

Lecture 4: Stabilization

There are many stabilization processes in topology, and often matters simplify in a stable limit. As a first example, consider the sequence of inclusions

$$(4.1) \quad S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \dots$$

where each sphere is included in the next as the equator. If we fix a nonnegative integer n and apply π_n to (4.1), then we obtain a sequence of groups with homomorphisms between them:

$$(4.2) \quad \pi_n S^0 \longrightarrow \pi_n S^1 \longrightarrow \pi_n S^2 \longrightarrow \dots$$

Here the homotopy group $\pi_n(X)$ of a topological space X is the set¹ of homotopy classes of maps $[S^n, X]$, and we must use basepoints, as described below. This sequence stabilizes in a trivial sense: for $m > n$ the group $\pi_n S^m$ is trivial. In this lecture we encounter a different sequence

$$(4.3) \quad \pi_n S^0 \longrightarrow \pi_{n+1} S^1 \longrightarrow \pi_{n+2} S^2 \longrightarrow \dots$$

whose stabilization is nontrivial. Here ‘stabilization’ means that with finitely many exceptions every homomorphism in (4.3) is an isomorphism. The groups thus computed are central in stable homotopy theory: the *stable homotopy groups of spheres*.

One reference for this lecture is [DK, Chapter 8].

Pointed Spaces

This is a quick review; look in any algebraic topology book for details.

Definition 4.4.

- (i) A *pointed space* is a pair (X, x) where X is a topological space and $x \in X$.
- (ii) A *map* $f: (X, x) \rightarrow (Y, y)$ of pointed spaces is a continuous map $f: X \rightarrow Y$ such that $f(x) = y$.
- (iii) A *homotopy* $F: \Delta^1 \times (X, x) \rightarrow (Y, y)$ of maps of pointed spaces is a continuous map $F: \Delta^1 \times X \rightarrow Y$ such that $F(t, x) = y$ for all $t \in \Delta^1 = [0, 1]$.

The set of homotopy classes of maps between pointed spaces is denoted $[(X, x), (Y, y)]$, or if basepoints need not be specified by $[X, Y]_*$.

Definition 4.5.

Let $(X_i, *_{i})$ be pointed spaces, $i = 1, 2$.

- (i) The *wedge* is the identification space

$$(4.6) \quad X_1 \vee X_2 = X_1 \amalg X_2 / *_1 \amalg *_2.$$

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¹It is a group for $n \geq 1$, and is an abelian group if $n \geq 2$.

(ii) The *smash* is the identification space

$$(4.7) \quad X_1 \wedge X_2 = X_1 \times X_2 / X_1 \vee X_2.$$

(iii) The *suspension* of X is

$$(4.8) \quad \Sigma X = S^1 \wedge X.$$

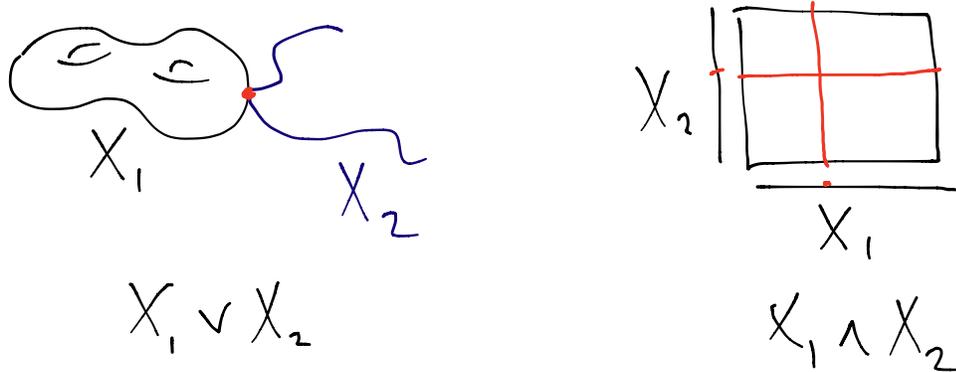


FIGURE 10. The wedge and the smash

For the suspension it is convenient to write S^1 as the quotient $D^1/\partial D^1$ of the 1-disk $[-1, 1] \subset \mathbb{A}^1$ by its boundary $\{-1, 1\}$.

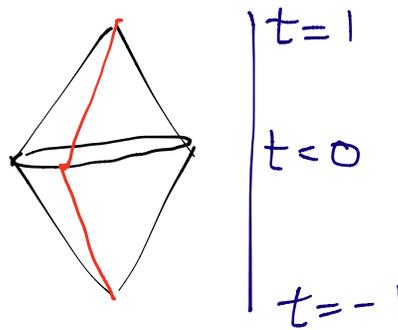


FIGURE 11. The suspension

Exercise 4.9. Construct a homeomorphism $S^k \wedge S^\ell \simeq S^{k+\ell}$. You may find it convenient to write the k -sphere as the quotient of the Cartesian product $(D^1)^{\times k}$ by its boundary.

Exercise 4.10. Suppose $f_i: X_i \rightarrow Y_i$ are maps of pointed spaces, $i = 1, 2$. Construct induced maps

$$(4.11) \quad \begin{aligned} f_1 \vee f_2: X_1 \vee X_2 &\longrightarrow Y_1 \vee Y_2 \\ f_1 \wedge f_2: X_1 \wedge X_2 &\longrightarrow Y_1 \wedge Y_2 \end{aligned}$$

Suppose all spaces are standard spheres and the maps f_i are smooth maps. Is the map $f_1 \wedge f_2$ smooth? Proof or counterexample.

Note that the suspension of a sphere is a smooth manifold, but in general the suspension of a manifold is not smooth at the basepoint.

Definition 4.12. Let $(X, *)$ be a pointed space and $n \in \mathbb{Z}^{\geq 0}$. The n^{th} homotopy group $\pi_n(X, *)$ of $(X, *)$ is the set of pointed homotopy classes of maps $[(S^n, *), (X, *)]$.

If we write S^n as the quotient $D^n/\partial D^n$ (or as the quotient of $(D^1)^{\times n}$ by its boundary), then it has a natural basepoint. We often overload the notation and use ‘ X ’ to denote the pair $(X, *)$. As the terminology suggests, the homotopy set of maps out of a sphere is a group, except for the 0-sphere. Precisely, $\pi_n X$ is a group if $n \geq 1$, and is an abelian group if $n \geq 2$. Figure 12 illustrates the composition in $\pi_n X$, as the composition of a “squeezing map” $S^n \rightarrow S^n \vee S^n$ and the wedge $f_1 \vee f_2$.

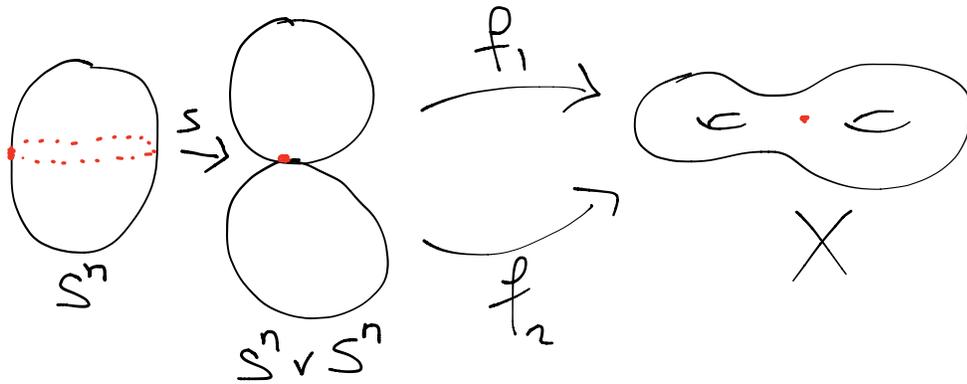


FIGURE 12. Composition in $\pi_n X$

We refer to standard texts for the proof that this composition is associative, that the constant map is the identity, that there are inverses, and that the composition is commutative if $n \geq 2$.

Stabilization of homotopy groups of spheres

We now study the Pontrjagin-Thom Theorem 3.9 in case $M = S^m$ is a sphere. First apply suspension, to both spaces and maps using Exercise 4.9 and Exercise 4.10, to construct a sequence of group homomorphisms

$$(4.13) \quad [S^m, S^q] \xrightarrow{\Sigma} [S^{m+1}, S^{q+1}] \xrightarrow{\Sigma} [S^{m+2}, S^{q+2}] \xrightarrow{\Sigma} \dots,$$

where $m \geq q$ are positive integers.

Theorem 4.14 (Freudenthal). *The sequence (4.13) stabilizes in the sense that all but finitely many maps are isomorphisms.*

The Freudenthal suspension theorem was proved in the late '30s. There are purely algebro-topological proofs. We prove it as a corollary of Theorem 4.44 below and the Pontrjagin-Thom theorem.

(4.15) *Basepoints.* We can introduce basepoints without changing the groups in (4.13).

Lemma 4.16. *If $m, q \geq 1$, then*

$$(4.17) \quad [S^m, S^q]_* = [S^m, S^q].$$

Proof. There is an obvious map $[S^m, S^q]_* \rightarrow [S^m, S^q]$ since a basepoint-preserving map is, in particular, a map. It is surjective since if $f: S^m \rightarrow S^q$, then we can compose f with a path R_t of rotations from the identity R_0 to a rotation R_1 which maps $f(*) \in S^q$ to $*$ in S^q . It is injective since if $F: D^{m+1} \rightarrow S^q$ is a null homotopy of a pointed map $f: S^m \rightarrow S^q$, then we precompose F with a homotopy equivalence $D^{m+1} \rightarrow D^{m+1}$ which maps the radial line segment connecting the center with the basepoint in S^m to the basepoint; see Figure 13. \square

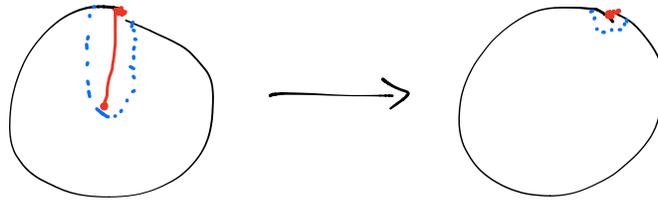


FIGURE 13. Homotopy equivalence of balls

So we can rewrite (4.13) as a sequence of homomorphisms of homotopy groups:

$$(4.18) \quad \pi_m S^q \xrightarrow{\Sigma} \pi_{m+1} S^{q+1} \xrightarrow{\Sigma} \pi_{m+2} S^{q+2} \xrightarrow{\Sigma} \dots$$

(4.19) *A limiting group.* It is natural to ask if there is a group we can assign as the “limit” of (4.18). In calculus we learn about limits, first inside the real numbers and then in arbitrary metric spaces, or in more general topological spaces. Here we want not a limit of elements of a set, but rather a limit of sets. So it is a very different—algebraic—limiting process. The proper setting for such limits is inside a mathematical object whose “elements” are sets, and this is a *category*. We will introduce these in due course, and then the limit we want is, in this case, a *colimit*.² We simply give an explicit construction here, in the form of an exercise.

Exercise 4.20. Let

$$(4.21) \quad A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

be a sequence of homomorphisms of abelian groups. Define

$$(4.22) \quad A = \operatorname{colim}_{q \rightarrow \infty} A_q = \bigoplus_{q=1}^{\infty} A_q / S$$

²Older terminology: *direct limit* or *inductive limit*.

where S is the subgroup of the direct sum generated by

$$(4.23) \quad (f_\ell \circ \cdots \circ f_k)(a_k) - a_k, \quad a_k \in A_k, \quad \ell \geq k.$$

Prove that A is an abelian group, construct homomorphisms $A_q \rightarrow A$, and show they are isomorphisms for $q \gg 1$ if the sequence (4.21) stabilizes in the sense that there exists q_0 such that f_q is an isomorphism for all $q \geq q_0$.

Definition 4.24. The limiting group of the sequence (4.13) is denoted

$$(4.25) \quad \pi_n^s = \operatorname{colim}_{q \rightarrow \infty} \pi_n S^{n+q}$$

and is the n^{th} stable homotopy group of the sphere, or n^{th} stable stem.

Colimits of topological spaces

Question: Is there a pointed space Q so that $\pi_n^s = \pi_n Q$?

There is another construction with pointed spaces which points the way.

Definition 4.26. Let $(X, *)$ be a pointed space. The (based) loop space of $(X, *)$ is the set of continuous maps

$$(4.27) \quad \Omega X = \{\gamma: S^1 \rightarrow X : \gamma(*) = *\}.$$

We topologize ΩX using the compact-open topology, and then complete to a compactly generated topology.

Definition 4.28. A Hausdorff topological space Z is *compactly generated* if $A \subset Z$ is closed iff $A \cap C$ is closed for every compact subset $C \subset Z$.

Compactly generated Hausdorff spaces are a convenient category in which to work, according to a classic paper of Steenrod [Ste]; see [DK, §6.1] for an exposition. A Hausdorff space Z has a compactly generated completion: declare $A \subset Z$ to be closed iff $A \cap C \subset Z$ is closed in the compact-open topology on Z for all compact subsets $C \subset Z$.

Exercise 4.29. Let X, Y be pointed spaces. Prove that there is an isomorphism of sets

$$(4.30) \quad \operatorname{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \operatorname{Map}_*(X, \Omega Y).$$

Here ‘ Map_* ’ denotes the set of pointed maps. If X and Y are compactly generated, then the map (4.30) is a homeomorphism of topological spaces, where the mapping spaces have the compactly generated completion of the compact-open topology. Metric spaces, in particular smooth manifolds, are compactly generated. You can find a nice discussion of compactly generated spaces in [DK, §6.1].

Use (4.30) to rewrite (4.18) as

$$(4.31) \quad \pi_n(S^0) \longrightarrow \pi_n(\Omega S^1) \longrightarrow \pi_n(\Omega^2 S^2) \longrightarrow \cdots$$

This suggests that the space Q is some sort of limit of the spaces $\Omega^q S^q$ as $q \rightarrow \infty$. This is indeed the case.

(4.32) *Colimit of a sequence of maps.* Let

$$(4.33) \quad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots$$

be a sequence of continuous inclusions of topological spaces. Then there is a limiting topological space

$$(4.34) \quad X = \operatorname{colim}_{q \rightarrow \infty} X_q = \coprod_{q=1}^{\infty} X_q / \sim$$

equipped with inclusions $g_q: X_q \hookrightarrow X$. Here \sim is the equivalence relation generated by setting $x_k \in X_k$ equivalent to $(f_\ell \circ \dots \circ f_k)(x_k)$ for all $\ell \geq k$. We give X the quotient topology. It is the strongest (finest) topology so that the maps g_q are continuous. More concretely, a set $A \subset X$ is closed iff $A \cap X_q \subset X_q$ is closed for all q . Then X is called the *colimit* of the sequence (4.33).

Exercise 4.35. Construct S^∞ as the colimit of (4.1). Prove that S^∞ is *weakly contractible*: for $n \in \mathbb{Z}^{\geq 0}$ any map $S^n \rightarrow S^\infty$ is null homotopic.

Exercise 4.36. Show that if each space X_q in (4.33) is Hausdorff compactly generated and f_q is a *closed* inclusion, then the colimit (4.34) is also compactly generated. Furthermore, every compact subset of the colimit is contained in X_q for some q .

(4.37) *The space QS^0 .* Now apply (4.32) to the sequence

$$(4.38) \quad S^0 \longrightarrow \Omega S^1 \longrightarrow \Omega^2 S^2 \longrightarrow \dots$$

This is, in fact, a sequence of inclusions of the form $X \hookrightarrow \Omega \Sigma X$, as illustrated in Figure 14. The limiting space of the sequence (4.38) is

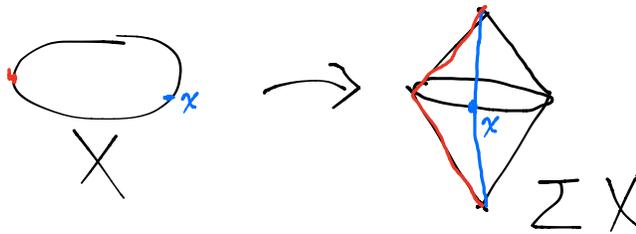


FIGURE 14. The inclusion $X \hookrightarrow \Omega \Sigma X$

$$(4.39) \quad QS^0 := \operatorname{colim}_{q \rightarrow \infty} \Omega^q S^q$$

and is the 0-space of the *sphere spectrum*.

Proposition 4.40. $\pi_n^s = \pi_n(QS^0)$.

Proof. More generally, for a sequence of closed inclusions of compactly generated Hausdorff spaces (4.33) we prove

$$(4.41) \quad \pi_n(\operatorname{colim}_{q \rightarrow \infty} X_q) \cong \operatorname{colim}_{q \rightarrow \infty} \pi_n X_q.$$

Let $X = \operatorname{colim}_{q \rightarrow \infty} X_q$. A class in $\pi_n X_q$ is represented by a continuous map $f: S^n \rightarrow X$, and by the last assertion you proved in Exercise 4.36 f factors through a map $\tilde{f}: S^n \rightarrow X_q$ for some q . This shows that the natural map $\operatorname{colim}_{q \rightarrow \infty} \pi_n X_q \rightarrow \pi_n X$ is surjective. Similarly, a null homotopy of the composite $S^n \xrightarrow{\tilde{f}} X_q \hookrightarrow X$ factors through some X_r , $r \geq q$, and this proves that this natural map is also injective. \square

Stabilization of framed submanifolds

By Theorem 3.9 we can rewrite (4.13) as a sequence of maps

$$(4.42) \quad \Omega_{n;S^m}^{\text{fr}} \xrightarrow{\sigma} \Omega_{n;S^{m+1}}^{\text{fr}} \xrightarrow{\sigma} \Omega_{n;S^{m+2}}^{\text{fr}} \xrightarrow{\sigma} \cdots$$

Representatives of these framed bordism groups are submanifolds of S^m . Write $S^m = \mathbb{A}^m \cup \{\infty\}$, and recall from the proof in Lecture §1 that each framed bordism class is represented by a framed submanifold $Y \subset \mathbb{A}^m$; we can arrange $\infty \notin Y$. This is the analog of passing to a sequence of pointed maps, as in Lemma 4.16.

We make two immediate deductions from the identification with (4.13). First, we must have that each $\Omega_{n;S^m}^{\text{fr}}$ is an abelian group. The abelian group law is the disjoint union of submanifolds of \mathbb{A}^m , effected by writing $\mathbb{A}^m = \mathbb{A}^m \amalg \mathbb{A}^m$ (similar to the collapse map in Figure 12). Second, the stabilization map σ in (4.42) is the map

$$(4.43) \quad (Y \subset \mathbb{A}^m) \longmapsto (0 \times Y \subset \mathbb{A}^1 \times \mathbb{A}^m)$$

and the new normal framing prepends the constant vector field $\partial/\partial x^1$ to the given normal framing of Y .

We can now state the stabilization theorem.

Theorem 4.44. *The map $\sigma: \Omega_{n;S^m}^{\text{fr}} \rightarrow \Omega_{n;S^{m+1}}^{\text{fr}}$ is an isomorphism for $m \geq 2n + 2$.*

As a corollary we obtain a precise estimate on the Freudenthal isomorphism, using the Pontrjagin-Thom identification.

Corollary 4.45. *The map $\Sigma: \pi_m S^q \rightarrow \pi_{m+1} S^{q+1}$ is an isomorphism for $m \leq 2q - 2$.*

We will not prove the precise estimate in Theorem 4.44, and so not the precise estimate in Corollary 4.45 either. Rather, we only prove Theorem 4.44 for sufficiently large m , where sufficiently large depends on n . This suffices to prove the stabilization.

Exercise 4.46. Show that the bound in Theorem 4.44 is optimal for $n = 1$.

The proof of Theorem 4.44 is based on the Whitney Embedding Theorem. We restrict to compact manifolds. Recall that for compact manifolds embeddings are easier to handle since they are injective immersions. An *isotopy of embeddings* $Y \hookrightarrow \mathbb{A}^N$ is a smooth map

$$(4.47) \quad \Delta^1 \times Y \longrightarrow \mathbb{A}^N$$

so that the restriction to $\{t\} \times Y$ is an embedding for all $t \in \Delta^1$. In other words, an isotopy of embeddings is a path of embeddings.

Theorem 4.48. *Let Y be a smooth compact n -manifold.*

- (i) *There exists an embedding $i: Y \hookrightarrow \mathbb{A}^{2n+1}$. Furthermore, if $i: Y \hookrightarrow \mathbb{A}^N$ is an embedding with $N > 2n + 1$, then there is an isotopy of i to an embedding into an affine subspace $\mathbb{A}^{2n+1} \subset \mathbb{A}^N$.*
- (ii) *If $i_0, i_1: Y \hookrightarrow \mathbb{A}^{2n+1}$ are embeddings, then their stabilizations*

$$(4.49) \quad \begin{aligned} \tilde{i}_k: Y &\longrightarrow \mathbb{A}^{2n+1} \times \mathbb{A}^{2n+1} \\ y &\longmapsto (0, i_k(y)) \end{aligned}$$

($k = 0, 1$) are isotopic.

- (iii) *Let X be a compact $(n+1)$ -manifold with boundary. Then there is an embedding $X \hookrightarrow \mathbb{A}_-^{2n+3}$ as a neat submanifold with boundary.*

Assertion (i) is the *easy* Whitney Embedding Theorem, and we refer to [GP] for a proof. The second statement in (i) follows from the proof, which uses linear projection onto an affine subspace to reduce the dimension of the embedding. Statement (iii) is stated as [Hi, Theorem 4.3]; perhaps in the mythical next version of these notes I'll supply a proof. In any case we do not need the statement with 'neat', and without 'neat' the proof is essentially the same as that of (i). We remark that the *hard* Whitney Embedding Theorem asserts that there is an embedding $Y \hookrightarrow \mathbb{A}^{2n}$.

Proof. We prove (ii). The desired isotopy $\tilde{i}_0 \rightarrow \tilde{i}_1$ is constructed as the composition in time of three isotopies:

$$(4.50) \quad \begin{aligned} (t, y) &\longmapsto (t i_0(y), i_0(y)), & 0 \leq t \leq 1; \\ (t, y) &\longmapsto (i_0(y), (2-t) i_0(y) + (t-1) i_1(y)), & 1 \leq t \leq 2; \\ (t, y) &\longmapsto ((3-t) i_0(y), i_1(y)), & 2 \leq t \leq 3; \end{aligned}$$

□

Exercise 4.51. Check that the map $[0, 3] \times Y \rightarrow \mathbb{A}^{4n+2}$ defined by (4.50) is an embedding. Now use the technique of the Whitney Embedding Theorem to project onto a subspace of dimension $2n + 2$ so that the composition is still an embedding. Can you use this to prove that, in fact, the stabilizations of i_0, i_1 to embeddings $Y \hookrightarrow \mathbb{A}^{2n+2}$ are isotopic? (That statement can be proved using an approximation theorem; see Exercise 10 in [Hi, p. 183].)

Exercise 4.52. A *parametrized knot* is an embedding $i: S^1 \rightarrow \mathbb{A}^3$. Exhibit two parametrized knots which are not isotopic. Can you *prove* that they are not isotopic? The proof above shows that they are isotopic when stabilized to embeddings $\tilde{i}: S^1 \rightarrow \mathbb{A}^6$. Prove that they are isotopic when stabilized to embeddings $\tilde{i}: S^1 \rightarrow \mathbb{A}^4$.

Sketch proof of Theorem 4.44. To show that $\sigma: \Omega_{n;S^m}^{\text{fr}} \rightarrow \Omega_{n;S^{m+1}}^{\text{fr}}$ is surjective, suppose $i_0: Y \hookrightarrow \mathbb{A}^{m+1}$ is an embedding. By Theorem 4.48(i) there is an isotopy $\Delta^1 \times Y \rightarrow \mathbb{A}^{m+1}$ to an embedding $i_1: Y \hookrightarrow \mathbb{A}^m \subset \mathbb{A}^{m+1}$. To show that σ is injective, suppose $j_0: Y \hookrightarrow \mathbb{A}^m$ is an embedding and $k_0: X \hookrightarrow \mathbb{A}^{m+1}$ is a null bordism of the composition $Y \xrightarrow{j_0} \mathbb{A}^m \subset \mathbb{A}^{m+1}$. Then Theorem 4.48(iii) implies there is an isotopy $k_t: \Delta^1 \times X \rightarrow \mathbb{A}^{m+1}$ with $k_1(X) \subset \mathbb{A}^m$ a null bordism of j_0 . \square

There is one problem: we have not discussed the normal framings. Briefly, in both the surjectivity and injectivity arguments there is an isotopy $\Delta^1 \times Z \rightarrow \mathbb{A}^{m+1}$, a normal bundle $\nu \rightarrow \Delta^1 \times Z$, and a framing of $\nu|_{\{0\} \times Z}$. We need two general results to get the desired framing of $\nu|_{\{1\} \times Z}$. First, we can extend the given framing over $\{0\} \times Z$ to the entire cylinder $\Delta^1 \times Z$, for example using parallel transport of a connection (so solving an ODE). Second, the restriction of ν to $\{1\} \times Z$ splits off a trivial line bundle, and we can homotop the framing to one which respects this splitting. This follows from a stability statement for homotopy groups of the general linear group. Perhaps these arguments will appear in that mystical future revision. . . .

References

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