Lecture 2: Orientations, framings, and the Pontrjagin-Thom construction

Thom's great contribution was to translate problems in geometric topology—such as the computation (Theorem 1.37) of the unoriented bordism ring—into a problem of homotopy theory. The correspondence works in both directions: facts about manifolds can sometimes be used to deduce homotopical information. This lecture ends with a first instance of that principle. The geometric side is the set of *framed* bordism classes of submanifolds of a fixed manifold M; the homotopical side is the set of homotopy classes of maps from M into a sphere. The theorem gives an isomorphism between these two sets. Here we introduce the basic idea; the proof will be given in the next lecture. We will build on these ideas in subsequent lectures and so translate the computation of bordism groups (Lecture 1) into homotopy theory.

Before getting to framed bordism we give a reminder on orientations and introduce the oriented bordism ring. Orientations are an example of a (stable) tangential structure; we will discuss general tangential structures and associated bordisms soon.

Orientations

(2.1) *Orientation of a real vector space.* Let V be a real vector space of dimension $n > 0$. A *basis* of V is a linear isomorphism $b\colon \mathbb{R}^n \to V$. Let $\mathcal{B}(V)$ denote the set of all bases of V. The group $GL_n(\mathbb{R})$ of linear isomorphisms of $\mathbb{R}^n$ acts *simply transitively* on the right of $\mathcal{B}(V)$ by composition: if $b\colon \mathbb{R}^n \to V$ and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ are isomorphisms, then so too is $b\circ g\colon \mathbb{R}^n \to V$. We say that $\mathcal{B}(V)$ is a *right $GL_n(\mathbb{R})$-torsor*. For any $b \in \mathcal{B}(V)$ the map $g \mapsto b\circ g$ is a bijection from $GL_n(\mathbb{R})$ to $\mathcal{B}(V)$, and we use it to topologize $\mathcal{B}(V)$. Since $GL_n(\mathbb{R})$ has two components, so does $\mathcal{B}(V)$.

Definition 2.2. An orientation of V is a choice of component of $\mathcal{B}(V)$.

(2.3) *Determinants and orientation.* Recall that the components of $GL_n(\mathbb{R})$ are distinguished by the determinant homomorphism

$$\det\colon GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^{\neq 0}; \tag{2.4}$$

the identity component consists of $g \in GL_n(\mathbb{R})$ with $\det(g) > 0$, and the other component consists of g with $\det(g) < 0$. On the other hand, an isomorphism $b\colon \mathbb{R}^n \to V$ does not have a numerical determinant. Rather, its determinant lives in the *determinant line* $\operatorname{Det} V$ of V. Namely, define

$$\operatorname{Det} V = \{\epsilon\colon \mathcal{B}(V) \to \mathbb{R} : \epsilon(b\circ g) = \det(g)^{-1}\epsilon(b)\ \text{ for all } b \in \mathcal{B}(V),\ g \in GL_n(\mathbb{R})\}. \tag{2.5}$$

Exercise 2.6. Here are some elementary facts about determinants and orientations.

(i) Construct a canonical isomorphism $\operatorname{Det} V \xrightarrow{\cong} \bigwedge^n V$ of the determinant line with the highest exterior power. The latter is often taken as the definition.

(ii) Prove that an orientation is a choice of component of $\operatorname{Det} V \setminus \{0\}$. More precisely, construct a map $\mathcal{B}(V) \to \operatorname{Det} V \setminus \{0\}$ which induces a bijection on components.

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(iii) Construct the "determinant" of an arbitrary linear map $b\colon \mathbb{R}^n \to V$ as an element of $\operatorname{Det} V$. Show it is nonzero iff b is invertible.
(iv) More generally, construct the determinant of a linear map $T\colon V \to W$ as a linear map $\det T\colon \operatorname{Det} V \to \operatorname{Det} W$, assuming $\dim V = \dim W$.
(v) Part (ii) gives two description of a canonical $\{\pm 1\}$-torsor[1] (=set of two points) associated to a finite dimensional real vector space. Show that it can also be defined as

$$\mathfrak{o}(V) = \{\epsilon\colon \mathcal{B}(V) \to \{\pm 1\} : \epsilon(b \circ g) = \operatorname{sign}\det(g)^{-1}\epsilon(b) \;\text{ for all } b \in \mathcal{B}(V),\; g \in GL_n(\mathbb{R})\}. \tag{2.7}$$

Summary: An orientation of V is a point of $\mathfrak{o}(V)$.

(2.8) *Orienting the zero vector space.* There is a unique zero-dimensional vector space 0 consisting of a single element, the zero vector. There is a unique basis—the empty set—and so by (2.5) the determinant line $\operatorname{Det} 0$ is canonically isomorphic to $\mathbb{R}$ and $\mathfrak{o}(V)$ is canonically isomorphic to $\{\pm 1\}$. Note that $\bigwedge^0(0) = \mathbb{R}$ as $\bigwedge^0 V = \mathbb{R}$ for *any* real vector space V. The real line $\mathbb{R}$ has a canonical orientation: the component $\mathbb{R}^{>0} \subset \mathbb{R}^{\neq 0}$. We denote this orientation as '+'. The opposite orientation is denoted '−'.

Exercise 2.9 (2-out-of-3)**.** Suppose

$$0 \longrightarrow V' \xrightarrow{i} V \xrightarrow{j} V'' \longrightarrow 0 \tag{2.10}$$

is a *short exact sequence* of finite dimensional real vector spaces. Construct a canonical isomorphism

$$\operatorname{Det} V'' \otimes \operatorname{Det} V' \longrightarrow \operatorname{Det} V. \tag{2.11}$$

Notice the order: *quotient before sub.* If two out of three of V, V', V'' are oriented, then there is a unique orientation of the third compatible with (2.11). This lemma is quite important in oriented intersection theory.

(2.12) *Real vector bundles and orientation.* Now let X be a smooth manifold and $V \to X$ a finite rank real vector bundle. For each $x \in X$ there is associated to the fiber V_x over x a canonical $\{\pm 1\}$-torsor $\mathfrak{o}(V)_x$—a two-element set—which has the two descriptions given in Exercise 2.6(ii).

Exercise 2.13. Use local trivializations of $V \to X$ to construct local trivializations of $\mathfrak{o}(V) \to X$, where $\mathfrak{o}(V) = \coprod_{x \in X} \mathfrak{o}(V)_x$.

The 2:1 map $\mathfrak{o}(V) \to X$ is called the *orientation double cover associated to* $V \to X$. In case $V = TX$ is the tangent bundle, it is called the *orientation double cover of* X.

Definition 2.14.

(i) An *orientation* of a real vector bundle $V \to X$ is a section of $\mathfrak{o}(V) \to X$.
(ii) If $o\colon X \to \mathfrak{o}(V)$ is an orientation, then the *opposite orientation* is the section $-o\colon X \to \mathfrak{o}(V)$.

[1] $\{\pm 1\}$ is the multiplicative group of square roots of unity, sometimes denoted μ_2.

(iii) An *orientation* of a manifold X is an orientation of its tangent bundle $TX \to X$.

Orientations may or may not exist, which is to say that a vector bundle $V \to X$ may be orientable or non-orientable. The notation '$-o$' in (ii) uses the fact that $\mathfrak{o}(V) \to X$ is a principal $\{\pm 1\}$-bundle: $-o$ is the result of acting $-1 \in \{\pm 1\}$ on the section o.

Exercise 2.15. Construct the *determinant line bundle* $\operatorname{Det} V \to X$ by carrying out the determinant construction (2.5) (cf. Exercise 2.6) pointwise and proving local trivializations exist. Show that a *nonzero* section of $\operatorname{Det} V \to X$ determines an orientation.

Our first bordism invariant

This subsection is an extended exercise in which you construct a homomorphism

$$\phi\colon \Omega_2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{2.16}$$

and prove that it is an isomorphism. (Recall that we computed $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$ in Proposition 1.32, and the proof depends on the fact that $\mathbb{RP}^2$ is not a boundary. In this exercise you will give a different proof of that fact.) An element of Ω_2 is represented by a closed 2-manifold Y. We must (i) define $\phi(Y) \in \mathbb{Z}/2\mathbb{Z}$; (ii) prove that if Y_0 and Y_1 are bordant, then $\phi(Y_0) = \phi(Y_1)$; (iii) prove that ϕ is a homomorphism; and (iv) show that $\phi(\mathbb{RP}^2) \neq 0$. Here is a sketch for you to complete. It relies on elementary differential topology à la Guillemin-Pollack and is a good review of techniques in intersection theory as well as the geometry of projective space.

(i) Choose a section s of $\operatorname{Det} Y \to Y$, where $\operatorname{Det} Y = \operatorname{Det} TY$ is the determinant line bundle of the tangent bundle. Show that we can assume that s is *transverse* to the zero section $Z \subset \operatorname{Det} Y$, where Z is the submanifold of zero vectors. Show that $s^{-1}(Z)$ is a 1-dimensional submanifold of Y. Define $\phi(Y)$ as the mod 2 intersection number of $s^{-1}(Z)$ with itself. Prove that $\phi(Y)$ is independent of the choice of s.

(ii) If X is a bordism from Y_0 to Y_1, show that $\operatorname{Det} X \to X$ restricts on the boundary to the determinant line of the boundary. You may want to use Exercise 2.9 and (1.12). Extend the section s constructed in (i) (for each of Y_0, Y_1) over X so that it is transverse to the zero section. What can you say now about the inverse image of the zero section in X and about its self-intersection?

(iii) This is easy: consider a disjoint union.

(iv) Since $\mathbb{RP}^2$ is the manifold of lines (= one-dimensional subspaces) in $\mathbb{R}^3$, there is a canonical line bundle $L \to \mathbb{RP}^2$ whose fiber at a line $\ell \subset \mathbb{R}^3$ is ℓ. Show that the determinant line bundle of $\mathbb{RP}^2$ is isomorphic to $L \to \mathbb{RP}^2$. (See **(2.17)** below.) Now fix the standard metric on $\mathbb{R}^3$ and define $s(\ell)$ to be the orthogonal projection of the vector $(1, 0, 0)$ onto ℓ. What is $s^{-1}(Z)$?

(2.17) *The tangent bundle to projective space.* In (iv) you are asked to "Show that the determinant line bundle of $\mathbb{RP}^2$ is isomorphic to $L \to \mathbb{RP}^2$." For that, let Q_ℓ denote the quotient vector space $\mathbb{R}^3/\ell$ for each line $\ell \subset \mathbb{R}^3$. The 2-dimensional vector spaces Q_ℓ fit together into a vector bundle $Q \to \mathbb{RP}^2$, and there is a short exact sequence

$$0 \longrightarrow L \longrightarrow \underline{\mathbb{R}^3} \longrightarrow Q \longrightarrow 0 \tag{2.18}$$

of vector bundles over $\mathbb{RP}^2$, where $\underline{U}$ denotes the vector bundle with constant fiber the vector space U. Claim: There is a natural isomorphism

$$T(\mathbb{RP}^2) \xrightarrow{\cong} \operatorname{Hom}(L, Q). \tag{2.19}$$

(There are analogous canonical sub and quotient bundles for any Grassmannian, and the analog of (2.19) is true.) To construct the isomorphism (2.19), fix $\ell \subset \mathbb{R}^3$ and a complementary subspace $W \subset \mathbb{R}^3$. Let ℓ_t, $-\epsilon < t < \epsilon$, be a curve in $\mathbb{RP}^2$ with $\ell_0 = \ell$. For $|t|$ sufficiently small we can write ℓ_t as the image of a unique linear map $T_t \in \operatorname{Hom}(\ell, W)$. Note $T_0 = 0$. The tangent vector to this curve of linear maps at time 0 is $\dot{T}_0 \in \operatorname{Hom}(\ell, W)$, and its image in $\operatorname{Hom}(\ell, \mathbb{R}^2/\ell)$ after composition with the isomorphism $W \hookrightarrow \mathbb{R}^3 \twoheadrightarrow \mathbb{R}^3/\ell$ is independent of the choice of complement W.

For the rest of (iv) I suggest tensoring (2.18) with L^* and applying the 2-out-of-3 principle (Exercise 2.9). You may also wish to show that the tensor square of a real line bundle is trivializable.

Oriented bordism

We repeat the discussion of unoriented bordism in Lecture 1, beginning with Definition 1.19, for manifolds with orientation. So in Definition 1.19 each of Y_0, Y_1 carries an orientation, as does the bordism X, and the embeddings θ_0, θ_1 are required to be orientation-preserving.

FIGURE 5. Some oriented bordisms of 0-manifolds

Figure 5 illustrates four different bordisms in which X is the oriented closed interval. The pictures do not explicitly indicate the decomposition $\partial X = (\partial X)_0 \amalg (\partial X)_1$ of the boundary into incoming and outgoing components, nor do we make explicit the collarings θ_0, θ_1. We make the convention that we read the picture from left to right with the incoming boundary components on the left. Thus, in the first two bordisms the incoming boundary $(\partial X)_0$ and outgoing boundary $(\partial X)_1$ each consist of a single point. In the third bordism the incoming boundary $(\partial X)_0$ consists of two points and the outgoing boundary $(\partial X)_1$ is empty. In the fourth bordism the situation is reversed. Check carefully that (1.20) and (1.21) are orientation-preserving. You will need to think through the orientation of a Cartesian product of manifolds, which amounts to the orientation of a direct sum of vector spaces, which is a special case of Exercise 2.9. (You will also need (**2.8**).)

(2.20) *Dual oriented bordism.* There is an important modification to Definition 1.22. Namely, the dual $Y^\vee$ to a closed oriented manifold Y is not equal to Y, as in the unoriented case (see Remark 1.24). Rather,

$$Y^\vee = -Y, \tag{2.21}$$

where $-Y$ denotes the manifold Y with the opposite orientation (Definition 2.14(ii)). The reversal of orientation ensures that $\theta_0^\vee$ and $\theta_1^\vee$ in (1.23) are orientation-preserving.

Exercise: Construct the dual to each bordism in Figure 5.

(2.22) *Oriented bordism defines an equivalence relation.* Define two closed oriented n-manifolds Y_0, Y_1 to be equivalent if there exists an oriented bordism from Y_0 to Y_1. As in Lemma 1.25 oriented bordism defines an equivalence relation. There is one small, but very important, modification in the proof of symmetry: if X is a bordism from Y_0 to Y_1, then $-X^\vee$ is a bordism from Y_1 to Y_0. (The point is to use the *orientation-reversed* dual.)

(2.23) *The oriented bordism ring.* We denote the set of oriented bordism classes of n-manifolds as Ω_n^{SO}. As in (1.35) there is an oriented bordism ring Ω^{SO}.

I will now summarize some facts about Ω^{SO}; see [St, M1, W], [MS, §17] for more details.

Theorem 2.24.

(i) [T] *There is an isomorphism*

$$\mathbb{Q}[y_4, y_8, y_{12}, \dots] \xrightarrow{\cong} \Omega^{SO} \otimes \mathbb{Q} \tag{2.25}$$

under which y_{4k} maps to the oriented bordism class of the complex projective space $\mathbb{CP}^{2k}$.

(ii) [Av, M2, W] *All torsion in Ω^{SO} is of order 2.*

(iii) [M2, No] *There is an isomorphism*

$$\mathbb{Z}[z_4, z_8, z_{12}, \dots] \xrightarrow{\cong} \Omega^{SO}/\text{torsion}. \tag{2.26}$$

(iv) [W] *The Stiefel-Whitney numbers* (1.41) *and Pontrjagin numbers*

$$\langle p_{j_1}(Y) \smile p_{j_2}(Y) \smile \cdots \smile p_{j_k}(Y)\, ,\, [Y]\rangle \in \mathbb{Z}, \tag{2.27}$$

determine the oriented bordism class of a closed oriented manifold Y. In particular, Y is the boundary of a compact oriented manifold iff all of the Stiefel-Whitney and Pontrjagin numbers vanish.

The generators in (2.26) are not complex projective spaces, but can be taken to be certain complex manifolds called *Milnor hypersurfaces*. The *Pontrjagin classes* are characteristic classes in integral cohomology, and they live in degrees divisible by 4. The Pontrjagin numbers of an oriented manifold are nonzero only for manifolds whose dimension is divisible by 4.

We will eventually give a proof of (i) and use it to prove Hirzebruch's signature theorem.

(2.28) *Low dimensions.*

$\Omega_0^{SO} \cong \mathbb{Z}$. The generator is an oriented point. Recall from **(2.8)** that a point has two canonical orientations: $+$ and $-$. For definiteness we take the generator to be pt_+, the positively oriented point.

$\Omega_1^{SO} = 0$. Every closed oriented 1-manifold is a finite disjoint union of circles S^1, and $S^1 = \partial D^2$.

$\Omega_2^{SO} = 0$. Every closed oriented surface is a disjoint union of connected sums of 2-tori, and such connected sums bound handlebodies in 3-dimensional space.

$\Omega_3^{SO} = 0$. This is the first theorem which goes beyond classical classification theorems in low dimensions. The general results in Theorem 2.24 imply that Ω_3^{SO} is torsion, but more is needed to prove that it vanishes.

$\Omega_4^{SO} \cong \mathbb{Z}$. The complex projective space $\mathbb{CP}^2$ is a generator. We will see in a subsequent lecture that the *signature* of a closed oriented 4-manifold defines an isomorphism $\Omega_4^{SO} \to \mathbb{Z}$.

$\Omega_5^{SO} \cong \mathbb{Z}/2\mathbb{Z}$. This is the lowest dimensional torsion in the oriented bordism ring. The nonzero element is represented by the Dold manifold Y^5 which is a fiber bundle $Y^5 \to \mathbb{RP}^1 = S^1$ with fiber $\mathbb{CP}^2$. (See the comment after Theorem 1.37.)

$\Omega_6^{SO} = \Omega_7^{SO} = 0$.

$\Omega_8^{SO} \cong \mathbb{Z} \oplus \mathbb{Z}$. It is generated by $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$.

More fun facts: $\Omega_n^{SO} \neq 0$ for all $n \geq 9$. Complex projective spaces and their Cartesian products generate $\Omega_4^{SO}, \Omega_8^{SO}, \Omega_{12}^{SO}$ but not Ω_{16}^{SO}.

Remark 2.29. The *cobordism hypothesis*, which is a recent theorem about the structure of *multi-categories* of manifolds, is a vast generalization of the theorem that Ω_0^{SO} is the free abelian group generated by pt_+.

Framed bordism and the Pontrjagin-Thom construction

Some of this discussion is a bit vague; we give precise definitions and proofs in the next lecture.

Fix a closed m-dimensional manifold M. Let $Y \subset M$ be a submanifold. Recall that on Y there is a short exact sequence of vector bundles

$$0 \longrightarrow TY \longrightarrow TM\big|_Y \longrightarrow \nu \longrightarrow 0 \tag{2.30}$$

where ν is defined to be the quotient bundle and is called the *normal bundle of Y in M*.

Definition 2.31. A *framing* of the submanifold $Y \subset M$ is a trivialization of the normal bundle ν.

Recall that a trivialization of ν is an isomorphism of vector bundles $\underline{\mathbb{R}^q} \to \nu$, where q is the codimension of Y in M. Equivalently, it is a global basis of sections of ν.

Framed submanifolds of M of codimension q arise as follows. Let N be a manifold of dimension q and $f\colon M \to N$ a smooth map. Suppose $p \in N$ is a *regular value* of f and fix a basis $e_1, \dots, e_q$ of T_pN. Then $Y := f^{-1}(p) \subset M$ is a submanifold and the basis $e_1, \dots, e_q$ pulls back to a basis of the normal bundle at each point $y \in Y$. For under the differential f_* at y the subspace $T_yY \subset T_yM$

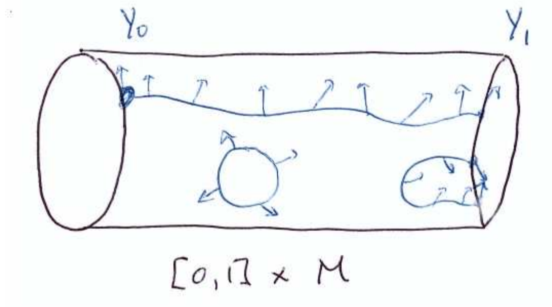

Figure 6. A framed bordism in M

maps to zero, whence f_* factors down to a map $\nu_y \to T_pN$. The fact that p is a regular value implies that the latter is an isomorphism.

Of course, regular values are not unique. In fact, Sard's theorem asserts that they form an open dense subset of N. If N is connected, then we will see that the inverse images $Y_0 := f^{-1}(p_0)$ and $Y_1 = f^{-1}(p_1)$ of two regular values $p_0, p_1 \in N$ are *framed bordant in M*. (See Figure 6.) This means that there is a framed submanifold with boundary $X \subset [0,1] \times M$ such that $X \cap \bigl(\{i\} \times M\bigr) = Y_i,\ i = 0,1$, where the framings match at the boundary. While we can transport the framing at p_0 to a framing at p_1 along the path, at least to obtain a homotopy class of framings, we need an *orientation* of N to consistently choose framings at all points of N. In other words, f determines a framed bordism class of framed submanifolds of M of codimension p as long as N is oriented (and connected). Denote the set of these classes as $\Omega^{\mathrm{fr}}_{m-q;M}$. We will also show that homotopic maps lead to the same framed bordism class, so the construction gives a well-defined map

$$[M,N] \longrightarrow \Omega^{\mathrm{fr}}_{m-q;M}. \tag{2.32}$$

Here $[M,N]$ denotes the set of homotopy classes of maps from M to N.

From now on suppose $N = S^q$. Then we construct an inverse to (2.32): *Pontrjagin-Thom collapse*. Let $Y \subset M$ be a framed submanifold of codimension q. Recall that any submanifold Y has a *tubular neighborhood*, which is an open neighborhood $U \subset M$ of Y, a submersion $U \to Y$, and an isomorphism $\varphi\colon \nu \to U$ which makes the diagram

$$\begin{tikzcd} \nu \arrow[rr, "\varphi"] \arrow[dr] & & U \arrow[dl] \\ & Y & \end{tikzcd} \tag{2.33}$$

commute. The framing of ν then leads to a map $h\colon U \to \mathbb{R}^q$. The collapse map $f_Y\colon Y \to S^q$ is

$$f_Y(x) = \begin{cases} \dfrac{h(x)}{\rho\bigl(|h(x)|\bigr)}, & x \in U; \\ \infty, & x \in N \setminus U. \end{cases} \tag{2.34}$$

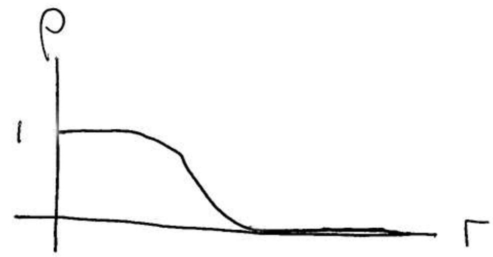

FIGURE 7. Cutoff function for collapse map

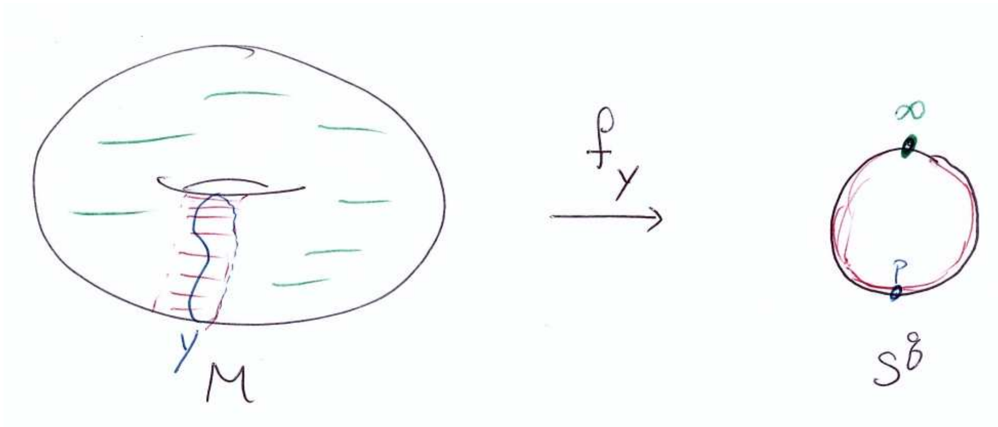

FIGURE 8. Pontrjagin-Thom collapse

Here we write $S^q = \mathbb{R}^q \cup \{\infty\}$ and we fix a cutoff function ρ as depicted in Figure 7. We represent a collapse map in Figure 8.

Theorem 2.35 (Pontrjagin-Thom)**.** *There is an isomorphism*

$$[M, S^q] \longrightarrow \Omega^{\mathrm{fr}}_{m-q;M} \tag{2.36}$$

which takes a map $M \to S^q$ to the inverse image of a regular value. The inverse map is Pontrjagin-Thom collapse.

There are choices (regular value, tubular neighborhood, cutoff function) in these construction. Part of Theorem 2.35 is that the resulting map (2.36) and its inverse are independent of these choices. We prove Theorem 2.35 in the next lecture.

The Hopf degree theorem

As a corollary of Theorem 2.35 we prove the following.

Theorem 2.37 (Hopf)**.** *Let M be a closed connected manifold of dimension m.*

(i) *If M is orientable, then there is an isomorphism*

$$[M, S^m] \longrightarrow \mathbb{Z} \tag{2.38}$$

given by the integer degree.

(ii) *If M is not orientable, then there is an isomorphism*

$$[M, S^m] \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{2.39}$$

given by the mod 2 degree.

By Theorem 2.35 homotopy classes of maps $M \to S^m$ are identified with framed bordism classes of framed 0-dimensional submanifolds of M. Now a 0-dimensional submanifold of M is a finite disjoint union of points, and a framed point is a point $y \in M$ together with a basis of T_yM.

We apply an important general principle in geometry: to study an object O introduce the *moduli space* of all objects of that type and formulate questions in terms of the geometry of that moduli space. In this case we are led to introduce the *frame bundle*.

(2.40) *The frame bundle.* For any smooth manifold M, define

$$\mathcal{B}(M) = \{(y, b) : y \in M,\ b \in \mathcal{B}(T_yM)\}. \tag{2.41}$$

Recall from **(2.1)** that b is an isomorphism $b\colon \mathbb{R}^m \to T_yM$. There is an obvious projection

$$\begin{aligned} \pi\colon \mathcal{B}(M) &\longrightarrow M \\ (y, b) &\longmapsto y \end{aligned} \tag{2.42}$$

We claim that (2.42) is a fiber bundle. There is more structure. Recall that each fiber $\mathcal{B}(M)_y = \mathcal{B}(T_yM)$ is a $GL_n(\mathbb{R})$-torsor. That is, the group $GL_n(\mathbb{R})$ acts simply transitively (on the right) on the fiber. So (2.42) is a *principal bundle* with structure group $GL_n(\mathbb{R})$.

Exercise 2.43. Prove that (2.42) is a fiber bundle. You can use the principal bundle structure to simplify: to construct local trivializations it suffices to construct local sections. Use coordinate charts to do so.

Each fiber of π has two components. Since M is assumed connected, the following is immediate from Definition 2.14 and covering space theory.

Lemma 2.44. *If M is connected and orientable, then $\mathcal{B}(M)$ has 2 components. If M is connected and non-orientable, then $\mathcal{B}(M)$ is connected.*

Proof. Let $\rho\colon \mathcal{B}(M) \to \mathfrak{o}(M)$ be the map which sends a basis of T_yM to the orientation of T_yM it determines. By Definition 2.2 ρ is surjective. We claim that ρ induces an isomorphism on components, and for that it suffices to check that if o_{y_0} and o_{y_1} are in the same component of $\mathfrak{o}(M)$, and if b_0, b_1 are bases of $T_{y_0}M, T_{y_1}M$ which induce the orientations o_{y_0}, o_{y_1}, then b_0 and b_1 are in the same component of $\mathcal{B}(M)$. Let $\gamma\colon [0,1]\to M$ be a smooth path with $\gamma(0)=y_0$ and $\gamma(1)=y_1$. Lift the vector field $\partial/\partial t$ on $[0,1]$ to a vector field on $\pi'\colon \gamma^*\mathcal{B}(M)\to[0,1]$, which we can do using a partition of unity since the differential of π' is surjective. Find an integral curve of this lifted vector field with initial point b_0. The terminal point of that integral curve lies in the fiber $\mathcal{B}(M)_{y_1}$ and is in the same component of the fiber as b_1, by the assumption that o_{y_0} and o_{y_1} are in the same component of $\mathcal{B}(M)$. □

Lemma 2.45. *If $Y_0=(y_0,b_0)$ and $Y_1=(y_1,b_1)$ are in the same component of $\mathcal{B}(M)$, then the framed points Y_0 and Y_1 are framed bordant in M.*

One special case of interest is where $y_0=y_1$ and b_0,b_1 belong to the same orientation.

Proof. Let $\gamma\colon [0,1]\to\mathcal{B}(M)$ be a smooth path with $\gamma(i)=(y_i,b_i)$, $i=1,2$. Let $X\subset[0,1]\times M$ be the image of the embedding $s\mapsto\bigl(s,\pi\circ\gamma(s)\bigr)$. The normal bundle at $\bigl(s,(\pi\circ\gamma)(s)\bigr)$ can be identified with $T_{\gamma(s)}M$, and we use the framing $\gamma(s)$ to frame X. □

Lemma 2.46. *Let $B\subset M$ be the image of the open unit ball in some coordinate system on M. Let $Y=\{y_0\}\amalg\{y_1\}$ be the union of disjoint points $y_0,y_1\in B$ and choose framings which lie in opposite components of $\mathcal{B}(B)$. Then Y is framed bordant to the empty manifold in B.*

Proof. We may as well take B to be the unit ball in $\mathbb{A}^m$, and after a diffeomorphism we may assume $y_0=(-1/2,0,\dots,0)$ and $y_1=(1/2,0,\dots,0)$. We may also reduce to the case where the framings are $\mp\partial/\partial x^1,\partial/\partial x^2,\dots,\partial/\partial x^m$; see the remark following Lemma 2.45. Then let $X\subset[0,1]\times B$ be the image of

$$s\longmapsto\Bigl(s(1-s);s-\frac12,0,\dots,0\Bigr) \tag{2.47}$$

where the framing at time s is

$$s(1-s)\frac{\partial}{\partial t}+(2s-1)\frac{\partial}{\partial x^1}\;,\;\frac{\partial}{\partial x^2}\;,\;\dots\;,\;\frac{\partial}{\partial x^m} \tag{2.48}$$

Here t is the coordinate on $[0,1]$. The m vectors in (2.48) project onto a framing of the normal bundle to X in $[0,1]\times M$, as is easily checked. □

Exercise 2.49. Assemble Lemma 2.44, Lemma 2.45, and Lemma 2.46 into a proof of Theorem 2.37.

Exercise 2.50. Use Theorem 2.37 to compute $[S^3,S^2]$ and $[S^4,S^3]$. As a warmup you might start with $[S^2,S^1]$, which you can also compute using covering space theory.

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