one by t filtering by p and the other by filtering with respect to q.



Because d points up, but δ to the left, the total D-cocycles of degree n lie c a line $p - q = n$ instead of the usual situation $p + q$ = total degree. Filterii with respect to p we find that:

$$E_1^{p,q} = H_q^\delta(A^p LU_*(M)) = \begin{cases} A^p LM & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases}$$

To see this let $\{\rho_a\}$ be a partition of unity subordinate to $\{U_a\}$. Then tl following sequence is exact:

$$0 \longleftarrow A^p L(M) \overset{\iota}{\longleftarrow} A^p LU_0(M) \overset{\delta}{\longleftarrow} A^p LU_1(M) \overset{\delta}{\longleftarrow} \cdots$$

where ι is induced by the inclusions $U_a \hookrightarrow M$. A contracting homotopy T is contructed using $\{\rho_a\}$. For $\varphi \in A^pLM$ and $X_1,\ldots,X_p \in LU_0(M)$, define $T : A^pLM \to A^pLU_0(M)$ by $(T\varphi)(X_1,\ldots,X_p) = \sum_a \varphi(\rho_a X_1|_{U_a},\ldots,\rho_a^{X_p}|U_a)$ where $\rho_a X_i|_{U_a}$ is extended to M to be 0 outside U_a. $T : A^pLU_k(M) \to A^pLU_{k+1}$ similarly, and we get $Id = T\delta + \delta T$.

Then $E_2^{p,q} = H_d^p H_q^\delta(A^*LU_*(M)) = H_{GF}^p(M)$ in the column $q = 0$, and 0 for $q > 0$. Then $E_\infty^{p,0} = E_2^{p,0} = H_{GF}^p(M)$ and $E_\infty^{p,q} = 0$ for $q > 0$. What is the spectral sequence going the other way (i.e., filtering with respect to q)?

$$'E_1^{p,q} = H_d^p(A^*LU_q(M)) = H_{GF}^p(U_qM)$$
$$'E_2^{p,q} = H_q^\delta H_d^p(A^*LU_*(M)) = H_q^\delta H_{GF}^p(U_*M).$$

In Section 25 we will show that $\oplus_{p-q=n}{}'E_2^{p,q}$ is finite dimensional for each n. Then $H_{GF}^n(M) = E_\infty^{n,0}$ must also be finite dimensional. (At this point, we are still bothered by a technical problem. Namely, it is not obvious that the spectral sequences, E_r and $'E_r$, both converge to $H_D(ALU(M))$ because the usual convergence criteria for spectral sequences do not hold in this case. In particular, any D-cocycle in the double complex $ALU(M)$ which has infinitely many non-zero entries on the line $p - q = n$ will escape detection by $'E_1 = H_d(ALU(M))$. (We will ignore this problem here and continue.)

25. The Künneth formula

If L_1 and L_2 are two finite dimensional Lie algebras then we have a Künneth Formula:

$$H^*(L_1 \oplus L_2) \simeq H^*(L_1) \otimes H^*(L_2)$$

and this relation follows directly from the fact that on the cochain level:

$$A^*(L_1 \oplus L_2) \simeq A^*(L_1) \otimes A^*(L_2).$$

Now the Gel'fand–Fuks theory, $L_1 = L(M)$ and $L_2 = L(N)$ (for manifolds M and N) are infinite dimensional. We replace A^* with continuous forms A_c^* and $\otimes$ with a completed tensor product $\widehat{\otimes}$.

$$A_c^*(L_1 \oplus L_2) \simeq A_c^*(L_1)\widehat{\otimes} A_c^*(L_2)$$

and $L_1 + L_2 = L(M \coprod N) =$ vector fields on the disjoint union of M and N. It is difficult to prove directly the Künneth relation. However, our earlier techniques can be extended to prove the Künneth formula for the special case of a disjoint union of $\mathbb{R}^n$'s. We have

$$H^*(\oplus_{j=1}^k L\mathbb{R}^n) = \otimes_{j=1}^k H^*(L\mathbb{R}^n),$$

so that

$$H_{GF}^*(\coprod_{j=1}^k \mathbb{R}^n) = \otimes_{j=1}^k H^*(a_n).$$

The idea of the proof is to apply the contraction f_t to all k copies of $\mathbb{R}^n$ at A point x in some copy of $\mathbb{R}^n$ is sent to tx in the same copy of $\mathbb{R}^n$. Our f may then blow up to order kn instead of n, but we use the same argume before to get down to weight 0.

Let $G = \widetilde{H_{GF}^*}(\mathbb{R}^n) = H_{GF}^*(\mathbb{R}^n)$ for $* > 0$ and 0 in dimension $* =$ reduced Gel'fand–Fuks cohomology. Then

$$\widetilde{H_{GF}^*}(\mathbb{R}^n \coprod \mathbb{R}^n) = G \oplus G \oplus (G \otimes G)_{(alternating\, part)} \oplus \cdots$$

In order that we interpret the δ maps of our complex, we must take thi scription functorial. Let Y be a disjoint union of $\mathbb{R}^n$'s. Consider the func

$$Y \to H_0(Y,G) \oplus H_0^\alpha(Y \times Y, G \otimes G) \oplus H_0^\alpha(Y \times Y \times Y, G \otimes G \otimes G),$$

where α means alternating under a permutation. Now since multiplicati Gel'fand–Fuks classes in any one component $\mathbb{R}^n$ of Y is trivial, we must re the above description by

$$Y \to H_0(Y,G) \oplus H_0^\alpha(Y \times Y, \Delta; G \otimes G) \oplus \cdots$$

where Δ is the diagonal of $Y \times Y$. $H_0(Y,G)$ represents those classes of $\widetilde{H_G^*}$ which can be represented by a GF class on some one component $\mathbb{R}^n$ (times 1's on the other components). $H_0^\alpha(Y \times Y, \Delta; G \otimes G)$ represent classes of $\widetilde{H_{GF}^*}(Y)$ which are an (alternating) product of two GF class some two components $\mathbb{R}^n$'s of Y (times 1's on the other components). come products of 3, etc. In general, for $Y = \mathbb{R}^n \coprod \cdots \coprod \mathbb{R}^n$,

$$H_{GF}^*(Y) = H_0(Y,G) + H_0^\alpha(Y^2, Y_1^2; G \otimes G) + H_0^\alpha(Y^3, Y_2^3; G \otimes G \otimes G)$$
$$\cdots + H_0^\alpha(Y^k, Y_{k-1}^k, \underbrace{G \otimes \cdots \otimes G}_{k-times}) + \cdots,$$

where $Y^k = Y \times \cdots \times Y$ (k times), and $Y_{k-1}^k = \{(y_1,\ldots,y_k) \in Y^k | y_i = y$ some $i \neq j\}$ generalizes $\Delta \subseteq Y^2$. This generalized diagonal expresses the that multiplying G-F classes on the same component $\mathbb{R}^n$ gives 0 since m plication in $H_{GF}^*(\mathbb{R}^n)$ is trivial. This is the Künneth formula made funct with respect to maps.

Recall our second spectral sequence which converges to $H_{GF}^*(M)$ and b with:

$$'E_1^{p,q} = H_{GF}^p(U_q(M))$$
$$'E_2^{p,q} = H_q^\delta H_{GF}^p(U_*(M))$$

But now we use our functorial description of $H_{GF}^*(U_q(M))$:

$$'E_1^{*,q} \simeq H_0(U_q(M);G) \oplus H_0^\alpha(U_q(M) \times U_q(M), \Delta; G \otimes G) \oplus \cdots$$
$$\downarrow\delta \qquad\qquad \downarrow\delta \qquad\qquad \downarrow\delta$$
$$'E_1^{*,q-1} \simeq H_0(U_{q-1}(M);G) \oplus H_0^\alpha(U_{q-1}(M) \times U_{q-1}(M), \Delta; G \otimes G) \oplus \cdots$$

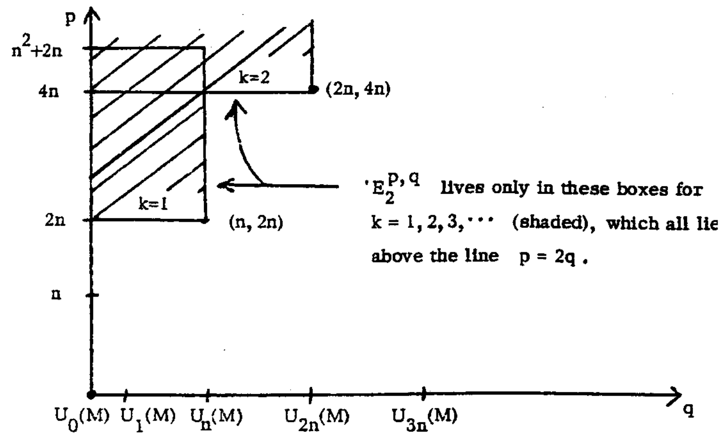
taking the δ-homology (Čech) we have

$$'E_2^{*,q} = H_q^\delta \widetilde{H^*_{GF}}(U_*M) = Hq(M;G) \oplus H_q^\alpha(M \times M, \Delta; G \otimes G) \otimes \cdots$$

since $U(M)$ is a good covering of M in the Čech theory. This is the original spectral sequence found in Gel'fand–Fuks. ([26] and [27].)

$$'E_2^{*,q} = \oplus_k H_q^\alpha(M^k, M^k_{k-1}; \otimes^k G).$$

These $'E_2$ terms appear only in certain rectangles (different rectangles for each summand k) shown on the following diagram for $k = 1, 2$:



Recall that $G^p \equiv \widetilde{H^p_{GF}}(\mathbb{R}^n) = 0$ whenever $p \leq 2n$ (or whenever $p > n^2 + 2n$). Therefore, $\otimes^k G = 0$ in dimensions $p \leq 2kn$. This gives the lower edge of our boxes. But we also have (for $k = 1$), $H^q(M, G) = 0$ in dimensions $q > n$ since M is n-dimensional. In general, $H^q(M^k, M^k_{k-1}; \otimes^k G) = 0$ whenever $q > kn = \dim M$. This gives the right edge of our boxes. The boxes all live above the line $p = 2q$, so any line $p - q =$ some constant r can only hit finitely many boxes. Since each box is finite dimensional, so is $\oplus'_{p-q=r} E_2^{p,q}$. Therefore, by the argument at the end of Section 24 (cf. [26] 8.7) we have:

Theorem 25.1. (Gel'fand–Fuks) *$H^r_{GF}(M)$ is finite dimensional for each r.*

26. The conjecture

The problem is to represent the Gel'fand–Fuks cohomology $H^*_{GF}(M)$ o manifold, M, as the usual cohomology of some topological space, Z. That is there a Z such that $H^*_{GF}(M) \simeq H^*(Z, \mathbb{R})$? For $M = \mathbb{R}^n$, we can set Z Y_n, which was constructed for Theorem 20.1. Indeed, $H^*_{GF}(\mathbb{R}^n) = H^*(\mathfrak{a}_n)$ $H^*(Y_n)$.

Secondly, we are led to a Künneth formula $H^*_{GF}(M \coprod N) = H^*_{GF}(M)$ $H^*_{GF}(N)$ for disjoint unions which we have proved for $M = N = \mathbb{R}^n$. This s of behavior appears in the cohomology of *function spaces*. That is $Z =$ Ma $\{M, Y\}$, for some fixed Y. For then Maps $\{M \coprod N, Y\}$ Maps $\{M, Y\} \times$ Ma $\{N, Y\}$ for disjoint unions, which suggests a Künneth formula. Futhermore, $M = \mathbb{R}^n$, $H^*(\text{Maps } \{\mathbb{R}^n, Y\}) = H^*(Y) = H^*_{GF}(\mathbb{R}^n)$ will hold if $Y = Y_n$. B for general M, we will have to take account of the gluing of the charts on by semi simplicial methods.

Consider the following situation: a fiber bundle with fiber Y:

$$\begin{array}{ccc} Y & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

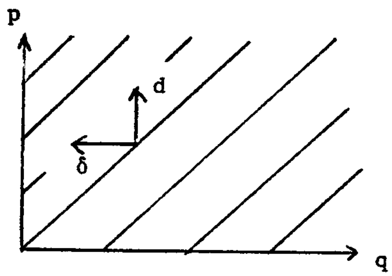
Instead of the function space Maps $\{M, Y\}$ we will consider the sections $\Gamma($ of the bundle. Now replace M by $U(M)$ for a good cover. Restricting t bundle to each of the opens $= \mathbb{R}^n$, $E|_{U_a} = U_a \times Y$ is trivial, since $U_a \simeq$ is contractible (same for $U_a \cap U_b$ etc.) Then $\Gamma(U_a) =$Maps $\{U_a, Y\} \sim$ Replacing M by $U(M)$, we get a co-semi-simplicial space

$$\Gamma(U_0(M)) \rightrightarrows \Gamma(U_1(M)) \;\substack{\longrightarrow \\ \rightrightarrows}\; \cdots$$

whose geometric realization has the homotopy type of $\Gamma(M)$. Now we ap the functor $S^* =$ singular cochains to get a semi-simplicial module for ea positive integer p.

$$S^p\Gamma_0 M \underset{\delta_1}{\overset{\delta_0}{\leftleftarrows}} S^p\Gamma U_1 M \underset{\delta_2}{\overset{\delta_0}{\leftleftarrows}} \cdots$$

The resulting double complex is half homology and half cohomology sin $d : S^p\Gamma U_q M \to S^{p+1}\Gamma U_q M$ is the singular coboundary and $\delta : S^p\Gamma U_q M$ $S^p\Gamma U_{q-1} M$ is the alternating sum of the δ_i maps. The total derivative $D = d$ has total degree n cochains along the $p - q = n$ lines:



Therefore, we need certain dimension restrictions to get convergence of the spectral sequence. The spectral sequence of Anderson [1] does exactly this for us. $H^*_D(S^*\Gamma UM) = H^*(\Gamma M)$ under the condition that the fiber Y be acyclic through dim M. $\Gamma(U(M))$ has the homotopy type of copies of the fiber Y, since the base $U_a, U_a \cap U_b$, etc. are all contractible. Therefore, the E_2 term is the same as the $'E_2^{p,q} = H_q^\delta H^p_{GF}(U(M))$ term of Sections 24 and 25, if we take fiber, Y, as a space with the same cohomology as $H^*(\mathfrak{a}_n) = H^*_{GF}(\mathbb{R}^n)$. We let $Y = Y_n$. How can we make a bundle over M with fiber Y_n? Let $F(M) \to M$ be the frame bundle of M (= the principal bundle associated to the tangent bundle of M). Since $GL(n,\mathbb{R}) \subseteq GL(n,\mathbb{C})$ acts on Y_n we can form Y_nM, the associated bundle with fiber Y_n by taking

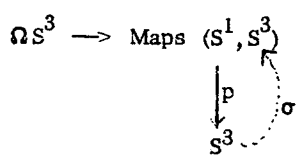
$$\begin{array}{ccc} Y_n & \longrightarrow & Y_nM \quad = F(M)X_{GL(n,\mathbb{R})}Y_n \\ & & \downarrow \\ & & M \end{array}$$

We are now in a position to state the

Conjecture 26.1. $H^*_{GF}(M) = H^*(\Gamma(Y_nM))$.

Exercise. Why is the conjecture true for $M = \mathbb{R}^n$?

Example. $M = S^1$. $Y_1 = S^3$. $F(M) = S^1 \times GL(1)$ is trivial. Therefore, $Y_1S^1 = S^1 \times S^3$. Hence, the sections, $\Gamma(Y_1S^1) = \text{Maps } (S^1, S^3)$. This function space has the cohomology of $H^*(S^3) \otimes H^*(\Omega S^3) = E(\omega) \otimes P(y)$ = Exterior algebra generated by ω in dim = 3, tensored with the polynomial algebra generated by y in dim =2. This is because we have a Serre fibration



where p is evaluation at a base point of S^1. The fiber is ΩS^3 = loop spa S^3. The section ∂ takes a point $x \in S^3$ to the constant map $S^1 \to \{x\}$. Serre spectral sequence ([58]) has E_2 term $H^*(S^3) \otimes H^*(\Omega S^3) = E(\omega) \otimes$ (S^3 is simply connected so we do not need local coefficients). ω persits to because of ∂. $H^*(\Omega S^3)$ is calculated similarly as in [58].

For $H^*_{GF}(S^n)$, $n \geq 2$ this computation gives a different answer than announced by Gel'fand–Fuks. However, in this case, the conjecture car verified by another method due to Haefliger to compute $H^*_{GF}(S^n)$.

27. Foliations

In the concluding sections, we wish to interpret $H^*(\mathfrak{a}_n)$ as characteri classes for foliations. Let $E \to M$ be a sub bundle of the tangent bu $TM \to M$ of an n-dimensional manifold M. E is integrable if and onl $\Gamma(E)$ = {sections of $E \to M$} is closed under bracket production [,]. Frobenius Theorem gives a local description:

Theorem 27.1. (Frobenius) *Locally, every integrable $E \subset TM$ is give the kernel of df where f is a submersion.*

A submersion is a map $f : M^n \to N^q$ between C^1 manifolds such t for all $x \in M$, $(df) : T_xM \to T_{f(x)}N$ is onto (in particular $q \leq n$). implicit function theorem tells us that locally (i.e., on some small chart), submersion looks like a projection $\mathbb{R}^n \to \mathbb{R}^q$. Now a projection $\mathbb{R}^n \to$ foliates $\mathbb{R}^n$, where the leaves are $\mathbb{R}^{n-q}$'s, the inverse images of points in Therefore, any integrable E is tangent to a foliation, defining a system of lc projections as follows: choose a covering $\{U_a\}$ of M such that

1) $f_a : U_a \to \mathbb{R}^q$ is a submersion (q = codimension of $E \subset TM$)
2) $E|_{U_a} = \ker (df_a)$

Then on $U_a \cap U_b$ we have two maps into $\mathbb{R}^q$, f_a and f_b. By the implicit funct theorem, f_b is a function of f_a (and vice versa) on some smaller neighborho So for any $x \in U_a \cap U_b$, there will be a smaller neighborhood U of x such th

$$f_b|_U = \gamma^U_{ba} \circ f_a|_U$$

where γ^u_{ba} is a local diffeomorphism of $\mathbb{R}^q$, $\gamma^U_{ba} : f_a(u) \to f_b(U)$. This is eas to state in terms of the germs. For all $x \in U_a \cap U_b$, there exists a germ o diffeomorphism γ^x_{ba} from $f_a(x)$ to $f_b(x)$ such that

$$f^x_b = \gamma^x_{ba} \circ f^x_a$$

where f^x_a and f^x_b are the germs of f_a and f_b at x. Again by the implicit functi theorem, we must have a cocycle condition:

$$\gamma^x_{ab} = \gamma^x_{ac} \circ \gamma^x_{cb} \quad \text{for} \quad x \in U_a \cap U_b \cap U_c.$$

This is just like bundle theory except that the structure (pseudo-) group (where γ^x_{ab} takes its values) is Γ_q, the germs of diffeomorphisms $\mathbb{R}^q \to \mathbb{R}^q$. If Γ_q is given the sheaf topology, the map $x \to \gamma^x_{ab}$ is continuous. We can imitate the classifying space construction for groups ([34] and [35]). For foliations, the differential, $d(\gamma^x_{ab})$ of the germ $\gamma^x_{ab} : \mathbb{R}^n \to \mathbb{R}^n$ is the $GL(q, \mathbb{R})$-valued cocycle of the normal bundle, TM/E, of the foliation.

Remark. This procedure extends the definition of a manifold so that the f_U's are submersions, not necessarily diffeomorphisms (charts). In this context, a codimension $q = n$ foliation of an n-dimensional manifold (where the "leaves" are just points) is the old definition of manifold, and the normal bundle to this trivial foliation is just the tangent bundle.

Now recall our procedure for characteristic classes for q-manifolds N^q via tautologous forms on the frame bundles. The mysterious classes were all in dimension $> q$, and, therefore, trivial in $H^*(N^q)$. However, these classes may be non-trivial on M^n with a codimension q foliation, $q < n$. A codim q foliation is like a q-manifold structure on M^n and we get a natural map:

$$H^*(\mathfrak{a}_q, so_q) \longrightarrow H^*(M)$$

Through dimension $2q, H^*(\mathfrak{a}_q, so_q)$ gives rise to the usual (Pontryagin, Euler) classes of the normal bundle TM/E of the foliation. (Via a Riemannian metric on M, the tangent bundle splits $TM = E \oplus TM/E$, and the normal bundle can be realized as vectors perpendicular to the foliation.) Technically, we must assume that the normal bundle be oriented, for then we can choose our γ^x_{ab} to be orientation preserving on $\mathbb{R}^q$. Otherwise we must replace SO_q above by O_q and we do not get an Euler class. Above dimension $2q$, $H^*(\mathfrak{a}_q, so_q)$ gives rise to exotic classes in $H^*(M)$.

For example, $H^*(\mathfrak{a}_1) = \mathbb{R}$ in dimensions 0 and 3. Since $SO_{(1)}$ is trivial class 1 in dim $=0$ and ω in dim $=3$. This is the Godbillon–Vey class for codimension 1 foliations (cf. [31]). An example of a foliated 3-manifold for which ω is non zero was found by Roussarie (see [6]). A large problem in the theory is to find examples of foliated manifolds with non-trivial exotic classes. Thurston has found many interesting examples of these (See [66], [68].)

Remark. In our earlier construction of canonical 1-forms on the higher frame bundles $\cdots \to F_2(M) \to F_1(M) \to M$, we consider the charts of diffeomorphisms $\mathbb{R}^n \to M$ (dim $M = n$). But we could equally well have taken our maps in the other direction, from open sets of M to $\mathbb{R}^n$.

For a fixed codim q foliation of M, consider the space G of germs of the local projections $f_a : u_a \to \mathbb{R}^q$ (which are constant along the leaves). We might as well assume that the target of each germ is $0 \in \mathbb{R}^q$ or compose with a translation to force this.

We can now take equivalence classes (jets) of the f_a's.

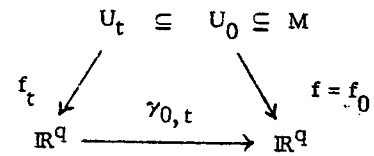
$F_0(M) =$ 0-jets $=$ M via the source map

$F_1(M) =$ 1-jets $=$ frame bundle of normal bundle of the foliation

$F_k(M) = k$-jets

$F_\infty(M) = \varprojlim F_R(M)$ via the projections $F_k(M) \to F_{k-1}(M)$.

In our earlier case, $q = n = \dim M$, $F_1(M) =$ frame bundle of M is t frame bundle of normal bundle of the foliation by points. Then as before, have tautologous forms ω^i on our new $F_1(M), \omega^i_j$ on $F_2(M)$, ω^i_{jk} on $F_3(M)$, a so forth, $i, j, k, \ldots = 1, 2, \ldots, q$. And we get a natural map $A^*_c(\mathfrak{a}_q) \to F_\infty($ given by $\delta^i \to \omega^i$, $\delta^i_j \to \omega^i_j$ etc. (Cf. [36].) For a tangent vector X on $F_R($ represented by $f_t : U_t \to \mathbb{R}^q$, $U_t \subset M$. We have the following picture:



Let X be a tangent vector at the point $j^k_x f_0 \in F_R(M)$, where $x \in M$ is source of $f_0 : U_0 \to \mathbb{R}^q$, $f_0(x) = 0$. X is represented by a curve $t \to j^k_{x(t}$ (where $f_t : U_t \to \mathbb{R}^q$; at $t = 0$ we get $f_0; f_t(x(t)) = 0$; and t is small enou so that $x(t) \in U_0$, $x(0) = x$). Let $\gamma_{0,t}$ be the diffeomorphism of $\mathbb{R}^q$ such tl $f_0 = \gamma_{0,t} \circ f_t$ holds in a neighborhood of $x(t)$. Then $t \to \gamma_{0,t}(p)$ defines a fam of C^1 curves through points p near $0 \in \mathbb{R}^q$. (For example, $\gamma_{0,t}(0) = f_0(x(t)$ the curve through the origin.) Let $Y = d\gamma_{0,t}/dt$ be the vector field tangent them. Since f_0, f_t, and $\gamma_{0,t}$ are defined only up to their k-jets at $0 \in \mathbb{R}^q$, Y defined only up to order $k-1$. Therefore, $\delta^i(Y), \delta^i_{j_1}(Y), \ldots, \delta^i_{j_1 \cdots j_{k-1}}(Y)$ defined and we define these to be the values of $\omega^i(X), \omega^i_j(X), \ldots, \omega^i_{j_1 \cdots j_{k-1}}($ on $F_k(M)$. Thus ω^i is defined on $F_1(M)$, ω^i and ω^i_j are defined on $F_2(M)$, e This is the map $A^*_c(\mathfrak{a}_n) \to$ tautological differential forms on $F_\infty(M)$.

$$\begin{array}{c} F_\infty(M) \\ \downarrow \\ \vdots \\ \downarrow \\ F_3(M) \sim \omega^i_{jk} \\ \downarrow \\ F_2(M) \sim \omega^i_j \\ \downarrow \\ F_1(M) \sim \omega^i \\ \downarrow \\ M \end{array}$$

The induced map in cohomology is $H^*(A_c(\mathfrak{a}_q)) \to H^*(F_\infty(M)) \to H^*(F_1(M))$ where the second arrow is via the sections $F_1(M) \to F_{i+1}(M)$ $i = 1, 2, \cdots$ which are defined uniquely up to homotopy. (The fibers of $F_{i+1}(M) \to F_i(M)$ are all affine spaces and thus contractible.) Now if the foliation of M has a trivialized normal bundle, then we have a section $M \to F_1(M)$ which pulls classes all the way down to $H^*(M)$. We get a map

$$H^*(A_c(\mathfrak{a}_q)) \to H^*(M).$$

Otherwise, we must divide out by the (maximal compact) subgroup $O(q)$ of $J \equiv$ infinite jets at the origin of diffeomorphisms $\mathbb{R}^q \to \mathbb{R}^q$ taking 0 to 0. If the normal bundle to the foliation is oriented, we can get away with $SO(q)$, since the structure group, J, of $F_\infty(M)$ can be reduced to the orientation preserving jets.

This defines (up to homotopy) a section:

$$M \longrightarrow F_1(M)/SO(q).$$

The natural map

$$H^*(\mathfrak{a}_q, so(q)) \to H^*(F_\infty(M)/SO(q)) \to H^*(M)$$

gives characteristic classes on M induced directly by the foliation. In the unoriented case, we get a map $H^*(\mathfrak{a}_q, O_q)) \to H^*(M)$. On $F_2(M)$ we have the formula (as in $A_c^*(\mathfrak{a}_q)$)

$$d\omega^i = -\sum \omega^i_k \wedge \omega^k$$

On $F_1(M)$ we do not have canonical ω^i_k forms, but the Frobenius theorem tells us that for an (integrable) foliation, there exists forms θ^i_k such that

$$d\omega^i = -\sum \theta^i_k \wedge \omega^k$$

The θ^i_k can be given as the pullbacks of the ω^i_k via a section $F_1(M) \to F_2(M)$. This section is equivariant with the action of $GL(n, \mathbb{R})$ if and only if the (θ^i_j) matrices transform by the adjoint action of $GL(n, \mathbb{R})$, that is the matrix (θ^i_j) is a connection on $F_1(M)$. Then the curvature is a matrix defined by

$$\Omega^i_j = d\theta^i_j - \theta^k_j \wedge \theta^i_k = \omega^k \wedge \theta^i_{jk}$$

where θ^i_{jk} is the pullback of ω^i_{jk} from $F_3(M)$ via a section. The invariant polynomials on the curvature matrix (Ω^i_j) are generated by $c_1, \dots, c_q$ which are (up to a scalar multiple) the coefficients of the characteristic polynomial of (Ω^i_j). The image of c_i in $H^{2i}(M)$ is the i-th Chern class of the normal bundle to the foliation. (See [6] for the definition of Chern classes via a curvature matrix.)

We obtain a refinement of the vanishing theorem (cf. [5]) that the ring of Pontryagin classes of the normal bundle to codimension-q foliation vanish in

dim $> 2q$ in $H^*(M)$. Let

$$W_q = E(h_1 h_2, \dots, h_q) \otimes p(c_1, c_2, \dots, c_q)$$
$$= \text{exterior alegbra in } \{h_i\} \text{ tensor the polynomial algebra in } \{c_i\}$$

We set $\dim h_i = 2i - 1$, $\dim c_i = 2i$. $d : W_q \to W_q$ is defined by $d(h_i)$ c_i, $d(c_i) = 0$. Recall that $H^*(W_q) = 0$. Let

$$WO_q = E(h_1, h_3, \dots, h_{2k+1}) \otimes P(c_1, c_2, \dots, c_q).$$

be the same except that we throw away all the even h_{2i}. $2k + 1 \leq q$ b $2k + 3 > q$. Let

$$W_q = W_q/\text{terms of } \dim > 2q \text{ in the } c_i\text{'s}.$$
$$WO_q = WO_q/\text{terms of } \dim > 2q \text{ in the } c_i\text{'s}.$$

We have seen (Theorem 20.1) that

$$H^*(\mathfrak{a}_q) = H^*(W_q)'$$

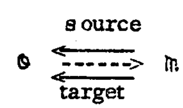
Also, it turns out (see [[10]) that

$$H^*(\mathfrak{a}_q, O(q) = H^*(WO_q)$$
$$H^*(\mathfrak{a}_q, so_q) = \begin{cases} H^*(WO_q), & q \text{ odd} \\ H^*(WO_q)[\chi]/(\chi^2 - c_q), & q \text{ even.} \end{cases}$$

In $H^*(\mathfrak{a}_q, so_q)$ we find the Pontryagin classes $c_2, c_4, \dots$ in dimensio 4,8,..., and, if q is even, the Euler class, χ, of the normal bundle. The Po tryagin classes all vanish in dimension $> 2q$ by the truncation. In $H^*(\mathfrak{a}_q)$ of the Chern classes $c_1, \dots, c_q$ (not just the odd ones) are coboundaries. Tl is reasonable since $H^*(\mathfrak{a}_q)$ gives characteristic classes in $H^*(M)$ when the nc mal bundle is trivial. Exotic classes (e.g., $h_1 c_q$) occur from dimensions $2q +$ through $q^2 + 2q$.

28. Haefliger's classifying space for foliations

We begin with the notion of a topological category. (Cf. [6], [56], [3 and [35].) This means that the objects, θ, and the morphisms, m, are bo toopological spaces and that the maps



and the composition of (composable) morphisms are all continuous. The dotte arrow takes an object, A, to the idenity morphism $A \to A$.

Example. Let G be a topological group. Let θ = points of G, and let m = left translations. For any two objects $g, h \in G$ there is exactly one morphism, $L_{hg} - 1$, between them. Therefore, $m \simeq G \times G$. The three maps

$G \leftleftarrows G \times G$	given by:
$g \longleftarrow (g,h)$	source
$x \longrightarrow (x,x)$	identity
$h \longleftarrow (g,h)$	target

are all continuous.

Another Example. Let $\theta =^*$ a single point, and let $m = G$ with composition as in G. We get three fairly trivial maps:

$$* \leftleftarrows \dashrightarrow G \qquad \text{where} \quad * \dashrightarrow \mathrm{id} \in G.$$

We now wish to define a topological category, Γ_q. The objects and morphisms are given by

$$\theta(\Gamma_q) = \text{points of } \mathbb{R}^q$$
$$m(\Gamma_q) = \text{germs of diffeomorphisms of } \mathbb{R}^q.$$

$m(\Gamma_q)$ has two possible topologies of interest. In the soft (C^∞) topology, two germs are close if their source and target, as well as derivatives up to some order, are close. But we will take here the sheaf topology: if $f : U \to V$ is a diffeomorphism with $x \in U$, let f^x denote the germ at x. A neighborhood of f^x is $\{f^y\}_{y \in U}$. This is a very non-Hausdorff topology and the induced topology on the germs with source x is discrete. The maps

$$\mathbb{R}^q = \theta \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows}} m$$

are local homeomorphisms and give m a (non-Hausdorff) smooth n-manifold structure. Recall how we constructed the classifying space BG from the semi-simplicial space NG

$$* \leftleftarrows G \leftleftarrows G \times G \cdots$$

Now consider the semi-simplicial space $N\Gamma_q$

$$m_0\Gamma_q \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows}} m_1(\Gamma_q) \leftleftarrows m_2(\Gamma_q) \leftleftarrows \cdots$$

where

$m_0\Gamma_q = \mathbb{R}^q$ = objects $\theta(\Gamma_q)$

$m_1(\Gamma_q) = m(\Gamma_q)$ = germs of diffeomorphisms of $\mathbb{R}^q$

$m_2(\Gamma_q)$ = pairs of composable germs (i.e, target one = source of other)

$m_3(\Gamma_q)$ = composable triples, etc.

The $k+1$ maps $m_k(\Gamma_q) \to m_{k-1}(\Gamma_q)$ are given by the same formulas as f NG.

$$\partial_i(g_1, \ldots, g_k) = \begin{cases} (g_2, \ldots, g_k) & i = 0 \\ (g_1, \ldots, g_i \circ g_{i+1}, \ldots, g_k) & 0 < i < k \\ (g_1, \ldots, g_{k-1}) & i = k. \end{cases}$$

The ordinary geometric realization is $|N\Gamma_q|$, but to get a classifying space w must use the unwound space $B\Gamma_q = |(N\Gamma_q)_z|$ (see Sections 4 and 6 above, an [6], Stasheff's appendix). Γ_q with the sheaf topology is the "structure grouр for foliations in the sense that the map

$$U_a \cap U_b \longrightarrow m(\Gamma_q)$$

given by

$$x \longrightarrow \gamma^x_{ab}$$

is continuous in the sheaf topology. A foliation on M induces a homotopy cla of maps

$$M \longrightarrow B\Gamma_q,$$

so that the characteristic classes for foliations are to be found in $H^*(B\Gamma_q$ Thus we have a map $H^*(\mathfrak{a}_q, O(q)) \to H^*(B\Gamma_q)$. However, we must identi foliations by a suitable notion of homotopy to make a homotopy class of ma $M \to B\Gamma_q$ represent a unique (homotopy class of) foliation. (See [6], [35].)

Actually, for any topological space, X, a map $X \to B\Gamma_q$ makes sens but what would a foliation on X be? We see that $B\Gamma_q$ actually classifi Haefliger structures (defined below), of which foliations are a special case. X is paracompact, we have ([35] p. 141):

Theorem 28.1. (Haefliger). *Homotopy classes of codim q Haefliger structures on X are in one-to-one correspondence with homotopy classes of continuous maps $X \to B\Gamma_q$.*

Definition. A codimension q Haefliger structure on a space X is defined by an open cover $\{U_a\}$ of X and continuous maps

$$f_a : U_a \longrightarrow \mathbb{R}^q$$

not necessarily submersions, and a continuous map

$$U_a \cap U_b \longrightarrow m(\Gamma_q)$$
$$x \longrightarrow \gamma^x_{ab}$$

such that

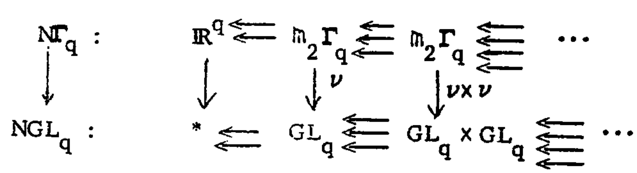
(1) $f^x_a = \gamma^x_{ab} f^x_b$
(2) $\gamma^x_{ab} = \gamma^x_{ac}\gamma^x_{cb}$ for $x \in U_a \cap U_b \cap U_c$

For a foliation, (2) follows from (1) since the f_a are submersions in this special case. Even if all the f_a are constant maps to $0 \in \mathbb{R}^q$, we get a non-trivial Haefliger structure if we take the γ^x_{ab} to be non-trivial. The normal bundle of a Haefliger structure is the vector bundle whose $GL(q,\mathbb{R})$-valued transition functions, g_{ab}, are given by $d(\gamma^x_{ab})$, the differential of γ^x_{ab} as a map $\mathbb{R}^q \to \mathbb{R}^q$.

The advantage of passing to Haefliger structures is that any continuous map, $N \to M$, between manifolds pulls back Haefliger structures on M to ones on N. For foliations, this only works if $N \to M$ is transverse to the leaves on M.

We will not in general have smooth leaves from a Haefliger structure. Locally the level sets (inverse images of points of $\mathbb{R}^q$) are defined, but these can have singularities. Nevertheless, these can be smoothed out in certain cases. By the Phillips–Gromov theorem, a Haefliger structure on an open (no compact component) manifold M whose normal bundle is a subbundle of the tangent bundle of M is homotopic to a smooth foliation ([35]). Thurston has a similar result for compact manifolds, when codim. of the foliation ≥ 2. [67]

Remark. The derivative of a germ $\nu : \Gamma_q \to GL(q,\mathbb{R})$ induces maps



and, therefore, induces a map

$$B\Gamma_q \xrightarrow{\nu} BGL_q$$

so that if $f : M \to B\Gamma_q$ classifies a foliation on M, then $\nu \circ f$ classifies t normal bundle. Then the vanishing theorem ([5]) says

Theorem 28.2. (Bott) *$\nu^* : H^*(BGL_q) \to H^*(B\Gamma_q)$ is 0 in dimensi $> 2q$.*

$H^*(BGL_q)$ and $H^*(B\Gamma_q)$ are equal to the total cohomology of the dou complexes $S^p(GL_q \times \cdots \times GL_q)$ (k factors) and $S^p(m_k(\Gamma_q))$, where S^p = singu p-cochains. (See Theorems 4.1, 4.2, 4.4.) Now I claim

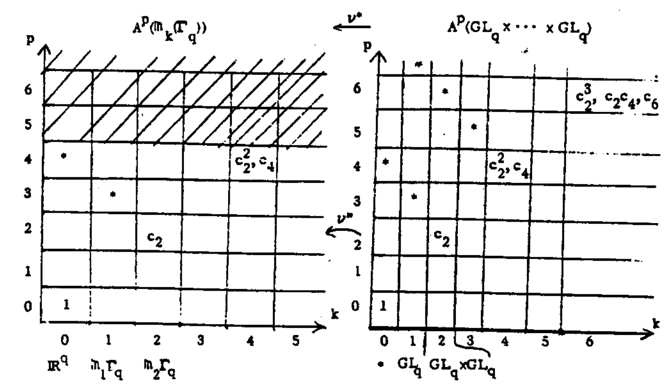
$$\nu^* : H^p(GL_q \times \cdots \times GL_q) \to H^p(m_k(\Gamma_q))$$

is 0 in dimension $p > q$. This is because ν^* factors through a map:

$$\begin{array}{ccc} H^p(GL_q \times \cdots \times GL_q) & \xrightarrow{\nu^*} & H^p(m_k(\Gamma_q)) \\ \simeq \downarrow & & \uparrow \\ H^p_{DR}(GL_q \times \cdots \times GL_q) & \longrightarrow & H^p_{DR}(m_k(\Gamma_q)) \end{array}$$

where H^p_{DR} = DeRham cohomology and the bottom arrow is zero for $p > q$ dim $m_k(\Gamma_q)$ (sheaf topology!).

Why do we then have a vanishing theorem for dim $> 2q$? Recall that 3-form on $m_2(\Gamma_q)$ counts as a form of total dimension 3+2 =5. For definitene we illustrate with q=4.



In the double complex, $A^p(m_k(\Gamma_q)) = 0$ for $p > q$ (in the shaded region). $A^p(GL_q \times \cdots \times GL_q)$, the Pontryagin classes appear on the diagonal as E terms which are extended to total $D = d \pm \delta$ cocycles by adding forms, *, ι and to the left only. (See Section 3.) For $q = 4$ as shown, the classes $c_2^3, c_2c_4,$

together with the forms to their upper left go to 0 (the shaded region). The highest degrees shown on the left which are non-zero are c_2^2 and c_4 in the (4,4) box with total degree 4+4 =8 =2q.

Appendix

Introduction

We briefly review some of the standard facts about Lie algebras. (See [59], [22].) We define the cohomology of a Lie algebra, g, with coefficients in a g-module, V, denoted by $H^*(g,V)$ or $H^*(g,\rho)$ where $\rho : g \to \text{End } V$ is the corresponding representation. If k is the base field (characteristic 0, but we only need $k = \mathbb{R}$ or $k = \mathbb{C}$) and g acts trivially on k, then $H^*(g,k)$ is just $H^*(g)$ as defined earlier. The basic results are (II.2) the following are equivalent:

(i) g has no Abelian ideals. (Abelian means $[X,Y] = 0$ for all X,Y.)
(ii) Every g-module, V, is a direct sum of simple submodules $V = \oplus V_i$, where each V_i has non-trivial submodules.

Such g are called *semi-simple* Lie algebras. Let $V^g = \{v \in V | \rho_X(v) = 0$ for all $X \in g\}$ be the *invariants* of V. Then we also show (IV. 2. 1) if g is semi-simple then $H^*(g,V) \simeq H^*(g,V^g) =$ as many copies of $H^*(g)$ as the dimension of V^g. (See [14] for details.)

In the application to Gel'fand–Fuks cohomology, we will want to use $g = gl(n,\mathbb{R})$, which is not semi-simple. However, $gl(n,\mathbb{R}) = sl(n,\mathbb{R}) \oplus Z$, where $sl(n,\mathbb{R})$ is semi-simple and $Z = \{X \in gl(n,\mathbb{R}) | [X,Y] = 0$ for all $Y \in gl(n,\mathbb{R})\}$ = the center of $gl(n,\mathbb{R})$ are only scalars times the identity matrix. If $x^1,\ldots,x^n$ are coordinates on $\mathbb{R}^n$, we can identify $gl(n,\mathbb{R})$ with the subalgebra with basis $\{(x_i)\partial/\partial x\}_{i,j=1}^n$ of a_n = formal vector fields on $\mathbb{R}^n$. The center, Z, is generated by the radial vector field $R = \sum(x_i)\partial/\partial x_i$, which corresponds to the identity matrix. Recall how R was used to reduce computations to "the zero-eigenspace of R" by the homotopy $\theta_R = \iota_R d + d\iota_R$ = the scalar λ on the λ-eigenspace. More precisely, we are concerned with $gl(n,\mathbb{R})$-modules, V, which break up as a direct sum $V = \oplus V_\lambda$ of eigenspaces for R, which acts on V_λ by scalar multiplication by λ. For $\lambda = 0$ we get the R-invariants $V^Z = V_0$. In this case also, we get (IV. 2. 2) $H^*(g,V) = H^*(g,V^g)$. Since V^g is often much smaller than V, we have achieved a reduction of the computations after we find what V^g is.

There are four paragraphs in this appendix:

I. Bilinear form of a representation
II. No Abelian ideals in $g \Leftrightarrow$ all g-modules are semi-simple
III. Special case: $sl(n,k)$
IV. Cohomology with coefficients in a module

I. In this section, we show that the bilinear form, B_ρ, of a faithful representatic of a semi-simple Lie algebra is non-degenerate.

Definitions. g will denote a finite dimensional Lie algebra over a field of characteristic 0. Define $C_g^1 = D^1 g = g$, and $C^r(g) = [g, C^{r-1}(g)]$, $D^r(g)$ $[D^{r-1}g, D^{r-1}g]$ for $r > 1$ an integer. Then g is *nilpotent* (resp. *solvable*) $C^n(g) = 0$ (resp. $D^n(g) = 0$) for some integer n. g is *semi-simple (s.s)* if has no Abelian ideals (other than 0). V will denote a finite dimensional vector space. A *representation* of g is a Lie algebra homomorphism $\rho : g$ · $\text{End}_k(V)$. ρ equips V with a g-*module* structure. ρ is *simple* if V contains ı non-trivial sub g-modules. ρ is *semi-simple* if V is a direct sum of simple sı g-modules. By finite dimensionality, this is equivalent to saying that for aı sub g-module W of V, $V \simeq W \oplus V/W$. $\rho : g \to \text{End}_k(V)$ is *faithful* if it one-to-one. If V is a g-module (via ρ) we often write $X \cdot v$ instead of $\rho_X(v)$ f $X \in g, v \in V$. A g-module structure is defined on V^* = dual of a g-module ' given by $(X \cdot f)(v) = -f(X \cdot v)$, for $X \in g$, $f \in V^*$, $v \in V$. If V and W a g-modules, then $V \oplus W$ and $V \otimes W$ are g-modules: For $X \in g$, $v \in V$, $w \in$ I we define $X\cdot(v \oplus w) = (X \cdot v) \oplus (X \cdot w)$ and $X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot u$ (These are just the rules for the derivative of sums and products when g is t Lie algebra of a Lie group acting on V and W.) In particular if V is a g-modu then so is the tensor algebra $\oplus(V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*)$. The alternati: and symmetric summands, $\Lambda^* V$ and $S^* V$ (and also $\Lambda^* V^*$ and $S^* V^*$) are, thı g-modules.

Theorem I.1. *If g is semi-simple and ρ is faithful, then the quadra form $B_\rho(X,Y) = Tr_V(\rho_X \rho_Y)$, for X and $Y \in g$, is non-degenerate.*

Proof. Let $b = \{X \in g | Tr(\rho_x \rho_Y) = 0$ for all $Y \in g\}$. To show $b =$ b is an ideal since $B_\rho([X,Y],Z) = B_\rho(X,[Y,Z])$ for all $X,Y,Z \in g$. We w show that b is solvable. But then if $b \neq 0$, the last non-zero derived, D^{n-1}(will be an Abelian ideal of g, contradicting semi-simplicity. We will show tł b is solvable by showing that $[b,b]$ is nilpotent. We use only the fact tł $B_\rho(X,Y) = 0$ for all X in b and for all $Y \in [b,b]$. (Cf. Cartan's criterion in a book on Lie algebras, e.g., [ ].)

To show that $[b,b]$ is nilpotent, it suffices to show that any $y \in [b,b]$ is nilı tent (via ρ) as an endomorphism of V. For then left and right multiplicatic L_y and R_y, are both nilpotent on End(V), and, therefore, so is $ad_y = L_y -$. (since $L_y R_y = R_y L_y$, $(L_y - R_y)^{m+n+1} = 0$ if $L_y^m = 0 = R_y^n$). Then by Enge theorem (proven after the present proof), $[b,b]$ is nilpotent because it has a fl $0 = b_0 \subset b_1 \subset b_2 \subset \cdots \subset b_r = [b,b]$ with $ad_X(b_i) \subseteq b_{i-1}$ for $i = 1,\ldots,r$, a for all $X \in [b,b]$.

Note that this theorem is linear in the sense that the hypothesis and t conclusion each remain true or remain false when the base field is extended restricted. Thus, we can assume k is algebraically closed. In fact, by choosi bases for g and V we can restrict k to a subfield k' so that (1) g and V are still defined over k' as a Lie algebra and module, and (2) k' is finitely generated over $\mathbb{Q} \subseteq k$. By embedding $k' \to \mathbb{C}$ and extending, we can even reduce to the case $k = \mathbb{C}$.

Lemma. *Let $y \in End_k(V)$. Then there exists a unique diagonalizable s and nilpotent n in $End_k(V)$ such that $y = s + n$ and $sn = ns$. Moreover, s and n are polynomials $S(y)$ and $N(y)$ of y with $N(0) = 0 = S(0)$.*

Proof. The characteristic polynomial of y in the indeterminant T is det $(T - y) = \Pi(T - \lambda_i)^{m_i}$, where λ_i are the eigenvalues of y with multiplicity m_i. For each i, let $V_i = \ker\ (y - \lambda_i)^{m_i} \subseteq V$ be the eigenspace of λ_i, $V = \oplus V_i$. Define $s|_{V_i} = \lambda_i$ and $n = y - s$ is $y - \lambda_i$ on V_i. Since $(y - \lambda_i)^{m_i} = 0$ on V_i, n is nilpotent. Also, $ns = sn$. If $y = s + n$ is any other such decomposition, then it is easy to check that s must be as above. Finally, let $S(T)$ be any polynomial satisfying $S(T) = \lambda_i \bmod (T - \lambda_i)^{m_i}$ and $S(0) = 0$. Let $N(T) = T - S(T)$. Then $N(0) = 0 = S(0)$ and $s = S(y)$ since on $V_i, (y - \lambda_i)^{m_i} = 0$ so that $S(y) = \lambda_i$ on V_i. Also $N(y) = y - S(y) = y - s = n$. Q. E. D. for Lemma.

Since ρ is faithful, we will occasionally forget to write it and identify elements of the Lie algebra with the corresponding endomorphisms of V. Our $Y \in [b, b]$ has a decomposition $Y = s + n$ as in the lemma where s (resp. n) is a diagonalizable (resp. nilpotent) endomorphism of V. We must show $s = 0$. I claim it is enough to show $tr(Y \circ \varphi(s)) = 0$ for all $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(k, \mathbb{Q})$ where $\varphi(s)$ is defined since s is diagonalizeable: If Y and s have eigenvalue λ_i with multiplicity m_i on $V_i \subseteq V$, then $\varphi(s)$ is defined to be $\varphi(\lambda_i)$ on V_i. Also $Y \circ \varphi(s)$ has eigenvalue $\lambda_i \varphi(\lambda_i)$ on V_i. Therefore:

$$0 = tr(Y \circ \varphi(s)) = \sum m_i \lambda_i \varphi(\lambda_i).$$

Applying φ again,

$$0 = \varphi(0) = \sum m_i \varphi(\lambda_i)\varphi(\lambda_i)$$

Since $\varphi(\lambda_i) \in \mathbb{Q}$ where φ is identity. But then $\varphi(\lambda_i) = 0$ for all i. Since $\varphi(\lambda_i) \in \mathbb{Q}$ was arbitrary, $\lambda_i = 0$ for all i. Thus $s = 0$, and $Y = n$ is nilpotent.

The trouble is that $\varphi(s) \in \mathrm{End}_k(V)$ may not be in $b \subseteq \mathrm{End}_k(V)$ (or even in g), so we cannot apply the definition of b directly. Since $Y \in [b, b]$, $Y = \sum_r [X_r, Z_r]$ for $X_r, Z_r \in b$.

$$tr(Y \circ \varphi(s)) = \sum_r tr([X_r, Z_r] \circ \varphi(s)) = \sum tr(X_r \circ [Z_r, \varphi(s)]).$$

Since $X_r \in b$ it suffices to show $[Z_r, \varphi(s)] \in b$ to get 0. In fact it is in $[b, b]$. Let ad denote the adjoint representation of $\mathrm{End}_k(V)$ on itself. Since ad is linear, $ad_Y = ad_s + ad_n$ (for $Y = n + s$). This is the canonical decomposition (in the lemma) of ad_Y because (1) $ad_n = L_n - R_n$ is nilpotent, (2) $ad_S = \lambda_i - \lambda_j$ on the eigenspace $V_i \otimes V_j^* \subseteq V \otimes V^* \simeq \mathrm{End}(V)$, and (3) $[ad_S, ad_n] = ad_{[s,n]} = ad_0 = 0$.

Thus $ad_s = P(ad_Y)$ where $P(T)$ is a polynomial with $P(0) = 0$. On $V_i \otimes V_j^*$ we have:

$$ad_{\varphi(s)} = \varphi(\lambda_i) - \varphi(\lambda_j) = \varphi(\lambda_i - \lambda_j) \quad \text{(by linearity of } \varphi)$$
$$= \varphi(ad_s) \quad \text{on} \quad V_i \otimes V_j^*$$

Thus, $ad_{\varphi(s)} = \varphi(ad_s) = Q(ad_s)$ where Q is a polynomial satisfying $Q(\lambda_i - \lambda_j) = \varphi(\lambda_i - \lambda_j)$. Then:

$$[\varphi(s), Z_r] = ad_{\varphi(s)}(Z_r) = \varphi(ad_s)(Z_r)$$
$$= Q(P(ad_Y))(Z_r) \in [b, b]$$

Since Q and P are polynomials, and $Y, Z_r \in b$. Q. E. D

Definition. For $\rho = ad =$ the adjoint representation of g on itself $B_{ad}(X, Y) = tr(ad_X ad_Y)$ is called the *Killing form* of g.

Corollary I.1.1. *g is semi-simple if and only if the Killing form is non degenerate.*

Proof. The kernel of ad is just the center of g, which is 0 if g is semi-simple (no Abelian ideals except 0). Apply I.1. Conversely, suppose $X \neq 0$ is in an Abelian ideal $\mathfrak{a} \subseteq g$. Let $Y \in g$. Then $ad_X \circ ad_Y$ takes g into $\mathfrak{a}$, and $\mathfrak{a}$ into 0. Thus, $(ad_X \circ ad_Y) = 0$. Since $Y \in g$ was arbitrary, the Killing form is degenerate: $B_{ad}(X, Y) = 0$ for all $y \in g$.

We have omitted the proof of the fact that g is nilpotent if and only if ad_X is nilpotent for each $X \in g$. This is a corollary of:

Theorem I.2. (Engel): *Let $\rho : g \to End(V)$ be a representation such that $\rho(X)$ is nilpotent for each $X \in g$. Then there exists a flag: $0 \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V$ such that $\rho(g) : V_i \to V_{i-1}$.*

Proof. The converse is trivial. The problem is to get one flag that works for all $X \in g$. By induction it suffices to show if $V \neq 0$ then there exists $v \in V, v \neq 0$ such that $\rho(g)(v) = 0$. Since the theorem only involves the image $\rho(g)$ of g, we assume $g \subseteq \mathrm{End}\ (V)$. For all $X \in g$, ad_X is nilpotent since $ad_X = L_X - R_X$ where left and right multiplication by X on $\mathrm{End}(V)$, L_X and R_X, are both nilpotent and commute.

Let $h \subset g$ be a maximal subalgebra, $h \neq g$. Then I claim h is an ideal and codim $(h) = 1$. We can assume the theorem for h, since it has smaller dimension than g, by an induction hypothesis. Since h acts on g/h by nilpotents, there exists an $X \in g - h$ such that for all $Y \in h, [Y, X] \in h$. Since $h + (X)$ is a larger subalgebra than $h, h + (X) = g$. h is an ideal of codim. 1 since $[h, X] \subseteq h$.

Finally, let $W = V^h = \{v \in V | X \cdot v = 0 \text{ for all } X \in h\}$. Then $g(W) \subset W$ since h is an ideal. $W \neq 0$ by induction since $\dim h < \dim g$. For our $X \in$

$g-h$, X is nilpotent so it kills something $\neq 0$ in W. That something is then killed by $g = h + (X)$. Q. E. D.

II. In this section, we show that if g is semi-simple, then every finite dimensional g-module is semi-simple, and conversely. First we prove the result for g as a module over itself via the adjoint action.

Theorem II. 1. *Let $\mathfrak{a}$ be an ideal of the semi-simple Lie algebra g. Let $\mathfrak{a}^\perp$ be the orthogonal complement of $\mathfrak{a}$ with respect to the non-degenerate bilinear form $B_{ad}(XY) = tr(ad_X ad_Y)$. That is, $\mathfrak{a}^\perp = \{x \in g | B_{ad}(X,Y) = 0$ for all $Y \in \mathfrak{a}\}$. Then $\mathfrak{a}^\perp$ is an ideal of g and $g = \mathfrak{a} \oplus \mathfrak{a}^\perp$.*

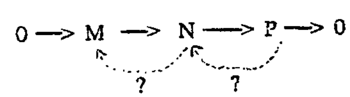
Proof. Let $X \in \mathfrak{a}^\perp$, $Y \in g$, $Z \in \mathfrak{a}$. Then $B([X,Y],Z) = B(X,[Y,Z]) = 0$ since $[Y,Z] \in \mathfrak{a}$. Since Z is arbitrary in $\mathfrak{a}$, $[X,Y] \in \mathfrak{a}^\perp$, so $\mathfrak{a}^\perp$ is an ideal. Extending a basis for $\mathfrak{a}$ to a basis for g and writing B_{ad} as a symmetric matrix (non-singular), $\mathfrak{a}^\perp$ is the image of the vector space complement to $\mathfrak{a}$ by B_{ad}^{-1}. So $\dim \mathfrak{a} + \dim \mathfrak{a}^\perp = \dim g$. So it suffices to show $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$. But ad is faithful, and $tr(ad_X ad_Y) = 0$ for all $X, Y \in \mathfrak{a} \cap \mathfrak{a}^\perp$. But this implies $\mathfrak{a} \cap \mathfrak{a}^\perp$ is solvable by the same proof that b was solvable in the proof of I.1, since in that proof all we used was the fact that for all $X, Y \in b$, $tr(\rho_X \rho_Y) = 0$, and we now take $\rho = ad$. (Cf. Cartan's Criterion.)

Corollary II. 1.1. *If g is semi-simple then $g = [g,g]$.*

Proof. g contains an ideal isomorphic to $[g,g]^\perp \simeq g/[g,g]$, which is Abelian.

Theorem II. 2. *Let g be a finite dimensional semi-simple Lie algebra. Then every finite dimensional representation is semi-simple.*

Proof. By finite dimensionality it suffices to show that every exact sequence of g modules and g-module homomorphisms splits:

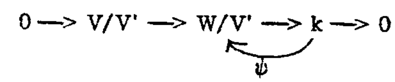


1) Reduction to the case $P = k$, the base field with the trivial g-module structure. Let $W = \{f \in \mathrm{Hom}_k(N,M) | f|_M$ is the scalar multiplication$\}$ and $V = \{f \in \mathrm{Hom}_k(N,M) | F|_M = 0\}$. For $f \in \mathrm{Hom}_k(N,M)$, $X \in g$, $X \circ f \in \mathrm{Hom}_k(N,M)$ is defined by $(X \cdot f)(n) = X(f(n)) - f(Xn)$ for $n \in N$. Then $X(W) \subset W$ (in fact $X(W) \subset V$) and $X(V) \subset V$. And we have an exact sequence of g-modules and g-homomorphisms.

$$0 \longrightarrow V \longrightarrow W \longrightarrow k \longrightarrow 0.$$

Assuming such sequences split, let f be the image of $1 \in k$ by a splitti (g-homomorphism) map. Since g acts trivially on k, g acts trivially on For all $X \in g$, $n \in N$, $0 = (X \cdot f)(n) = X(f(n))$ implies $f \in \mathrm{Hom}_g(N, M$ is the required splitting map since $f|_M = 1$. It remains to show $0 \to V$ · $W \to k \to 0$ splits.

2) Reduce to the case V is simple. If $V' \subset V$ is a ($\neq 0$) sub g-module, then



splits by an inductive hypothesis on the dimension of W (one supposes th theorem for modules of smaller dimension). Let $V^0 \subseteq W$ be the inver image of $\psi(k) \subseteq W/V'$. Then $V' \hookrightarrow V^0$. Also $0 \to V' \to V^0 \to k \to 0$ spli by the same inductive hypothesis on dimension. This gives the require map $k \to W$.

3) Reduce to the case ρ faithful on V. Let $h = \{X \in g | X \cdot (V) = 0\}$ be th kernel of $\rho|_V$. Then h is semi-simple, for an Abelian ideal of h is one $g = h \oplus h^\perp$. Now for $X \in h$, $X \cdot (W) \subset V$ because

$$\begin{array}{ccccc} & & W & \longrightarrow & k \\ & & \downarrow \rho_X & & \downarrow 0 \\ V & \hookrightarrow & W & \longrightarrow & k \end{array}$$

commutes, i.e., $X \cdot W \to 0$ in k so it comes from V. Also $h(V) = 0$. Bu h is semi-simple so $h = [h,h]$ takes W to 0 and we can replace g by $g/$ throughout.

Now let B_ρ be the (non-degenerate) form for $\rho : g \hookrightarrow \mathrm{End}(V)$ (we assum ρ to be faithful). Let $\{Y_i\}$ and $\{Z_i\}$ be dual bases of g so that $B_\rho(Y_i, Z_j)$ = $Tr(\rho_{Y_i}\rho_{Z_j}) = \delta_{ij}$, the Kronecker delta. The *Universal enveloping* alegbra U_g, of g is the (associative) tensor algebra $k \oplus g \oplus (g \otimes g) \oplus \cdots$ divided by th relations generated by $X \otimes Y - Y \otimes X - [X,Y]$, for $X, Y \in g$. If V is a g-modu (via ρ), then V is also a $U(g)$-module: For $X = X_1 \otimes X_2 \otimes \cdots \otimes X_r \in U(g$ $v \in V$, $\rho_X(v) = \rho_{X_1} \circ \rho_{X_2} \circ \cdots \circ \rho_{X_r}(v)$. Conversely, any module of the associati algebra $U(g)$, is also a module of the Lie algebra g because of the relations i $U(g)$. In particular $U(g)$, itself is a g-module. The invariants, $U(g)^g$ is ju the center of $U(g)$. Define the *Casimir* element $K = K_\rho = \sum_i Y_i \otimes Z_i \in U(g$ Note that for all $X \in g$, $X = \sum_i B_\rho(Z_i, X) Y_i = \sum_i B_\rho(Y_i, X) Z_i$. *$K$ is wel defined,* for if $\{R_i\}$ and $\{S_i\}$ are another pair of dual bases then $\sum_j R_j \otimes S_j$ = $\sum_{i,j} B(Z_i, R_j) Y_i \otimes S_j = \sum_{i,j} Y_i \otimes B(R_j, Z_i) S_j = \sum_i Y_i \otimes Z_i$. *$K$ is in the cente* *of* $U(g)$ because for all $X \in g$,

$$X \otimes K - K \otimes X = \sum_i (X \otimes Y_i \otimes Z_i - Y_i \otimes X \otimes Z_i + Y_i \otimes X \otimes Z_i$$
$$- Y_i \otimes Z_i \otimes X) = \sum_i ([X, Y_i] \otimes Z_i + Y_i \otimes [X, Z_i])$$
$$= \sum_i (B_\rho([X, Y_i], Z_i) Y_i \otimes Z_i + Y_i \otimes B_\rho(Y_i, [X, Z_i]) Z_i) = 0$$

since $B_\rho([Y, X], Z) = B_\rho(Y, [X, Z])$ for all $X, Y, Z \in g$.

Consequently, the *Casimir* operator, $\rho(K) = \sum_i \rho_{Y_i} \circ \rho_{Z_i}$ commutes with the action of g on W. Now $\rho(K)|_V$ is either 0 or an isomorphism of V since ker $(\rho(k)|_V)$ is a sub g-module of V and V is simple. But $\rho(K)|_V \neq 0$ since $Tr_V(\rho(K)) = Tr(\sum_i \rho_{Y_i} \circ \rho_{Z_i}) = \sum_i B_\rho(Y_i, Z_i) = \dim g \neq 0$. Therefore, $\rho(K)|_V$ is an isomorphism of V. Since $\rho(K)(W)$ goes to 0 in k (where g acts trivially), $\rho(K)(W) \subseteq V$. Then $\rho(K)|_V)^{-1} \circ \rho(K)$ is the required splitting map:

$$0 \to V \to W \to k \to 0$$

Q. E. D.

Converse II. 2.1. *Suppose every representation of g is semi-simple. Then g is semi-simple.*

Proof. g is a g-module under the adjoint representation. If g has any Abelian ideals then g is a direct sum $g = g_1 \oplus \cdots \oplus g_r$ of simple submodules (ideals), where, say g_1 is Abelian, therefore, 1-dimensional. $g_1 \simeq k$, the base field. Define the representation $\rho : g \to \mathrm{End}_k(k^2)$ by:

$$a \xrightarrow{\rho} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

for $a \in g_1 \simeq k$, and $\rho = 0$ on $g_2, \ldots, g_r$. Then ρ is not a semi-simple representation, since $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ generates a submodule, V, of k^2 without a complementary submodule, W, such that $k^2 = V \oplus W$ as g-modules.

III. We compute the Killing form of $sl(n, k)$ and show that $sl(n, k)$ is semi-simple.

Exercise. Let M, N be finite dimensional k-vector spaces. Let $x \in \mathrm{End}(M), Y \in \mathrm{End}(N)$. Define $X \otimes Y \in \mathrm{End}(M \otimes N)$ by $(X \otimes Y)(m \otimes n) = X(m) \otimes Y(n)$ for $m \in M$, $n \in N$. Then $Tr_{M \otimes N}(X \otimes Y) = Tr_M(X) Tr_N(Y)$.

Let $V = k^n$. Any $X \in \mathrm{End}(V)$ acts on V by the canonical representation, ρ. X acts on the dual V^* by $(\rho_X^*(b^*))(v) = -b^*(X \cdot v)$ for $b^* \in V^*$, $v \in V$.

So we write $\rho_X^* b^* = -b^* X$. X acts on $V \otimes V^*$ $=\mathrm{End}(V)$ as a derivation $X \cdot (a \otimes b^*) = Xa \otimes b^* - a \otimes b^* X = [X, a \otimes b^*]$, where (by choosing a basis fo V) we write $a \in V$ as a column, $b^* \in V^*$ as a row, and X as a matrix. This i just the usual adjoint representation, ad, if we identify $a \otimes b^* \in V \otimes V^*$ wit the $n \times n$ matrix $ab^* \in \mathrm{End}(V)$.

Computation. Suppose $X, Y \in sl(n, k) \subseteq \mathrm{End}(V)$ have trace $=0$. Then

$$Tr_{V \otimes V^*}(ad_X ad_Y) = Tr_{V \otimes V^*}(\rho_X \otimes 1 + 1 \otimes \rho_X^*)(\rho_Y \otimes 1 + 1 \otimes \rho_Y^*)$$
$$= Tr_{V \otimes V^*}(\rho_X \rho_Y \otimes 1 + \rho_X \otimes \rho_Y^* + \rho_Y \otimes \rho_X^*$$
$$+ 1 \otimes \rho_X^* \rho_Y^*) = Tr_V(\rho_X \rho_Y) Tr_{V^*}(1) + Tr_V(\rho_X) Tr_{V^*}(\rho_Y^*)$$
$$+ Tr_V(\rho_Y) Tr_{V^*}(\rho_X) + Tr_V(1) Tr_{V^*}(\rho_X^* \rho_Y^*)$$
$$= Tr(XY) \cdot n + 0 + 0 + nTr(Y^t X^t)$$
$$= 2n\, Tr(XY) = 2n\, B_\rho(X, Y)$$

On the other hand, $V \otimes V^* \simeq sl(n, k) \oplus Z$ where Z = center of $V \otimes V^*$ $=\mathrm{End}(V$ is generated by the identity matrix. For all $X \in \mathrm{End}\ (V), ad_X(Z) = 0$.

$$Tr_{V \otimes V^*}(ad_X ad_Y) = Tr_{sl(n,k)}(ad_X ad_Y) + Tr_Z(ad_X ad_Y) = B_{ad}(X, Y) + 0$$

Therefore, we have

Proposition III.1. *The Killing form, $B_{ad}(X, Y) = 2nB_\rho(X, Y) = 2nTr(XY)$ is non-degenerate, and $sl(n, k)$ is semi-simple. The correspondin Casimir elements also differ by a scalar, $K_\rho = 2n \cdot K_{ad}$.*

IV. In this section, we define Lie algebra cohomology with coefficients in module and describe conditions under which we can replace the coefficien module by the submodule of invariants.

Notations and basic lemmas. Let g be a finite dimensional Lie algebr and V a finite dimensional vector space over a field k of characteristic 0. Le $\rho : g \to \mathrm{End}(V)$ be a Lie homomorphism, giving V a g-module structure $\Lambda^* g^* = \oplus_{n=0}^{\infty} (\Lambda^n g^*)$ is the exterior algebra of the dual, g^*, of g. The adjoin representation of g on itself extends to a representations of g on $\Lambda^* g$ (see Sectio I). We also denote the dual representation on $\Lambda^* g^*$ by $ad : g \to \mathrm{End}(\Lambda^* g^*)$. I Section 14 we called this action θ, but calling it "ad" instead will remind u where it comes from and will distinguish it from the other Greek letter, ρ, use in this section. Explicitly, for $\omega \in \Lambda^q g^*$ and $X_1, \ldots, X_q \in g$, we have

$$(ad_X \omega)(X_1, \ldots, X_q) = -\{\omega([X, X_1], X_2, X_3, \ldots, X_q) + \omega(X_1, [X, X_2], X_3, \ldots, X_q) + \ldots\}.$$

For each $X \in g$, ad_X is a derivation of degree 0 on $\Lambda^* g^*$. $ad_X : \Lambda^q g^* \to \Lambda^q g^*$ and $ad_X(\omega \wedge \eta) = (ad_X \omega) \wedge \eta + \omega \wedge ad_X \eta$. For each $X \in g$, $\iota_X : \Lambda^q g^* \to \Lambda^{q-1} g$ is defined by

$$(\iota_X \omega)(X_1, \ldots, X_{q-1}) = \omega(X, X_1, \ldots, X_{q-1})$$

for $\omega \in \Lambda^q g^*$, $X_1, \ldots, X_{q-1} \in g$. Then ι_X is seen to be an anti-derivation on $\Lambda^* g^*$ of degree -1 (i.e., lowering degree by 1), and $ad_X \iota_Y = \iota_Y ad_X + \iota_{[X,Y]}$. Define $d : \Lambda^q g^* \to \Lambda^{q+1} g^*$ as the anti-derivation of $\Lambda^* g^*$ such that $d|_{\Lambda^0 g^*} \equiv 0$ and $d|_{\Lambda^1 g^*}$ is defined by $(d\omega)(X, Y) = -\omega([X, Y])$ for $X, Y \in g$, $\omega \in \Lambda^1 g^*$. In general, for $\omega \in \Lambda^q g^*$, $X_1, \ldots, X_{q+1} \in g$,

$$d\omega(X_1, \ldots, X_{q+1}) = \sum_{i<j} (-1)^{i+j} ([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{q+1}).$$

Since d and ι_X (for $X \in g$) are anti-derivations of degrees +1 and -1, respectively, $d\iota_X + \iota_X d$ is a derivation of degree 0 of $\Lambda^* g^*$. Since $ad_X = d\iota_X + \iota_X d$ on $\Lambda^0 g^*$ and $\Lambda^1 g^*$, this formula is true on $\Lambda^* g^*$. For $\tilde{x} \in \Lambda^1 g^*$, define $e(\tilde{x})$ on $\Lambda^* g^*$ to be exterior multiplication (on the left) by $\tilde{x}$. That is $e(\tilde{x})(\omega) = \tilde{x} \wedge \omega$, $(\omega \in \Lambda^* g^*)$.

Lemma IV. 1. *For $Z \in g$,*

$$ad_Z = -\sum_{j=1}^{n} e(\tilde{x}_j) \iota_{[Z, X_j]}$$

where $X_1, \ldots, X_n$ is any basis of g and $\tilde{x}_1, \ldots, \tilde{x}_n$ the dual basis for $\Lambda^1 g^*$.

Proof. Apply both sides to $f \in \Lambda^q g^*$ and check on q linearly independent elements of g. We can choose these from our basis $X_1, \ldots, X_n$, and by renumbering, we can suppose we have chosen the first q elements.

$$\begin{aligned} -\sum_{j=1}^{n} & e(\tilde{x}_j)(\iota_{[Z,X_j]} f)(X_1, \ldots, X_q) \\ &= -\sum_{j=1}^{q} (\iota_{[Z,X_j]} f)(X_1, \ldots, \widehat{X}_j, \ldots, X_n)(-1)^{j-1} \\ &= -\sum_{j=1}^{q} f([Z, X_j], X_1, \ldots, \widehat{X}_j, \ldots, X_n)(-1)^{j-1} \\ &= -\sum_{j=1}^{q} f(X_1, \ldots, [Z, X_j], \ldots, X_n)(-1)^{j-1}(-1)^{j-1} \\ &= (ad_Z f)(X_1, \ldots, X_q) \end{aligned}$$

Definitions. g acts on $\Lambda^* g^* \otimes V$ by $\mathcal{L}_X \equiv ad_X \otimes 1 + 1 \otimes \rho_X$, for $X \in g$. (Cf. definition of tensor product of g-modules in I.) Explictly, for $\omega \otimes v \in \Lambda^* g^* \otimes V$, $\mathcal{L}_X(\omega \otimes v) = ad_X \omega \otimes v + \omega \otimes \rho_X v$.

Define $\delta_V : \Lambda^q g^* \otimes V \to \Lambda^{q+1} g^* \otimes V$ to be $\sum_{j=1}^{n} e(\tilde{x}_j) \otimes \rho_{X_j}$, where $X_1, \ldots, X_n$ is a basis for g, and $\tilde{x}_1, \ldots, \tilde{x}_n \in g^*$ is the dual basis. δ_V is independent of this choice of basis.

Define $\delta = d \otimes 1 + \delta_V : \Lambda^q g^* \otimes V \to \Lambda^{q+1} g^* \otimes V$. Then δ satisfies

$$\mathcal{L}_X = \delta i_X + \iota_X \delta,$$

for all $X \in g$, where we have written ι_X for $\iota_X \otimes 1$ on $\Lambda^* g^* \otimes V$.

Define $H^*(g, \rho) = H^*(g, V)$ to be the cohomology of the complex:

$$0 \longrightarrow \Lambda^0 g^* \otimes V \xrightarrow{\delta} \Lambda^1 g^* \otimes V \xrightarrow{\delta} \Lambda^2 g^* \otimes V \longrightarrow \cdots$$

For example, check that $H^0(g, \rho) \simeq V^g \equiv \{v \in V | \rho_X v = 0 \text{ for all } X \in g\}$.

Theorem IV. 2. *Let g be a semi-simple finite dimensional Lie algebra, let V be a finite dimensional g-module such that $V^g = 0$. Then $H^q(g, V)$ for all q.*

Proof. (See [14].) Since g is semi-simple, $V = \oplus V_i$, a sum of simpl modules, and $\rho|_{V_i} \neq 0$. Since $H^*(g, \oplus V_i) \simeq \oplus_i H^*(g, V_i)$, it suffices to tak to be simple. Let $h \subseteq g$ be the complementary ideal to ker ρ, $g = h \oplus k$ $h \neq 0$ since $\rho \neq 0$. h is semi-simple since any Abelian ideal of h is also or g. Since $\rho|_h$ is faithful, I. 1 tells us that there exist dual bases $\{Y_i\}$ and · of h such that $B_\rho(Y_i, Z_j) = \delta_{ij}$, the Kronecker delta. In the proof of II. : defined the Casimir element $K = K_\rho = \sum_{i=1}^{\dim h} (Y_i \otimes Z_i) \in U(h) \subseteq U(g)$ showed that K is well-defined and is in the center of $U(h)$. Since K is the the center of $U(g)$, the Casimir operator $\rho(K) = \sum \rho_{Y_i} \circ \rho Z_i$ commutes · ρ_X for all $X \in g$. Since V is simple, ker $\rho(K)$ is a submodule of V. There $\rho(K)$ is either 0 or an isomorphism of V. But $\rho(K) \neq 0$, since its tra $\sum_i tr(\rho_{Y_i} \circ \rho_{Z_i}) = \dim h \neq 0$. Thus $\rho(K)$ is an isomorphism of V.

Let $\Gamma = 1 \otimes \rho(K)$ on $\Lambda^* g^* \otimes V$. Γ commutes with $ad_X \otimes 1$ and $1 \otimes \rho_X$ fc $X \in g$. Thus, Γ commutes with $\mathcal{L}_X = ad_X \otimes 1 + 1 \otimes \rho_X$. Also, Γ commutes · $d \otimes 1$ and $\delta_v = \sum_j e(\tilde{x}_j) \otimes \rho_{X_j}$ (for $\{X_j\}$ a basis of g and $\{\tilde{x}_j\}$ the dual ba Therefore, Γ commutes with $\delta = d \otimes 1 + \delta_v$. $\delta\Gamma = \Gamma\delta$ implies $\Gamma^{-1}\delta = \ell$ implies Γ^* and Γ^{-1*} are defined on $H^*(g, \rho)$, and $\Gamma^* \circ \Gamma^{-1*} = (\Gamma \circ \Gamma^{-1})^* =$ But we will show that Γ is homotopic to zero: $\Gamma^* = 0^*$. Hence, $H^*(g, \rho) =$

Let $T = \sum_{i=1}^{\dim h} \iota Z_i \otimes \rho_{Y_i}$. I claim that $\Gamma = \delta T + T\delta$.

$$\begin{aligned} \delta T + T\delta &= \delta(\sum_i \iota_{z_i} \otimes \rho_{Y_i}) + (\sum_i \iota_{Z_i} \otimes \rho_{Y_i})\delta \\ &= \delta(\sum \iota_{Z_i} \otimes \rho_{Y_i}) + (1 \otimes \rho Y_i)(-\delta(\iota_{z_i} \otimes 1) + \mathcal{L}Z_i), \\ \text{since } \mathcal{L}_{Z_i} &= \delta\, \iota\, Z_i + \iota_{Z_i}\delta = \sum (d \otimes 1 + \delta_v)(\iota_{Z_i} \otimes \rho_{Y_i}) \\ &\quad - \sum (1 \otimes \rho_{Y_i})(d \otimes 1 + \delta_v)(\iota_{Z_i} \otimes 1) + \sum (1 \otimes \rho_{Y_i})(\mathcal{L}Z_i) \end{aligned}$$

Since $d \otimes 1$ and $1 \otimes \rho_{Y_i}$ commute, we get some cancellation, and, since $\mathcal{L}_{Z_i} = 1 \otimes \rho_{Z_i} + ad_{Z_i} \otimes 1$, we obtain:

$$\begin{aligned}\delta T + T\delta = \delta_V(\sum \iota_{Z_i} \otimes \rho_{Y_i}) - \sum (1 \otimes \rho_{Y_i})\delta_V(\iota_{Z_i} \otimes 1) \\ + \sum (1 \otimes \rho_{Y_i})(1 \otimes \rho_{Z_i} + ad_{Z_i} \otimes 1) = \delta_V(\sum \iota_{Z_i} \otimes \rho_{Y_i}) \\ - \sum (1 \otimes \rho_{Y_i})\delta_V(\iota_{Z_i} \otimes 1) + \Gamma + \sum (ad_{Z_i} \otimes \rho_{Y_i}).\end{aligned}$$

Now express δ_V in terms of a basis $\{X_i\}$ of g with dual basis $\{X_i\}$. Also, express ad_{Z_i} in terms of the same basis via Lemma IV. 1.

$$\begin{aligned}\delta T + T\delta - \Gamma &= \sum_{i,j} (e(\tilde{x}_j)\iota_{Z_i} \otimes \rho_{X_j}\rho_{Y_i} - \sum_{i,j} (e(\tilde{x}_j)\iota_{Z_i} \otimes \rho_{Y_i}\rho_{X_j}) \\ &- \sum_{i,j} (e(\tilde{x}_j)\iota_{[Z_i,X_j]} \otimes \rho_{Y_i}) \\ &= \sum_{i,j} e(\tilde{x}_j) \otimes 1\{\iota_{Z_i} \otimes (\rho_{X_j}\rho_{Y_i} - \rho_{Y_i}\rho_{X_j}) - \iota_{[Z_i,X_j]} \otimes \rho_{Y_i}\} \\ &= \sum_{i,j} e(\tilde{x}_j) \otimes 1 \{\iota_{Z_i} \otimes \rho_{[X_j,Y_i]} - \iota_{[Z_i,X_j]} \otimes \rho_{Y_i}\} = 0\end{aligned}$$

because

$$\begin{aligned}\sum_i \iota_{[Z_i,X_j]} \otimes \rho_{Y_i} &= \sum_{i,k} \iota_{B([Z_i,X_j],Y_k)Z_k} \otimes \rho_{Y_i} \\ &= \sum_{i,k} \iota Z_k \otimes \rho_{B([Z_i,X_j],Y_k)Y_i} \\ &= \sum_{i,k} \iota_{Z_k} \otimes \rho_{B(Z_i,[X_j,Y_k])Y_i} \\ &= \sum_k \iota_{Z_k} \otimes \rho_{[X_j,Y_k]} \\ &= \sum_i \iota_{Z_i} \otimes \rho_{[X_j,Y_i]}.\end{aligned}$$

Q. E. D.

Corollary IV. 2. 1. *If g is semi-simple, and V is a finite dimensional g-module, then $H^*(g,V) = H^*(g,V^g)$, where $V^g \subseteq V$ are the g-invariants, $V^g = \{v \in V | X \cdot v = 0$ for all $X \in g\}$.*

Proof. $V = V^g \oplus W$, $W^g = 0$, $H^*(g,W) = 0$, and $H^*(g,V) \simeq H^*(g,V^g) \oplus H^*(g,W)$.

Remark. Let $g = gl(n,k) = sl(n,k) \oplus Z$, where $sl(n,k)$ is semi-simple and Z is the center, generated by the identity matrix R. We wish to extend the above corollary to $g = gl(n,k)$. However, we need the additional condition that the g-module V break up as a sum $V = \oplus_\lambda V_\lambda$ of eigenspaces for $R \in Z \subseteq g$ with $\rho_{R|V_\lambda}$ = multiplication by $\lambda \in k$. We also assume V_0 (but not necessarily V)

is finite dimensional. $\mathcal{L}_R = ad_R \otimes 1 + 1 \otimes \rho_R = 0 + 1 \otimes \rho_R$ is multiplicati by λ on $\Lambda^* g^* \otimes V_\lambda$. But $\mathcal{L}_R = i_R\delta + \delta i_R$ is homotopic to zero. Therefo $H^*(g,V) = \oplus_\lambda H^*(g,V_\lambda) = H^*(g,V_0)$. Since $sl(n,k)$ is semi-simple, V_0 brea up as a sum of $sl(n,k)$-modules (in fact as $gl(n,k)$-modules since $\rho_R = 0$ ‹ V_0), $V_0 = V_0^{sl(n,k)} \oplus W = V^{gl(n,k)} \oplus W$, where $W^{sl(n,k)} = W^{gl(n,k)} = 0$. Sin $R \in \ker(\rho|_W)$, a complementary ideal h can be chosen $\subseteq sl(n,k)$. Just in IV. 2, we get a Casimir operator ($\rho|_W$ restricted to h is faithful) whi is an isomorphism of W, commutes with δ and is homotopic to the identit $H^*(g,W) = 0$. Finally, we have in this case,

Corollary IV. 2.2. $H^*(g,V) = H^*(g,V^g)$.

For more details, see the section on reductive algebras and the spectral sequen relative to a reductive subalgebra in [39].

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CENTRO INTERNAZIONALE MATEMATICO ESTIVO
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DIFFERENTIAL OPERATORS ON MANIFOLDS

Some Aspects of Invariant Theory in Differential Geometry

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