<u>The manifold of embeddings of a closed manifold</u>

by

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The present note is concerned with some basic properties of the set $E(M,N)$ of all smooth embeddings $M \longrightarrow N$ where M and N are (finite-dimensional) paracompact C^∞-manifolds. For technical reasons, we assume M to be compact without boundary, so that the currently available techniques of the theory of manifolds of smooth maps are applicable to our case. Since $E(M,N)$ is known to be open in the manifold of all smooth maps $M \rightarrow N$ (cf.e.g.[M]), it is a Fréchet manifold. The group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M operates in the obvious manner on this manifold and our main result is that under this operation, $E(M,N)$ is a principal $\mathrm{Diff}(M)$-bundle over the quotient $U(M,N) = E(M,N)/\mathrm{Diff}(M)$. In particular $U(M,N)$ - which is the set of all submanifolds of type M in N - carries a natural differentiable structure and $E(M,N) \longrightarrow U(M,N)$ is a submersion. A differentiable structure was obtained on $U(M,N)$ for compact M with boundary some time ago (cf.e.g.[Ko]), so that now natural manifold structures are available on all $U(M,N)$ for compact manifolds M. The non-compact case so far remains quite intractable because of some fundamental difficulties arising already in connection with the choice of an appropriate topology on, say, $C(M,N)$ (=the smooth maps $M \rightarrow N$), of reasonable "candidates" for parameter spaces and with differential calculus.

The motivation for our investigation stems from geometrodynamics ([MW] , [W] and further references in section 6), the "3+1 approach" to relativity, especially in the formulation it obtained in the framework of "hyperspace" (Kuchar's terminology, cf.[K] , [HKT] and their bibliographies). In fact, "hyperspace" essentially is a special case of manifold of the type of $E(M,N)$, arrived at as follows: M,N are (orientable) manifolds of dimension 3,4 resp. and one considers now the set of all space-like embeddings $M \longrightarrow N$ where N carries an Einstein structure and where "space-like" means that the metric induced on the image is Riemannian. Since the set $E_o(M,N)$ of such embeddings is open in $E(M,N)$, it is a Fréchet manifold and is saturated under the right action of $\mathrm{Diff}(M)$, hence again a principal bundle over the manifold $U_o(M,N)$ of all space-like "slices" of type M in N. "Motions" of such a slice $i(M)$ then correspond to (smooth) curves through i in $E_o(M,N)$ - which thus represent the kinematics of submanifolds of type M in N, as well as the dynamics in the presence of suitable constraints. The details of these matters, using the notations and results of this note, will be dealt with elsewhere and we limit our exposition here to a few brief remarks in section 6).

After introducing some notations and preliminary results in 1, we establish the topological bundle structure of $E(M,N)$ in sections 2 to 4 by proving the existence of "enough" local sections of $E(M,N) \longrightarrow U(M,N)$ and we add the necessary differentiability considerations in section 5, using a form of differential calculus developed in [F] and [G] (cf. also [O]) in order to obtain precise information on the differentiability of the various constructions of the preceding sections.

1. The finite-dimensional manifolds $M,N,Z,\ldots$ to be considered in the sequel are understood to be paracompact C^∞-manifolds, so that they always admit smooth partitions of unity. Where used, the term "differentiable" will mean "smooth", i.e. "of class C^∞". It is no loss of generality to assume the manifolds $M,N,\ldots$ connected.

Let now M be compact. For each finite k, $C^k(M,N)$ denotes the set of C^k-maps $M \longrightarrow N$ equipped with the usual C^k-topology (cf.e.g. [M]). Of importance to us is the well-known result that composition

$$C^k(N,Z) \times C^k(M,N) \xrightarrow{\beta} C^k(M,Z) \ ,$$

$\beta(f,g) = f\circ g$, is a continuous map. Moreover, if $\mathrm{Diff}^k(M)$ denotes the group of C^k-diffeomorphisms of M, inversion in $\mathrm{Diff}^k(M)$ also is continuous, so that $\mathrm{Diff}^k(M)$ is a topological group, cf.[M], and is an open subset of $C^k(M,M)$ for $k \geqslant 1$. Since the latter is known to be a Banach manifold, the same holds for $\mathrm{Diff}^k(M)$; however, it is not a differentiable group in this structure. For the sake of clarity, we shall give some details concerning the corresponding Fréchet-structure on spaces of smooth maps below.

$E^k(M,N)$ denotes the set of all C^k-embeddings $M \longrightarrow N$; again, for $k \geqslant 1$, this is open in $C^k(M,N)$ - in fact, C^1-open - and thus again is a Banach manifold: the proof that E^k is open is analogous to the one of prop.5 in [M] .

Similar remarks apply in the case of analogously defined structures on spaces of smooth maps - where we also simplify the notations by setting $C(M,N) = C^\infty(M,N)$, $\mathrm{Diff}(M) = \mathrm{Diff}^\infty(M)$, $E(M,N) = E^\infty(M,N)$, etc. The manifold structures now are Fréchet, not Banach, and are obtained as follows:

Choose a smooth spray $TN \longrightarrow T^2N$ on N and denote its expontential map by exp. It is known ([L]) that there exists an open neighbourhood O' of the zero-section $0_N \subset TN$ such that exp induces diffeomorphisms of the $O' \cap T_yN$ onto open neighbourhoods of the points $y \in N$ ("normal charts"). There is more: define the map Exp from TN to $N \times N$ by:

$$\mathrm{Exp}(v_y) = (y, \exp_y(v_y)) \ , \ v_y \in T_yN \ .$$

Then there is an open neighbourhood $\mathcal{O}$ of 0_N such that Exp induces a (smooth) diffeomorphism of $\mathcal{O}$ onto an open neighbourhood of the diagonal $\Delta \subset N \times N$.

Given now $f \in C(M,N)$, we denote by $\mathcal{U}(f)$ the set of all smooth maps $g:M \longrightarrow N$ such that $(f \times g)(\Delta_M) \subset \mathrm{Exp}(\mathcal{O})$ or, which is the same thing, $(f \times \mathrm{id}_N)(\langle g\rangle) \subset \mathrm{Exp}(\mathcal{O})$, $\langle g\rangle \subset M \times N$ the graph of g. The set $\mathcal{U}(f)$ is an open neighbourhood of f in $C(M,N)$ with respect to the usual C^∞-topology, M being compact.

Next, let

$$\Gamma_f(TN) = \{l \in C(M,TN) \mid \tau_N \circ l = f\} \ ,$$

$\tau_N : TN \longrightarrow N$ the projection, be the set of all "lifts" of f to TN. Clearly, this is a vector space under pointwise operations and it becomes a Fréchetspace under the C^∞-topology. The space $\Gamma_f(TN)$ is naturally isomorphic to $\Gamma(f^*TN)$, the space of smooth sections of the pull-back bundle f^*TN. Let $W(f) = \{l \in \Gamma_f(TN) \mid l(M) \subset \mathcal{O}\}$; this is an open neighbourhood of o and the exponential map induces a homeomorphism $W(f) \longrightarrow \mathcal{U}(f)$ by assigning to $l \in W(f)$ the map g defined by: $g(x) = \exp_{f(x)}(l(x))$, also written $g = \exp_f(l)$.
We denote the inverse map by $\Phi_f : \mathcal{U}(f) \longrightarrow W(f)$. The pair $(\mathcal{U}(f),\Phi_f)$ is a <u>standard chart</u> at f. One shows that these charts define on $C(M,N)$ a Fréchet-manifold structure whose underlying topology is the C^∞-topology; for all details, we refer to [F] and [G] , in particular, for the precise sense in which this structure is "differentiable" (ILB-differentiable in a strong sense!). We also note that the construction goes back to [E].

Since $E(M,N)$ is $(C^1\text{-})$open in $C(M,N)$, it inherits an induced manifold structure; this will be referred to as the natural manifold structure of $E(M,N)$.

Similarly, Diff(M) is open in C(M,M) and therefore again a Fréchet manifold. As opposed to the C^k-case for finite k, however, the group operations in Diff(M) are (ILB-)differentiable: Diff(M) is a smooth Fréchet Liegroup. Again, the details may be found in [F] , [G] and in [E-M] , [O] . Note that the tangent space $\Gamma_e(TM)$ to Diff(M) at the identity e simply is the algebra $\Gamma(TM)$ of smooth vector fields on M. The construction of charts (at e, say) still requires the choice of a spray on M and the earlier arguments, since the classical "exponential map", assigning to a vector field its flow, will not, in general, be surjective onto any neighbourhood of e, cf.e.g.[O] .

2. Henceforth, M is assumed to be closed (compact without boundary). The group Diff(M) acts in a natural way on the right on E(M,N) by: $(f,g) \longrightarrow f\circ g$ and the action is continuous by what was mentioned earlier. By e.g. [G] , it even is (ILB-)differentiable in a sense which can be made precise quite explicitly. For reasons of convenience, we shall write $\beta(f,g)$ for $f\circ g$. The partial maps defined by β are

$$\beta_f : \mathrm{Diff}(M) \longrightarrow E(M,N) , \quad \text{for a fixed } f \in E(M,N);$$

$$\beta_g : E(M,N) \longrightarrow E(M,N) , \quad \text{for fixed } g \in \mathrm{Diff}(M) .$$

Clearly, β_g is a diffeomorphism of E(M,N) whose inverse is $\beta_{g^{-1}}$. Moreover, the injectivity of the $f \in E(M,N)$ immediately implies that the right action of Diff(M) is free: $f\circ g_1 = f\circ g_2$ implies $g_1 = g_2$ by cancellation of f . Thus, β_f certainly is a bijection of Diff(M) onto the orbit of f. If we set $L = f(M) \subset N$, a closed submanifold of N, then the orbit of f consists precisely of all diffeomorphisms of M onto L: indeed, it is easy to see that $i,j \in E(M,N)$ are equivalent under Diff(M) if and only if im(i) = = im(j). In fact, the orbit of f may also be identified with E(M,L): one inclusion is clear and for the converse, if h is an embedding of M into f(M) = L, then h(M) is compact, hence closed; it also is open by the inverse function theorem since T_xh is an isomorphism of T_xM onto $T_{h(x)}L$ for each $x \in M$. Thus, M being connected, the assertion follows. We shall see below that the identification of the orbit of $i \in E(M,N)$ with E(M,i(M)) is topological as well, even differentiable.

Next, we note that the orbit $i\cdot\mathrm{Diff}(M) = \tilde{i}$ of $i \in E(M,N)$ is closed in E(M,N): let (g_α) be a net in Diff(M) such that $i\cdot g_\alpha$

converges in $E(M,N)$ to, say, f. Since $\beta_i(g_\alpha)(M) = i(M)$ for every α and since $\beta_i(g_\alpha)$ converges in particular for the C^0-topology we conclude $f(M) \subset i(M)$, thus $f(M) = i(M)$ as before. Further details will be added below (prop.3,cor.).

Lastly, we note that the quotient $U(M,N)$ of $E(M,N)$ mod. the action of Diff(M) simply is the set of all submanifolds "of type M" of N. This set carries the quotient topology. With this, we state the main result of this note:

Theorem: Let M be closed, $\dim(M) < \dim(N)$. Then $E(M,N)$ is a (topological) principal $\mathrm{Diff}(M)$-bundle over $U(M,N)$.

Moreover, $U(M,N)$ carries a natural (ILB-)differentiable structure for which the fibration $E(M,N) \xrightarrow{u} U(M,N)$ is smooth.

The proof of the theorem is divided into several steps which will follow in the next sections, but is essentially classical: it consists of establishing the existence of "enough" local sections of the quotient map u and, eventually, the necessary differentiability considerations. The first part of the argument is topological in nature and will establish the first (main) part of the theorem.

3. We choose, once and for all, a quadratic structure G on N ("pseudo-Riemannian" or Riemannian structure). For $i \in E(M,N)$, we denote by ν_i the "geometric" normal bundle of i defined by means of G: the fibre of ν_i at $p = i(x) \in i(M)$ is the orthogonal in T_pN of $T_xi(T_xM)$. Clearly, then,

(1) $$TN|i(M) = Ti(TM) \oplus \nu_i \, .$$

We now have the

Prop.1: For $i \in E(M,N)$, the map

$$\Omega_{Ti}: \Gamma(TM) \longrightarrow \Gamma_i(TN)$$

which maps a vector field X to $Ti \circ X$ is a linear and topological isomorphism of $\Gamma(TM)$ onto a topological direct summand, namely the subspace $\{l \in \Gamma_i(TN) \mid l(M) \subset Ti(TM)\}$.

For the proof one observes, firstly, that the indicated subspace is a topological summand - using the orthogonal projection onto $Ti(TM)$ in the standard Ω-lemma. Secondly, one easily verifies that Ω_{Ti} is a linear bijection onto this subspace which also is continuous. The closed graph theorem now implies the assertion.

In TN, we use the exponential maps exp and Exp defined by the metric G (so that the normal charts are geodesic charts for G). Given the open neighbourhood $\mathcal{O}$ of the zero-section O_N as used in 1., there now are open neighbourhoods $\mathcal{O}_M$ of the zero-section of TM and $\mathcal{O}_{\nu_i}$ of the zero-section of ν_i such that $Ti(\mathcal{O}_M) + \mathcal{O}_{\nu_i} \subset TN|i(M)$ still is open.
Then:

Prop.2: Given $i \in E(M,N)$, $\mathcal{O}_{\nu_i}$ and $\mathcal{O}_M$ may be chosen so as to satisfy the conditions

(1) $\exp|\mathcal{O}_{\nu_i} : \mathcal{O}_{\nu_i} \longrightarrow \exp(\mathcal{O}_{\nu_i})$ is a diffeomorphism,

(2) $\exp(\mathcal{O}_{\nu_i}) \subset N$ is open.

For the proof, let $pr_2 : N \times N \longrightarrow N$ be the second projection, so that $pr_2 \circ \mathrm{Exp} = \exp$, and let $o_{\nu_i(p)} \in \nu_i(p)$ be the zero-element of the fibre. Exp yields a diffeomorphism of $\mathcal{O}$ onto some neighbourhood of the diagonal in $N \times N$ and, moreover, $T(pr_2 \cdot \mathrm{Exp}|\mathcal{O} \cap \nu_i)$, mapping $T_{o_{\nu_i(p)}}\nu_i$ to T_pN, $p \in i(M)$, is an isomorphism, being a linear injection between vector spaces of the same finite dimension. The assertion now follows from Prop.7.3,p.69, of [G-G].

In addition, we assure that N possesses a Finsler structure $\| \ \| : TN \longrightarrow R$. Because of the compactness of M, we may assume that there is $\varepsilon > o$ such that

$$\mathcal{O}_{\nu_i} = \{v \in \nu_i \mid \|v\| < \varepsilon\} .$$

The open submanifold $\exp(\mathcal{O}_{\nu_i})$ then is the "ε-tube around i(M)" and will be denoted by S_i^ε. The image $\exp(\nu_i(p))$ is the fibre at $p \in i(M)$ in S_i^ε.

In the next step, we construct a map $P_i : S_i^\varepsilon \longrightarrow i(M)$ with $P_i|i(M) = \mathrm{id}.$: S_i^ε is embedded into the fibre-product $i(M) \times_{i(M)} S_i^\varepsilon$ under the map ι_1 which maps $\exp_p(v) \in S_i^\varepsilon$, $v \in \nu_i(p) \cap \mathcal{O}_{\nu_i}$, to $(p, \exp_p(v)) = \mathrm{Exp}(v)$. ι_2 denotes the inclusion of the fibre-product into $N \times N$. Set

$$P_i = \tau_N \circ \mathrm{Exp}^{-1} \circ \iota_2 \circ \iota_1 ,$$

a smooth map $S_i^\varepsilon \longrightarrow i(M)$. Now $(\mathrm{Exp}^{-1} \circ \iota_2 \circ \iota_1)(p) = o_{\nu_i(p)}$ for $p \in i(M)$, so that $P_i|i(M) = \mathrm{id}.$: P_i is a retraction for the inclusion ι of $i(M)$ into S_i^ε, mapping an open neighbourhood of $i(M)$, namely S_i^ε, onto $i(M)$ along geodesics.

Clearly, $\Omega_{P_i} \circ \Omega_\iota = \Omega_{P_i \circ \iota}$ = id. on $C(M,i(M))$. This implies that $C(M,i(M))$ is mapped homeomorphically onto its image in $C(M,S_i^\varepsilon)$, an open subset of $C(M,N)$. Therefore, if we interpret $C(M,i(M))$ as a subspace of $C(M,N)$, the induced topology will coincide with the C^∞-topology of $C(M,I(M))$. Thus:

<u>Prop.3:</u> The elements of $E(M,i(M))$ are precisely the maps $i \circ g$, $g \in \in \mathrm{Diff}(M)$, i.e. $E(M,i(M))$, interpreted as a subset of $E(M,N)$, is the orbit of i. Moreover, the C^∞-topology of $E(M,i(M)$ coincides with the topology induced from $E(M,N)$.

<u>Cor.:</u> With the earlier notations, β_i is a homeomorphism of $\mathrm{Diff}(M)$ onto the orbit of i in $E(M,N)$ and this orbit is closed.

To obtain the corollary, note that $\beta_i^{-1} = \Omega_{i^{-1}}$ is continuous on $E(M,i(M))$ by the usual Ω-lemma. The remaining claims already were established (cf.also 2.).

Suppose now that the map $j:M \longrightarrow S_i^\varepsilon$ satisfies $P_i \circ j \in E(M,i(M))$. Then j is an <u>embedding</u> whose inverse is given by $(P_i \circ j)^{-1} \circ (P_i | j(M))$. In addition,

$$T_x j(T_x M) \oplus T_v \exp_{P_i(j(x))}(\nu_{P_i(j(x))}) = T_{j(x)} S_i^\varepsilon \text{ , where } j(x) = \exp_{P_i(j(x))}(v)$$

The first part is easily verified directly; for the second assertion one notes that $TP_i(T_j(T_x M)) = T_{P_i(j(x))} S_i^\varepsilon = TP_i(T_{j(x)} S_i^\varepsilon$, from which the rest follows by a dimension argument.

In an obvious sense, the embeddings j with $P_i \circ j \in E(M,i(M))$ are transversal to the fibres of P_i; the set of these embeddings is denoted by $E^t(M,S_i^\varepsilon)$. $E(M,i(M))$ is open in $C(M,i(M))$ and Ω_{P_i} is continuous. Thus, we obtain

<u>Prop.4:</u> The complete inverse image of $E(M,i(M)) \subset C(M,i(M))$ under Ω_{P_i} is $E^t(M,S_i^\varepsilon)$ which therefore is open in $C(M,S_i^\varepsilon)$ as well as in $E(M,N)$.

The composition

$$E(M,i(M)) \xrightarrow{\Omega_\iota} E^t(M,S_i^\varepsilon) \xrightarrow{\Omega_{P_i}} E(M,i(M))$$

is the identity and, in particular, Ω_{P_i} is a quotient map.

4. We now are going to investigate the quotient $E(M,N) \longrightarrow U(M,N)$ in more detail: we wish to show that u admits "enough" local sections and that, consequently, $U(M,N)$ possesses a (topological) manifold structure over a Fréchet space; the questions of differentiability will be discussed below.

As before, we let $(\mathcal{U}_i, \Phi_i)$ be a natural chart at $i \in E(M,N)$, $\Phi_i = \exp_i^{-1} | \mathcal{U}_i$, taking values in $\Gamma_i(TN)$. We may assume, moreover, that $\mathcal{U}_i$ is constructed using the neighbourhood $\mathcal{O} = Ti(\mathcal{O}_M) + \mathcal{O}_{\nu_i}$ of the zero-section of $TN|i(M)$.

Recall that $(i \circ g)(M) = i(M)$ for $g \in \mathrm{Diff}(M)$. In particular $\Omega_{P_i}^{-1} \circ \beta_i$ is well-defined. With this remark, we now obtain the

Prop.5: Given $i \in E(M,N)$ there exists a subset $\mathcal{V}_i \subset E(M,N)$ with the following properties:

(a) $i \in \mathcal{V}_i$ and $u|\mathcal{V}_i$ is injective;

(b) for every open neighbourhood W of the identity in $\mathrm{Diff}(M)$, the set $\mathcal{V}_i \cdot W$ is open.

In particular, $u(\mathcal{V}_i) \subset U(M,N)$ is an open neighbourhood of $\hat{i} = u(i)$ and $s_i = (u|\mathcal{V}_i)^{-1}$ is a continuous section of u such that $s_i(\hat{i}) = i$.

Proof: Recall that $E^t(M, S_i^\varepsilon)$ is open. If W is an open neighbourhood of the identity in $\mathrm{Diff}(M)$, then $U = \Omega_{P_i}^{-1}(\beta_i(W)) \subset E^t(M, S_i^\varepsilon)$ is open and contains i. We set

$$\mathcal{V}_i = \{ j \in U | P_i \circ j = i \} = \Omega_{P_i}^{-1}(i) .$$

Clearly, $i \in \mathcal{V}_i$ and, furthermore, $u|\mathcal{V}_i$ is injective: if $j, j' \in \mathcal{V}_i$ and $u(j) = u(j')$, then $j' = j \circ g$ for some $g \in \mathrm{Diff}(M)$, therefore $i = P_i \circ j' = P_i \circ j \circ g = i \cdot g$, hence $g = \mathrm{id}.$ and $j' = j$.

Next we claim that $U = \mathcal{V}_i \cdot W$: Firstly, suppose that $P_i \circ j \in \beta_i(W)$, i.e. that $P_i \circ j = i \circ g$. Then $j \circ g^{-1} \in E^t(M, S_i^\varepsilon)$, $P_i \circ (j \circ g^{-1}) = i$, i.e. $j \circ g^{-1} \in \mathcal{V}_i$ and therefore $j \in \mathcal{V}_i \cdot W$. Conversely, let $j \in \mathcal{V}_i$, $g \in W$. Then $P_i \circ j = i$, i.e. $P_i \circ (j \circ g) = i \circ g \in \beta_i(W)$ and therefore $j \circ g \in U$.

It is clear that $U \cdot \mathrm{Diff}(M)$ is open in $E(M,N)$. The identity just established says that $\mathcal{V}_i \cdot \mathrm{Diff}(M) = \mathcal{V}_i \cdot W \cdot \mathrm{Diff}(M) = U \cdot \mathrm{Diff}(M)$. In particular, $u(\mathcal{V}_i)$ is an open neighbourhood of $\hat{i}$. The proposition now follows.

Let now O_i be the open neighbourhood $u(V_i)$ of $\hat{i}$, $s_i = (u|V_i)^{-1}$ the section just obtained. The fibres of u over the points of O_i meet V_i in precisely one point and this now means that the composition map $(j,g) \longrightarrow j\circ g$: $V_i \times \mathrm{Diff}(M) \longrightarrow u^{-1}(O_i)$ has an inverse Z. Composition is continuous and the continuity of Z may be derived from the assertion (b) of the proposition or, somewhat more directly, as follows: Let $j \in u^{-1}(O_i) = V_i\cdot\mathrm{Diff}(M)$. Then $Z(j) = (j_o,g)$ where j_o is the common point of V_i and the orbit of j. Since $P_i \circ j_o = i$, this implies that g is determined by the equation $P_i\circ j = i\circ g$, i.e. that $g = \beta_i^{-1}\circ\Omega_{P_i}(j)$ and this is continuous in j. Similarly $j_o = j\circ g^{-1}$ is continuous in j by the properties of composition and inversion. These remarks establish the

Prop.6: With the notations just introduced, the fibration $E(M,N \xrightarrow{u} U(M,N)$ is trivial over O_i, a trivilization being given as usual by the map $(y,g) \longrightarrow s_i(y)g$ of $O_i \times \mathrm{Diff}(M)$ onto $u^{-1}(O_i)$. Thus, $E(M,N) \xrightarrow{u} U(M,N)$ is a (topological) principal $\mathrm{Diff}(M)$-bundle over $U(M,N)$.

Evidently, this proves the topological assertions of the theorem of section 2.

Lastly, we determine the (topological) manifold structure of $U(M,N)$: The charts will be defined by the sets O_i of prop.6 and the maps $\Phi_i\circ s_i$ of O_i into $\Gamma_i(TN)$. There remains the description of the image $\Phi_i(V_i)$ of O_i under $\Phi_i\circ s_i$: The condition $P_i\, j = i$ means that for each $x \in M$, $j(x)$ lies in the fibre of S_i^{ε} over $i(x)$. This in turn simply says that j is given by $\exp_i\circ\, l$ for some lift l of i which lies in ν_i , cf. 3. In other words: Φ_i maps V_i (homeomorphically) onto some open neighbourhood of $o \in \Gamma_i(\nu_i)$ which therefore is the parameter space of $U(M,N)$ for the chart $(O_i,\Phi_i\circ s_i)$.

The sets O_i are metrizable, hence in particular completely regular. This may be used to show that $U(M,N)$ is Hausdorff. We omit the elementary argument.

5. In this section, we briefly outline in what sense the various constructions of the preceding sections will be differentiable, retaining the earlier notations. Once again, we refer to [E-M], [F],[G] and [Ø] for all technical details (cf.also [B]). We already pointed out that the operation of $\mathrm{Diff}(M)$ on $E(M,N)$ is Γ-differentiable, as also are composition and inversion in $\mathrm{Diff}(M)$; the respective manifold structures are "$HC^{\infty}_{\sigma_o}$" ([F],[G]), i.e. Γ-differentiable in a rather strong sense. The "Ω-lemma", known from the theory of Banach

manifolds of maps (e.g.[B] still holds: "Ω-constructions" do not lead to any loss of differentiability.

One observes that the sets $V_i = s_i(O_i)$ constructed in 4. are submanifolds of $E(M,N)$ carrying a global chart Φ_i with values in $\Gamma_i(\nu_i)$, a topological summand of $T_iE(M,N) = \Gamma_i(TN)$.

The images $O_i = s_i(V_i)$ are manifolds diffeomorphic to the V_i under s_i by construction. One also notes that the open neighbourhood W of the identity in $\mathrm{Diff}(M)$ used in prop.5 is arbitrary and may, in particular, be chosen to be $\mathrm{Diff}(M)$: the projection P_i makes sense in all of $u^{-1}(O_i)$. Thus, if t is any section of u over O_i, we can construct a map $g: O_i \longrightarrow \mathrm{Diff}(M)$ by solving the equation $P_i \circ t = i \cdot g$, i.e. by means of $g = \beta_i^{-1} \circ \Omega_{P_i}(t)$. By the Ω-lemma, this map will be Γ-differentiable in the same sense as t and, moreover, $t = s_i \cdot g$, as one readily sees.

Now set $L = i(M)$. The "ε-tube" S_i^ε introduced earlier clearly does not depend on i, only on L, so that it will also be denoted by S_L^ε. Similarly, ν_i only depends on L and is also denoted by ν_L : $Ti(TM)$simply is the bundle of those elements of TN which are tangent to (curves in) L. Lastly, the exponential map used in the construction of S_L^ε is given by the quadratic structure of N and again is independent of i, so that finally P_i depends only on L and will be denoted by P_L. The data $L, \nu_L, S_L^\varepsilon$ and P_L therefore are common to all embeddings j on the orbit of i. As a consequence, one sees that the "natural" transversal slices V_j satisfy $V_{i\cdot g} = V_i \cdot g$, i.e. are translates of each other and thus are Γ-diffeomorphic (even in the sense of HC_σ^∞, as an adaption of the argument for the differentiability of right$^\circ$translations in $\mathrm{Diff}(M)$ will show). In particular, the differentiable structure of O_i may be obtained using any one of the natural sections s_j as long as j lies on the orbit of i.

Suppose now that $O_1 = O_{i_1}$ and $O_2 = O_{i_2}$ are given such that $O_1 \cap O_2 \neq \emptyset$; we assume that i_1 and i_2 do not lie on the same orbit. Set $L = i_1(M)$, $K = i_2(M)$. Finally, s_1 and s_2 are the natural sections obtained after choosing the representatives i_1, i_2 of the two orbits. Then $s_2(O_1 \cap O_2) = s_2(O_2) \cap u^{-1}(O_1 \cap O_2) =$ $= V_2 \cap u^{-1}(O_1 \cap O_2)$. For $j \in V_2$, $P_K \circ j = i_2$ and therefore $j = \exp_K \circ l_2$ for some $l_2 \in \Gamma_{i_2}(\nu_K)$. In addition, if $j \in s_2(O_1 \cap O_2)$, we also have that $P_L \circ j = i_1 \cdot g(\hat{\jmath})$ for some $g(\hat{\jmath}) \in \mathrm{Diff}(M)$ as above. Solving for g, we obtain

$$g(\hat{\jmath}) = (\beta_{i_1}^{-1}\,(P_L \circ \exp_K \circ l_2);$$

by the Ω-lemma, this is Γ-differentiable in l_2, i.e. in $\hat{\jmath}$. Once again, writing y for the points $\hat{\jmath}$ of $O_1 \cap O_2$, we obtain

$$s_2(y) = s_1(y) \cdot g(y)$$

on $O_1 \cap O_2$. This implies that the parameter transformation $(\Phi_1 \circ s_1)\,(\Phi_2 \circ s_2)^{-1}$ is Γ-differentiable in the same sense as composition and inversion are. Thus, $U(M,N)$ carries a natural differentiable structure. In fact, with this structure, $U(M,N)$ is the quotient of $E(M,N)$ mod. u: Given the map $f: E(M,N) \longrightarrow H$ which factors through u, $f = h \circ u$, f will be smooth if and only if h is.

The map u is a submersion: u is Γ-differentiable, for if $i \in E(M,N)$, we may choose an open neighbourhood of i of the form $V_i \cdot W$ as before (so that $u(V_i \cdot W) = O_i$) and then $s_i \circ u$ is the "projection" of $V_i \cdot W$ onto V_i constructed earlier; this map is Γ-differentiable and $T(s_i \circ u)$ maps $\Gamma_i(TM)$ onto the tangent space at i to V_i, i.e. onto $\Gamma_i(\nu_i)$. Since u is a diffeomorphism of V_i onto O_i, $T_i u$ will be an isomorphism of $\Gamma_i(\nu_i)$ onto the tangent space at $\hat{\imath}$.

Since $u \circ \beta_i = \hat{\imath}$, we have $Tu \bullet T\beta_i = o$ and, in particular, $T_i u \bullet T_e \beta_i = o$, so that $\mathrm{im}(T_e\beta_i) = \Gamma_i(Ti(TM) \subset \ker(T_i u)$. Now $\Gamma_i(TN) = \Gamma_i(Ti(TM)) \oplus \oplus\, \Gamma_i(\nu_i)$ and $T_i u | \Gamma_i(\nu_i)$ is an isomorphism. One concludes from this that $\ker(T_i u) = \Gamma_i(T_i(TM))$. With this, the vertical bundle $\ker(Tu)$ is determined. We shall return later to the problem of the existence of horizontal bundles, i.e. of principal connections on the bundle $E(M,N)$.

6. In this final section, we give brief outlines of some of the constructions which will be of importance in the applications mentioned in the introduction. The details will appear elsewhere.

I) So far, the tangent bundle of $E(M,N)$ has been tacitly identified with the space of all smooth maps $l: M \longrightarrow TN$ which satisfy $\tau_N \bullet l \in E(M,N)$. It can be verified directly that this mapping space is, in fact, a bundle over $E(M,N)$ with the expexted transition maps and local trivializations and is trivial over the natural charts introduced in 1.; we refer to [EL] and [F] for some details on this point.

This construction agrees in an obvious manner with the "geometric" one which interprets a tangent vector at a point as an equivalence class of (locally defined) smooth curves through the point, etc. It also agrees with the classical construction of "contravariant vectors" at a point, cf. [E] , [L] . One observes that the assertions are not entirely trivial since, among other things, it has to be shown that the natural manifold structure of the set $\{l: M \longrightarrow TN \mid \tau_N \circ l \in E(M,N)\} = TE(M,N)$ agrees with the one obtained from the local trivializations mentioned. The natural manifold structure is the obvious one: $TE(M,N)$ is an open subset of $C(M,TN)$, $E(M,N)$ being open in $C(M,N)$. Note that the classical motivation for the use of $\Gamma_i(TN)$ as the tangent space $T_iE(M,N)$ is contained in the following observation: Let $\sigma: I \longrightarrow E(M,N)$, $I \subset R$ an open interval containing o, be a smooth curve such that $\sigma(o) = i$. Then σ corresponds to a smooth map $\sigma: I \times M \longrightarrow N$ under $\sigma(t,x) = \sigma(t)(x)$, etc. The tangent vector $\sigma(o)$ now corresponds to the map $x \longrightarrow \frac{\partial\sigma}{\partial t}(o,x)$ of M into TN which clearly is a lift of $i = \sigma(o)$.

The projection $TE \longrightarrow E$ now simply is Ω_{τ_N} , i.e. the map $l \longrightarrow \tau_N \circ l$ of TE onto E whose fibre at i is precisely $\Gamma_i(TN)$. In the same manner, $T^2E(M,N)$ is the manifold of smooth maps $f: M \longrightarrow T^2N$ with the property that $\tau_N^{(2)} \circ f \in TE(M,N)$, $\tau_N^{(2)}: T^2N \longrightarrow TN$ the usual projection.

The notion of a <u>spray</u> on $E(M,N)$ is introduced as it is on Banach manifolds (for which we refer to [L]): denoting multiplication on the fibres of $TE(M,N)$, $T^2E(M,N), \ldots$ by μ_λ $(\lambda \in R)$, the definition is the following:

A spray on $E(M,N)$ is a smooth section ξ of $T^2E(M,N)$ (a vector field on $TE(M,N)$!) which satisfies the additional conditions

(a) $T\tau_N \circ \xi$ = id. on TE (ξ is a second order differential equation on E!)

(b) $\xi \circ \mu_\lambda = \mu_\lambda \cdot T\mu_\lambda \circ \xi$.

Due to the absence of simple inverse function theorems in the general differential calculus for Fréchet spaces, there do not seem to be any reasonable, general existence theorems for flows of such sprays (or for their "base integral curves"). However, the following elementary considerations often are sufficient for practical purposes:

It is immediate from the definition that if $S: TN \longrightarrow T^2N$ is a spray on N, then Ω_S is a spray on E, called an "Ω-spray" on E for simplicity's sake. For these, one has a local existence theorem for a flow and, in particular, for base integral curves (geodesics) in E.

Let S be a spray on N. Then, given $i \in E(M,N)$ and $l \in \Gamma_i(TN)$, there is $\varepsilon > o$ such that there exists a unique geodesic $\sigma : (-\varepsilon,\varepsilon) \longrightarrow E(M,N)$ for Ω_S with the properties that $\sigma(o) = i$ and $\dot{\sigma}(o) = l$.

The argument uses the compactness of M, thus of $i(M) \subset N$, and is quite standard: for $p = i(x) \in i(M)$, there are $\varepsilon_x > o$, a neighbourhood $V_x = i(U_x)$ in $i(M)$ and a map s_x from $(-\varepsilon_x,\varepsilon_x) \times U_x$ into N such that $s_x(o,y) = i(y)$, that $t \longrightarrow s_x(t,y)$ is the unique S-geodesic through $i(y)$ with $\dot{s}_x(o,y) = l(y) \in T_{i(y)}N$, and that no two such geodesics meet. Finitely many U_x now cover M and one chooses ε equal to the smallest one of the corresponding numbers ε_x. One obtains $s:(-\varepsilon,\varepsilon) \times M \longrightarrow N$ satisfying the conditions mentioned on all of its domain. Lastly, one defines σ by: $\sigma(t)(x) = s(t,x)$ and obtains the desired geodesic through $i = \sigma(o)$ in $E(M,N)$.

II) Assume once more that N carries a quadratic structure G. For each $i \in E(M,N)$, let again ν_i be the normal bundle of i with respect to G, so that $TN|i(M) = Ti(TM) \oplus \nu_i$ and one has the induced splitting $T_iE(M,N) = \Gamma_i(Ti(TM)) \oplus \Gamma_i(\nu_i)$ which we know to be topological. Using e.g. the "product neighbourhood" $\mathcal{V}_i \cdot W$ introduced earlier one can show that both $\bigcup \Gamma_i(Ti(TM))$ and $\bigcup \Gamma_i(\nu_i)$ are subbundles of $TE(M,N)$, the former being the vertical bundle $VE(M,N)$ of the principal bundle $E(M,N) \longrightarrow U(M,N)$, the latter being a summand $HE(M,N)$ of VE. One remarks, by the way, that both subbundles are invariant under the right action of $\mathrm{Diff}(M)$ and that, as a consequence, $HE(M,N)$ defines a principal connection on $E(M,N)$. We shall return to this point elsewhere.

The splitting $TE = VE \oplus HE$ also has the following immediate consequence: Suppose that ξ is a spray on $E(M,N)$ which admits local solutions through i and let $\sigma: I \longrightarrow E(M,N)$ be a geodesic, $o \in I$ and $\sigma(o) = i$. Then, for each $t \in I$, the tangent vector $\dot{\sigma}(t) \in T_{\sigma(t)}E$ admits a decomposition into its vertical and horizontal components:

$$\dot{\sigma}(t) = v_1(t) + v_2(t)$$

with $v_1(t) \in \Gamma_{\sigma(t)}(T\sigma(t)(TM))$ and $v_2(t) \in \Gamma_{\sigma(t)}(\nu_{\sigma(t)})$. In this case, there exists a time-dependent vector field X_t on M such that

$$v_1(t) = T\sigma(t) \circ X_t ,$$

the "shift" (of M) corresponding to the motion of $L = i(M) = \sigma(o)(M)$

determined by σ .

In addition, we now assume that $n = \dim(N) = \dim(M) + 1 = m + 1$ and that both M and N are <u>orientable</u>. Then the normal bundle ν_i of $i \in E(M,N)$ is a <u>trivial line bundle</u> : that it is of rank 1 is clear and the orientability assumption means that $\Lambda^m TM$, $\Lambda^n TN$ are trivial line bundles. Moreover, $TN|i(M) = Ti(TM) \oplus \nu_i$ implies, as usual, that

$$\Lambda^n(TN|i(M)) = \Lambda^n TN|i(M) = \bigoplus_{r+s=n} (\Lambda^r Ti(TM) \oplus \Lambda^s \nu_i) = \Lambda^m Ti(TM) \oplus \nu_i ;$$

the triviality of ν_i therefore is clear. In particular, ν_i admits non-vanishing global sections. This is of great importance in the following special case which yields a framework for the geometrodynamical view of general relativity:
Let $\dim(M) = 3$, $\dim(N) = 4$, G of index 1 (Lorentz structure) and suppose again that M and N are orientable. Denote by $E_o(M,N)$ the set of all <u>space-like</u> embeddings $M \longrightarrow N$:

$$i \in E_o(M,N) \text{ iff. } i^*G \text{ is positive-definite.}$$

Then $E_o(M,N)$ is an open submanifold of $E(M,N)$, saturated under $\mathrm{Diff}(M)$, and thus is a principal $\mathrm{Diff}(M)$-bundle over the open submanifold $U_o(M,N) \subset U(M,N)$ which consists of all space-like submanifolds of type M in N . Given $i \in E_o(M,N)$, ν_i now admits a global section T such that $G(T,T) = -1$ and any other section is of the form fT for some smooth real-valued function f on M . Therefore, given a geodesic σ through i as before, the decomposition of $\dot\sigma(t)$ now is of the form

$$\dot\sigma(t) = T\sigma(t)\circ X_t + N_t T .$$

In this formula, T stands for a section of TN , defined in some neighbourhood of $L = i(M)$, e.g. some "ε-tube" S_i^ε , such that T takes values in ν_j for $j \in E^t(M,S_i^\varepsilon)$ and that it has no zeroes.
The time-dependent function N_t on M is called the <u>lapse</u> (function) in Geometrodynamics. For these notions, we refer to [D], [DW], [ADM], [HKT], [K], [FM] and the bibliographies of these papers. Note that the approach of [FM], directly following [D], [ADM] and [DW], is formulated essentially on the manifold $\mathcal{M}$ of Riemannian structures on M (or on "superspace" $\mathcal{M}/\mathrm{Diff}(M)$, which is not a manifold), while the model used in [K], [HKT], etc. is quite different: Since the term "hypersurface" in [HKT] means "smooth embedding of M into N", "hyperspace" of these papers simply is $E_o(M,N)$, described using

local coordinates in M and in N. A more detailed description of "hyperspace" in our current frame-work will appear elsewhere, as will the connection with the "superspace approach": this part is based on the observation that there is a natural smooth map $\varphi:E_o(M,N) \longrightarrow \mathcal{M}$ given by $\varphi(i) = i^*G$. This allows us, e.g., to pull back the (DeWitt) Lagrangian from $\mathcal{M}$ and describe those geodesics of its spray on $\mathcal{M}$ which are "realizable" for given M and N, i.e. which correspond to curves in $E_o(M,N)$.

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