These notes are based on lectures given at UT Austin in Spring 2023 and Harvard University in Fall 2023. The course is designed to facilitate engagement with quantum physics. In Part 1 the setting is close to traditional physics: real time quantum mechanics and quantum field theory. It ends with axioms for Wick-rotated (imaginary time) quantum field theory. Part 2 is mostly in the Wick-rotated setting and focuses on parts of the theory that are most mathematically developed: topological field theory, invertible field theory, supersymmetric field theory, and applications to symplectic geometry and gauge theory. The notes are rough in parts. I offer this preliminary version in the hope that (1) it is useful, and (2) you will provide feedback and suggestions.

You will see the application of mathematical ideas, techniques, and viewpoints to a subject outside of mathematics. Such applications draw freely on whatever mathematics is needed with no walls between subdisciplines (analysis, algebra, geometry, representation theory, etc.). As such the course places large demands on your mathematical knowledge and sophistication. I assume exposure to fundamental material at least at the level of basic graduate courses, and often beyond. While I review many mathematical ideas, I do not not develop them in detail in this course. I give references for you to learn more, and I hope that you will delve in at some point, if not immediately. Most likely you will not emerge from this course with a working knowledge of all of this mathematics, but you will emerge with a deeper perspective on how different parts of mathematics work together. Similar remarks apply to the physics. My focus is the geometric structure of quantum theory, not on physics applications per se, though there are many of those as well. Bringing quantum field theory into mathematics is an ongoing longterm project; this course is a snapshot of aspects of my current understanding.

Recall the classic Three-Step Procedure of applications of mathematics to problems external to mathematics: (1) build a mathematical model, (2) solve the mathematics questions which arise in the model, and (3) apply the solution back to the external problems. Recall too that within physics theories refine over time: relativistic mechanics replaces Newtonian mechanics at large scales, just as quantum mechanics replaces Newtonian mechanics at small scales. So too the models (axiom systems) in these lectures can be expected to apply to some range of problems, but not to all. Furthermore, as we penetrate further into problems external to mathematics, we refine and amend our models. In any case, a major goal of the interaction between mathematics and quantum theory is to articulate and develop new mathematical structures, and these too are dynamic until they settle. (This is true of any notion in mathematics that we now think of as stare decisis: think of the early history of the notion of a topological space, or a smooth manifold, or almost any of our current mathematical objects.) All this is to reassure you that the unsettled feeling you may experience in this course is expected. Embrace it! Unsettledness is the steady state of a researcher.

*Date: January 29, 2024.*
Finally, mathematics is very powerful in many ways. One of them is its conceptualization of structure. The edifice of definitions, axioms, and theorems is a foundation for all of our investigations. If we can build this for a field outside of mathematics, then we have a solid basis from which to proceed, even if that edifice evolves over time. It is in this spirit that we begin this course in the next section with an abstract conceptualization of states, observables, and mechanical systems.

There are many topics missing in these notes, some of which I hope to fill in in a future version. The most important of these is the relationships between classical and quantum theories, including quantization.

I appended homework problems to these notes.

My views on this material, both general and specific, are heavily influenced by many mathematicians and physicists over many years. I give many references for further reading and investigation. Nonetheless, some insights I’ve learned from others have sneaked into these notes without attribution.

I warmly welcome your comments/corrections/suggestions: email me at dafr@math.harvard.edu.

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Lecture 1: States and observables

We begin with very general axioms that capture the notions of ‘state’ and ‘observable’. These axioms apply to classical mechanics, statistical mechanics, and—our main foci—quantum mechanics and quantum field theory. As with all axiom systems in this course, they are not meant to be in final form, but rather serve as a guide for engagement with physics. The intuition of many mathematicians and physicists are summarized here, among them Dirac [D], von Neumann [vN1], Irving Segal [S] and Mackey [Ma1, §2-2]. I highly recommend the leisurely account in [FY].

In mechanics one has motion, which is a flow of states and observables: a representation of the additive group $\mathbb{R}$ of time translation. Later in the course we discuss systems equipped with a representation of a larger symmetry group, in particular relativistic systems.

We illustrate the axioms in the familiar case of a particle or finite set of particles moving in Euclidean space. Then we describe states and observables in abstract quantum mechanical systems. Here we summarize the structure quickly; in the next several lectures we spell it out more thoroughly and give many concrete examples of classical and quantum mechanical systems. What is a glaring omission in these notes is a systematic discussion of the relation between classical and quantum systems, though we do discuss quantization of free theories in later lectures.

Sometimes background mathematical material is included at the end of a written lecture so as not to interrupt the main exposition. You should consult it for unfamiliar terms and concepts. In this lecture we include background on affine and convex spaces §1.4.1, symplectic manifolds §1.4.2,
and on bounded operators in Hilbert space §1.4.3. The spectral theorem is stated and used in this lecture; in Lecture 4 we discuss it and several of its cousins in more detail.

1.1 States, observables, and motion

A physical system consists of states and observables and a pairing between them. For example, fix a positive integer $k$ and consider a system of $k$ particles moving in a Euclidean space. A state is the collection of trajectories of the particles, or less canonically, the collection of positions and velocities at some fixed time. An observable is a real-valued function on collections of trajectories. Example: the distance between the 2nd and 13th particles at 4:45 AM on January 10, 2023. The following axiomatization of states and observables is due to many people. The original references [D, vN1] are the origin. A thorough mathematical account is in [Ma1, §2-2], and the first lectures of [FY] are a helpful exposition.

Axiom System 1.1. Data of states and observables (DOSO) consists of:

1. A real convex space $S$. Elements of $S$ are states, extreme points of $S$ are pure states, and non-extreme points are mixed states. Let $\mathcal{P}S \subset S$ denote the subspace of pure states.

2. A complex topological vector space $O$ equipped with a real structure $A \mapsto A^*$, a dense subspace $O^{\infty} \subset O$, a complex Lie algebra structure on $O^{\infty}$, and a map

$$\text{Borel}(\mathbb{R}; \mathbb{R}) \times O_{\mathbb{R}} \rightarrow O_{\mathbb{R}}$$

$$f , A \mapsto f(A)$$

A real point of $O$—an element of $O_{\mathbb{R}}$—is called an observable.

3. A measurement pairing

$$O_{\mathbb{R}} \times S \rightarrow \text{Prob}(\mathbb{R})$$

$$A , \sigma \mapsto \sigma_A$$

where $\text{Prob}(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$.

4. A map from $O^{\infty}_{\mathbb{R}}$ to a one-parameter group of automorphisms of $S$ and a map from $O^{\infty}_{\mathbb{R}}$ to a one-parameter group of automorphisms of $O$.

There are many compatibilities among this data, for example the following. For $f \in \text{Borel}(\mathbb{R}; \mathbb{R})$, $A, A_1, A_2 \in O$, $\sigma, \sigma_i \in S$, and positive real numbers $x^i$ summing to 1 we require

$$(x^i\sigma_i)_A = x^i(\sigma_i)_A$$

$$\sigma_{A_1 + A_2} = \sigma_{A_1} * \sigma_{A_2}$$

$$\sigma_{f(A)} = f_* \sigma_A$$

$$[A_1, A_2]^* = [A_1^*, A_2^*], \quad A_1, A_2 \in O^{\infty},$$

\footnote{A real structure on a complex vector space is an antilinear involution. Its fixed points are called real points.}

\footnote{Borel$(X; \mathbb{R})$ denotes the vector space of real-valued Borel functions on the topological space $X$.}
where ‘∗’ in the second equation is the convolution product of measures. For $H \in \mathcal{O}_R^\infty$, let

\begin{align*}
\sigma &\mapsto \sigma^{(t)} \\
A &\mapsto A^{(t)}
\end{align*}

(1.5)

denote the generated one-parameter groups of automorphisms of $\mathcal{S}$ and $\mathcal{O}$, respectively. Then

\begin{align*}
\frac{dA^{(t)}}{dt} + [H, A^{(t)}] &= 0, & A \in \mathcal{O}_R^\infty, & t \in \mathbb{R}, \\
\sigma^{(t)}_A &= \sigma_{A^{(t)}}, & \sigma \in \mathcal{S}, & A \in \mathcal{O}, & t \in \mathbb{R}.
\end{align*}

(1.6)

Finally, the expectation value

\begin{align*}
\mathbb{E}_\sigma(A) &= \int_\mathbb{R} \lambda d\sigma_A(\lambda)
\end{align*}

(1.8)

defines a real-valued pairing $\mathcal{S} \times \mathcal{O}_R \rightarrow \mathbb{R}$ that separates points.

The last statement means: (1) if $\sigma_1, \sigma_2 \in \mathcal{S}$ satisfy $\mathbb{E}_\sigma_1(A) = \mathbb{E}_\sigma_2(A)$ for all $A \in \mathcal{O}_R$, then $\sigma_1 = \sigma_2$; and (2) if $A_1, A_2 \in \mathcal{O}_R$ satisfy $\mathbb{E}_\sigma(A_1) = \mathbb{E}_\sigma(A_2)$ for all $\sigma \in \mathcal{S}$, then $A_1 = A_2$.

We make several remarks about Axiom System 1.1.

**Remark 1.9.**

1. Equation (1.7) expresses the compatibility between (1) the Schrödinger picture, in which states evolve, and (2) the Heisenberg picture, in which operators evolve.

2. In this general setting, the measurement process is probabilistic. In classical mechanics, in a pure state $\sigma$, which describes a definite trajectory, the result $\sigma_A$ of measuring an observable $A$ is a point measure, so no probability is necessary. In a mixed state, which describes a statistical mechanical state, the measure $\sigma_A$ has positive variance. In quantum mechanics, even in a pure state, the result of measurement usually has positive variance.

3. There is a map

\begin{align*}
\text{Prob}(\mathcal{S}_0) &\rightarrow \mathcal{S}
\end{align*}

(1.10)

that averages pure states over a probability measure to produce a general state.

4. A symmetry of DOSO is the data of automorphisms of $\mathcal{S}$ and of $\mathcal{O}$ that preserve the structural data. This is typically a very large group.

**Definition 1.11.** Suppose given DOSO as in Axiom System 1.1. A **motion** is the data of compatible 1-parameter groups of automorphisms of $\mathcal{S}$ and of $\mathcal{O}$. A **mechanical system** is DOSO together with a fixed motion.
Denote the chosen one-parameter groups as

\( \sigma \mapsto \sigma^{(t)} \)

\( A \mapsto A^{(t)} \)

Then the compatibility condition (1.7) is that these flows are adjoint under the measurement pairing (1.3), as in (1.7):

\( \sigma^{(t)} A = \sigma A^{(t)} \)

**Remark 1.14.**

1. In many cases the flows are generated by an observable via the map (4) in Axiom System 1.1. If so, the *negative* of that observable is called the *Hamiltonian* of the system. Our sign convention for this and other choices are elaborated in [DF1].

2. The flows (1.12) are actions of the group \( \mathbb{R} \) of time translations.

3. Time is an affine line and time translation is its group of translations. Here, for simplicity, we use the standard affine line \( \mathbb{A}^1 \) for time (it should also have the standard Euclidean structure).

4. The symmetry group of a mechanical system is a subgroup of the symmetry group of the DOSO. We require that symmetries preserve the flows (1.12) up to reversal of time; see [forward xref].

Often a state \( \sigma \in S \) is fixed as part of the data of a mechanical system, and it may be special.

**Definition 1.15.** Fix a mechanical system. A state \( \sigma \in S \) that satisfies \( \sigma^{(t)} = \sigma \) for all \( t \in \mathbb{R} \) is called a *stationary state*.

### 1.2 Classical mechanics

We illustrate how Axiom System 1.1 and Definition 1.11 encode a classical or statistical mechanical system. For concreteness consider a single particle moving in standard Euclidean space \( \mathbb{E}^d \) of some dimension \( d \in \mathbb{Z}^>0 \). (See Definition 1.33. We revisit this example in more detail below in Example 3.3.) We also assume a potential energy function \( V: \mathbb{E}^d \to \mathbb{R} \). For convenience, write \( M = \mathbb{E}^d \). Let \( m \in \mathbb{R}^>0 \) be the mass of the particle.

Let \( \mathcal{F} = \text{Map}(\mathbb{R}, M) \) be the space of all smooth motions \( x: \mathbb{R} \to \mathbb{E}^d \) in \( M = \mathbb{E}^d \); it is the space of *possible* trajectories of a particle. 3 Physical or classical trajectories satisfy Newton’s law:

\( m\ddot{x} = -\nabla V. \)

(This is an equality of functions \( \mathbb{R} \to \mathbb{R}^d \).) We assume that solutions to (1.16) exist for all time, given an initial position and velocity. Then the set \( N \subset \mathcal{F} \) of physical trajectories forms a finite

3We can endow \( \mathcal{F} \) with the structure of an infinite dimensional manifold, but we do not elaborate here. We could also work with finite dimensional families of smooth motions, which suffices for all of our considerations.
dimensional manifold. Note that for any time $t_0 \in \mathbb{R}$, the map

$$N \rightarrow \mathbb{E}^d \times \mathbb{R}^d$$

(1.17)

$$x \mapsto (x(t_0), \dot{x}(t_0))$$

is a diffeomorphism. (This description breaks time-translation symmetry.)

Now we proceed to the data (1)–(4) in Axiom System 1.1. Define the space $\mathcal{PS} = N$ of pure states to be the manifold of classical trajectories. Then $\mathcal{S} = \text{Prob}(N)$ is the space of probability measures on $N$; see Example 1.34. A mixed state $\sigma \in \mathcal{S} \setminus \mathcal{PS}$ describes a statistical state in which the particle does not undergo a fixed classical motion, but rather its motion is probabilistic.

Define an observable to be a function on $N$. It may be the restriction of a function on $F$, but a possible extension beyond classical trajectories is not relevant to the data (2) in Axiom System 1.1. We must specify what kinds of functions we allow:

$$\mathcal{O} = \text{Borel}(N; \mathbb{C})$$

$$\mathcal{O}^\infty = C^\infty(N; \mathbb{C})$$

(1.18)

The real structure $A \mapsto A^*$ is complex conjugation, hence $\mathcal{O}_\mathbb{R} = \text{Borel}(N; \mathbb{R})$ consists of real-valued Borel functions.

The measurement pairing (1.3) is simply the pushforward measure: if $A \in \text{Borel}(N; \mathbb{R})$ and $\sigma \in \text{Prob}(N)$, then

$$\sigma_A = A_* \sigma$$

(1.19)

is a probability measure on $\mathbb{R}$. Note that if $\sigma \in \mathcal{PS}$ is a pure state supported at $x \in N$, then $\sigma_A$ is a point measure supported at $A(x)$. In that case, the measurement is deterministic.

It remains to explain how smooth observables—that is, smooth functions on $N$—generate motion. The motion we need in Axiom System 1.1(4) is that of states and observables, but in fact those motions are induced from a motion on $N$. And the latter comes from a symplectic structure on $N$. See §1.4.2 below for some basics about symplectic manifolds, and Lecture 2 for a more leisurely account. In fact, a pair $(N, H)$ consisting of a symplectic manifold $N$ and a function $H \in C^\infty(N; \mathbb{R})$ determines the data of a mechanical system. For the case of a particle moving in Euclidean space $\mathbb{E}^d$, the symplectic structure on the space $N$ of classical trajectories is derived from a more fundamental structure: a variational principle or action principle for the classical equation of motion (1.16). This is a Lagrangian formulation of the classical mechanical system; the general case of a pair $(N, H)$ is the Hamiltonian formulation. All of this, including the derivation of the symplectic structure in the Lagrangian framework, is explained in detail in Lecture 3. Here we simply exhibit the symplectic 2-form on $N$.

Remark 1.20. More generally, a pair $(N, \varphi_t)$ of a symplectic manifold and a flow of symplectic diffeomorphisms (symplectomorphisms) determines the data of a mechanical system. Not every such system admits a Hamiltonian function: see Example 3.12.
In fact, we define a 2-form $\omega$ on the entire mapping space $\mathcal{F}$, implicitly assuming it is a smooth manifold. The restriction of $\omega$ to $N$ is closed and nondegenerate, i.e., is a symplectic form. Write $\delta$ for the de Rham differential on $\mathcal{F}$, which is also a (partial) differential on the product manifold $\mathcal{F} \times \mathbb{R}$. Let

$$e: \mathcal{F} \times \mathbb{R} \longrightarrow M$$

$$(x, t) \longmapsto x(t)$$

be the evaluation map. Then

$$\omega = m(\delta \dot{e} \wedge \delta e)$$

is a 2-form on $\mathcal{F} \times \mathbb{R}$. The restriction of $\omega$ to $N \times \{t_0\} \subset N \times \mathbb{R}$ is closed, nondegenerate, and is independent of $t_0 \in \mathbb{R}$. We write the pullback of $\omega$ under the identification (1.17). Let $x^1, \ldots, x^d$ be the standard coordinates on $\mathbb{R}^d$, and let $\dot{x}^1, \ldots, \dot{x}^d$ be the standard coordinates on $\mathbb{R}^d$. Then the symplectic form is

$$\omega = \sum_{i=1}^{d} m \dot{x}^i \wedge dx^i.$$

**Remark 1.24.** A symplectic manifold $N$ together with a smooth function $H: N \rightarrow \mathbb{R}$ is a dynamical system. The theory of ordinary differential equations provides a local flow on $N$ that integrates the symplectic gradient vector field. Existence and uniqueness of the flow are the main theorems, and they follow from fixed point theorems used to solve the flow equations. That is relatively easy. What is more interesting are the dynamical questions: Are there periodic orbits? Are there limit cycles? Is the system chaotic? Observe that the basic existence and uniqueness concerns small times, whereas the dynamical questions are at large times.

Classical field theory (as opposed to classical particle theory) has similar small and large time questions. In field theory there is space as well as time, and the local theory involves small time differences and small distances, said simply as short range. (In a relativistic field theory, a universal constant—the speed of light—relates time differences and lengths.) The short range questions involve partial differential equations, such as wave equations or equations from fluid dynamics. Then small time existence and uniqueness is already a delicate issue. And again the long range questions are the most interesting. Example: Is there existence for all time, or do singularities develop?

Quantum field theory has a similar dichotomy of questions, which is sometimes called kinematics vs. dynamics. The dynamical questions at long range, i.e., at large times and large distances—the two are the same in relativistic systems—are most interesting and have the most powerful mathematical ramifications.

**1.3 Quantum mechanics**

We turn now to quantum mechanics in the framework of Axiom System 1.1. Our exposition here is quite formal, presented with no history or motivation. I strongly suggest reading elsewhere, e.g.,
The basics are in the classic texts of Dirac [D] and von Neumann [vN1]. Other mathematical axiom systems for quantum mechanics may be found in [Str, §1.3] and [Ta, §2]. In subsequent lectures we flesh this out and give many examples of quantum mechanical systems.

In our approach here we restrict to bounded observables. This is the approach taken in many treatments; see [Ha], for example. On the other hand, the self-adjoint operator that generates the flow in states and observables may be an unbounded operator.

Our presentation relies on the background material in §1.4.3.

Quantum theory has a universal constant of nature, Planck’s constant $\hbar$, which has units

$$\hbar = \text{mass} \cdot \text{length}^2 \text{time}.$$  

Recall that energy has units $\text{mass} \cdot \text{length}^2 / \text{time}^2$, so the units of $\hbar$ are those of energy $\cdot$ time.

The DOSO of a quantum mechanical system is derived from a separable complex Hilbert space $\mathcal{H}$, which may be finite or countably infinite dimensional. The space $\mathcal{P}_S$ of pure states is the projective space $\mathbb{P}\mathcal{H}$; a pure state is a line in $\mathcal{H}$. The space $\mathcal{S}$ of all states is the space of nonnegative self-adjoint trace class operators $S$ with $\text{trace}(S) = 1$. Remark 1.59 tells how $\mathcal{P}_S$ is a subset of $\mathcal{S}$, and in fact $\mathcal{P}_S \subset \mathcal{S}$ is the subspace of extreme points, as it must be according to Axiom System 1.1(1).

Define $\mathcal{O} = \mathcal{O}^\infty = \text{End} \mathcal{H}$ as the complex vector space of bounded linear operators on $\mathcal{H}$. The adjoint map $A \mapsto A^*$ is a real structure on $\mathcal{O}$. Hence $\mathcal{O}_\mathbb{R}$ is the space of bounded self-adjoint operators on $\mathcal{H}$. Set $i = \sqrt{-1}$ and define the Lie algebra structure on $\mathcal{O}$ as

$$[A_1, A_2] = -\frac{i}{\hbar} (A_1 \circ A_2 - A_2 \circ A_1),$$

Functions of observables (1.2) are defined via the functional calculus (Definition 1.52).

We turn now to measurement (1.3). It is most familiar for pure states. Thus suppose $L \subset \mathcal{H}$ is a line, i.e., a one-dimensional subspace. Fix a unit vector $\psi \in L$. Let $A \in \mathcal{O}_\mathbb{R}$ be a bounded self-adjoint operator, and let $\pi_A$ be the associated self-adjoint projection-valued measure on $\mathbb{R}$. Then the result of pairing the pure state $L$ and the observable $A$ is the probability measure on $\mathbb{R}$

$$E \mapsto \langle \psi, \pi_A(E) \psi \rangle, \quad E \text{ a Borel subset of } \mathbb{R}. $$

This is independent of the choice of unit vector $\psi \in L$. A general state is a nonnegative self-adjoint trace class operator $S$ with unit trace, and then (1.27) generalizes to

$$E \mapsto \text{trace}(\pi_A(E) \circ S), \quad E \text{ a Borel subset of } \mathbb{R}.$$

Finally, the motion generated by an observable $H \in \mathcal{O}_\mathbb{R}^\infty$ is defined by an application of functional calculus, as in (1.54). Namely, let

$$U_t = e^{-itH/\hbar}, \quad t \in \mathbb{R},$$
be the 1-parameter group of unitary transformations of \( \mathcal{H} \) generated by \( H \). Then the motions (1.5) are conjugation by \( U_t \): for \( S \in \mathcal{S} \) and \( A \in \mathcal{O} \) set

\[
S^{(t)} := U_t \circ S \circ U_{-t} \\
A^{(t)} := U_{-t} \circ A \circ U_t
\]

1.4 Mathematical background

1.4.1 Affine and convex spaces. A vector space is a group under vector addition. Infinite dimensional vector spaces typically come with a topology—the compatible topology on a finite dimensional vector space is unique—and underlying a topological vector space is a topological group. Examples include Hilbert spaces, Banach spaces, Fréchet spaces, nuclear spaces, etc. Standard textbooks on functional analysis, such as [RS], develop the theory of topological vector spaces. Here we need only the very basic definitions, and we defer to the textbooks for a detailed development of the theory.

Affine space is the arena of flat geometry. We allow both finite and infinite dimensional spaces. Global parallelism in these geometries is expressed in terms of the translation action of a topological vector space.

Definition 1.31. Let \( V \) be a real topological vector space.

1. An affine space \( A \) over \( V \) is a topological space \( A \) equipped with a continuous simply transitive \( V \)-action. We write the result of acting \( \xi \in V \) on \( a \in A \) as \( a + \xi \in A \).
2. A subset \( S \subset A \) is convex if for all finite ordered tuples \( \sigma_0, \ldots, \sigma_k \in S \) and \( x^0, \ldots, x^k \in \mathbb{R}^{\geq 0} \) with \( x^0 + \cdots + x^k = 1 \) we have \( x^i \sigma_i = x^0 \sigma_0 + \cdots + x^k \xi_k \in S \).
3. A point \( \sigma \in S \) is extreme if whenever \( \sigma = x \sigma' + (1-x) \sigma'' \) for \( x \in [0,1] \), \( \sigma', \sigma'' \in S \), then \( \sigma = \sigma' \) or \( \sigma = \sigma'' \).

A vector space is an affine space (over itself). We cannot add points in an affine space, but we can average them. See [F1, Lectures 1–2] for an elementary exposition of some affine geometry, with a focus on finite dimensions. For every \( n \in \mathbb{R}^{\geq 0} \) there is a standard real affine space \( \mathbb{A}^n \) whose points are \( n \)-tuples \((a^1, \ldots, a^n)\) of real numbers; the standard real vector space \( \mathbb{R}^n \) acts on \( \mathbb{A}^n \) as usual by component-wise addition:

\[
(a^1, \ldots, a^n) + (\xi^1, \ldots, \xi^n) = (a^1 + \xi^1, \ldots, a^n + \xi^n), \quad (a^1, \ldots, a^n) \in \mathbb{A}^n, \quad (\xi^1, \ldots, \xi^n) \subset \mathbb{R}^n.
\]

Definition 1.33. A Euclidean space is an affine space over a real inner product space.

For an infinite dimensional Euclidean space, the underlying inner product space might be assumed complete (a real Hilbert space). For each finite dimension \( n \) there is a standard Euclidean space \( \mathbb{E}^n \). Its underlying affine space is \( \mathbb{A}^n \); the group of translations is \( \mathbb{R}^n \) with the standard dot product.
Example 1.34. Here is an important example of an infinite dimensional affine space. Let $X$ be a set equipped with a $\sigma$-algebra of measurable sets. The space $\text{Meas}(X)$ of (finite signed) real measures is a topological vector space. The subspace $\text{Prob}(X) \subset \text{Meas}(X)$ of probability measures is a convex subspace. The point measure $\mu_x$ at $x \in X$, defined by

$$(1.35) \quad \mu_x(E) = \begin{cases} 1, & x \in E; \\ 0, & x \notin E, \end{cases}$$

is an extreme point, and all extreme points have this form.

1.4.2 Symplectic manifolds. Informally, a symplectic manifold is a smooth manifold $N$ equipped with a smoothly varying nondegenerate skew-symmetric bilinear form $\omega$ on its tangent spaces, and $\omega$ is required to satisfy an integrability condition. Thus $\omega \in \Omega^2_N$ is a differential 2-form with the property that for all $p \in N$ the skew-symmetric form $\omega_p : T_pN \times T_pN \to \mathbb{R}$ is nondegenerate. The integrability condition is that $\omega$ be closed:

$$(1.36) \quad d\omega = 0.$$

Riemannian vs. symplectic: (symmetric positive-definite) vs. (skew-symmetric nondegenerate); (no integrability condition) vs. (integrability condition). We develop this analogy in Lecture 2. See [GS, MS] for much more on symplectic manifolds. (There are many more textbooks to recommend.)

The skew form $\omega$ induces an isomorphism

$$(1.37) \quad \mathcal{X}_N \cong \Omega^1_N \quad \xi \mapsto -\iota_{\xi}\omega$$

of the topological vector space of vector fields with the topological vector space of differential 1-forms; the notation indicates the contraction of the vector field $\xi$ into the symplectic 2-form $\omega$. (Note that because of the skew-symmetry there is a sign at stake; it is fixed by the formula). Also, there is a symplectic gradient map

$$(1.38) \quad \Omega^0_N \xrightarrow{d} \Omega^1_N \xrightarrow{\cong} \mathcal{X}_N \quad f \mapsto df \mapsto \xi_f$$

Recall that vector fields generate flows, at least locally. Hence a function on a symplectic manifold generates a 1-parameter group of diffeomorphisms, at least locally (in time and space).

Functions on a symplectic manifold form a Lie algebra under the Poisson bracket:

$$(1.39) \quad \{f_1, f_2\} = \xi_{f_1} \cdot f_2 = \omega(\xi_{f_1}, \xi_{f_2}).$$
1.4.3 Bounded operators on Hilbert space. The reference [RS] is an excellent functional analysis text, among many others.

The Hermitian inner product on $\mathcal{H}$ is complex antilinear in the first variable and complex linear in the second variable:

$$\langle \lambda_1 \psi_1, \lambda_2 \psi_2 \rangle = \overline{\lambda_1} \lambda_2 \langle \psi_1, \psi_2 \rangle, \quad \lambda_1, \lambda_2 \in \mathbb{C}, \quad \psi_1, \psi_2 \in \mathcal{H}. \tag{1.40}$$

See [DF1] for a justification of this choice.

**Definition 1.41.** Let $\mathcal{H}$ be a separable complex Hilbert space.

1. The **projective space** $\mathbb{P}\mathcal{H}$ is the space of lines (one-dimensional subspaces) in $\mathcal{H}$.
2. $\text{End}\mathcal{H}$ is the algebra of **bounded linear operators** $A: \mathcal{H} \to \mathcal{H}$. The adjoint of a bounded operator $A$ is the bounded linear operator $A^*$ characterized by

$$\langle A^* \psi_1, \psi_2 \rangle = \langle \psi_1, A \psi_2 \rangle, \quad \psi_1, \psi_2 \in \mathcal{H}. \tag{1.42}$$

Adjunction is an involution

$$\text{End}\mathcal{H} \longrightarrow \text{End}\mathcal{H} \quad \quad A \longmapsto A^* \tag{1.43}$$

and it satisfies

$$\lambda A^* = \overline{\lambda} A^* \tag{1.44}$$

$$(A_1 \circ A_2)^* = A_2^* \circ A_1^*$$

for all $\lambda \in \mathbb{C}$, $A, A_1, A_2 \in \text{End}\mathcal{H}$. The map (1.43) is a real structure on the vector space underlying $\text{End}\mathcal{H}$.

3. A **self-adjoint operator** is a real point of $\text{End}\mathcal{H}$, i.e., $A \in \text{End}\mathcal{H}$ such that $A^* = A$.
4. A **projection** is an operator $P \in \text{End}\mathcal{H}$ that satisfies $P^2 = P$. A **self-adjoint projection** additionally satisfies $P^* = P$.
5. The **rank** of a projection $P$ is $\dim P\mathcal{H}$, which may be finite or infinite.

The dimension of $\mathcal{H}$ is either finite or countably infinite. I recommend thinking through this material first in the finite dimensional case. The vector space $\text{End}\mathcal{H}$ is topologized using the operator norm. With that topology it is a topological $*$-algebra. If $P$ is a projection, then so too is $1 - P$ (where ‘1’ denotes the identity map $\text{id}_{\mathcal{H}}$), and there is a direct sum decomposition

$$\mathcal{H} = P\mathcal{H} \oplus (1 - P)\mathcal{H} \tag{1.45}$$

in which each subspace is closed in $\mathcal{H}$. If $P$ is a self-adjoint projection, then (1.45) is an orthogonal decomposition. Identify $\mathbb{P}\mathcal{H}$ with the space of rank one self-adjoint projections. It may be topologized as a subspace of $\text{End}\mathcal{H}$.

The following spectral theorem, due to von Neumann, is the basic structure theorem for bounded self-adjoint operators.

---

4The same holds in other topologies as well; of course, this must be checked in each case.
Theorem 1.46. Let \( \mathcal{H} \) be a separable complex Hilbert space, and suppose \( A \in \text{End} \mathcal{H} \) is a bounded self-adjoint operator. Then there is a self-adjoint projection-valued measure \( \pi_A \) on \( \mathbb{R} \) such that

\[
A = \int_{\mathbb{R}} \lambda \, d\pi_A(\lambda).
\]

The spectral measure\(^5\) \( \pi_A \) assigns to each Borel subset \( E \subset \mathbb{R} \) a self-adjoint projection \( \pi_A(E) : \mathcal{H} \to \mathcal{H} \), and this assignment satisfies \( \pi_A(\emptyset) = 0 \), \( \pi_A(\mathbb{R}) = \text{id}_{\mathcal{H}} \). Furthermore, if \( E' \cap E'' = \emptyset \), then \( \pi_A(E') \) commutes with \( \pi_A(E'') \).

Example 1.48. Set \( \mathcal{H} = L^2(\mathbb{R}; \mathbb{C}) \) (with respect to Lebesgue measure), and suppose \( f \in L^\infty(\mathbb{R}; \mathbb{R}) \) is a real-valued function. Let \( M_f : \mathcal{H} \to \mathcal{H} \) be the linear operator that multiplies an \( L^2 \) function by \( f \). Then for a Borel subset \( E \subset \mathbb{R} \), the spectral measure of \( M_f \) evaluated on \( E \) is the composition

\[
\pi_{M_f}(E) : L^2(\mathbb{R}; \mathbb{C}) \to L^2(f^{-1}E; \mathbb{C}) \to L^2(\mathbb{R}; \mathbb{C})
\]

of restriction followed by extension by zero.

Example 1.50. Let \( \Lambda \subset \mathbb{R} \) be a bounded discrete subset, and suppose given an orthogonal direct sum decomposition

\[
\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda
\]

in which each \( \mathcal{H}_\lambda \subset \mathcal{H} \) is a closed subspace. Define a bounded self-adjoint operator \( A_\Lambda : \mathcal{H} \to \mathcal{H} \) by requiring \( A_\Lambda \big|_{\mathcal{H}_\lambda} = \lambda \). I leave the description of the spectral measure to the reader.

The next definition goes by the name ‘functional calculus’.

Definition 1.52. Let \( A \) be a bounded self-adjoint operator on a separable complex Hilbert space \( \mathcal{H} \), and suppose \( f : \mathbb{R} \to \mathbb{R} \) is a bounded Borel function. Then \( f(A) \in \text{End} \mathcal{H} \) is defined by specifying its spectral measure to be the pushforward \( f_* \pi_A \). Thus

\[
f(A) = \int_{\mathbb{R}} \lambda \, d(f_* \pi_A)(\lambda) = \int_{\mathbb{R}} f(\lambda) \, d\pi_A(\lambda).
\]

An important application of functional calculus constructs a 1-parameter group of unitary automorphisms of \( \mathcal{H} \) from a bounded self-adjoint operator \( H \) on \( \mathcal{H} \). Namely, fix\(^6\) \( h \in \mathbb{R} \) and define

\[
U_t = e^{-itH/h}, \quad t \in \mathbb{R},
\]

which is the functional calculus for the 1-parameter group of functions \( f_t : \lambda \mapsto e^{-it\lambda/h} \).

We use the spectral calculus to define the spectrum as well as the notion of positivity. There are more standard alternative definitions.

---

\(^5\)The term ‘spectral measure’ seems to be used instead for the real-valued measures \( \langle \psi, \pi_A(\psi) \rangle \) associated to each \( \psi \in \mathcal{H} \). Tant pis!

\(^6\)In quantum theory, \( h \) is Planck’s constant. It has units \( \text{(mass)} \times \text{(length)}^2 \times \text{(time)}^{-1} \), which are the units of \( tH \) if \( H \) has units of energy. The constant is put in so that the exponent in (1.54) is dimensionless. Of course, in a mathematical context we usually omit \( h \) and suppose \( H \) is dimensionless.
**Definition 1.55.** Let $A$ be a bounded self-adjoint operator on a separable complex Hilbert space $\mathcal{H}$.

1. The spectrum of $A$ is the support\(^7\) of $\pi_A$.
2. The operator $A$ is nonnegative if its spectrum is a subset of $\mathbb{R}_{\geq 0}$.

The operator $A_\Lambda$ in Example 1.50 has spectrum $\Lambda$. In this case we say $A$ has pure point spectrum, and each $\lambda \in \Lambda$ is an eigenvalue of $A_\Lambda$ of multiplicity $\dim \mathcal{H}_\lambda$. The subspace $\mathcal{H}_\lambda \subset \mathcal{H}$ is the corresponding eigenspace. If the function $f$ in Example 1.48 is continuous, then the operator $M_f$ has no eigenvalues; its spectrum is continuous.

Finally, we introduce trace class operators, restricting to self-adjoint operators as a simplification.

**Definition 1.56.** Let $A$ be a bounded self-adjoint operator on a separable complex Hilbert space $\mathcal{H}$. Then $A$ is a self-adjoint trace class operator if it has the form $A_\Lambda$ as in Example 1.50, each $\mathcal{H}_\lambda$ is finite dimensional, and $\sum_{\lambda \in \Lambda} |\lambda| \dim \mathcal{H}_\lambda < \infty$. The trace of $A$ is then

\begin{equation}
(1.57) \quad \text{trace}(A) = \sum_{\lambda \in \Lambda} (\dim \mathcal{H}_\lambda) \lambda.
\end{equation}

**Example 1.58.** Let $\mathcal{H}' \subset \mathcal{H}$ be a finite dimensional subspace, and let $P: \mathcal{H} \to \mathcal{H}$ be the orthogonal projection with image $\mathcal{H}'$. Then $P$ is self-adjoint trace class, and $\text{trace}(P) = \dim \mathcal{H}'$.

**Remark 1.59.** Building on the remark following (1.45), we identify the projective space $\mathbb{P}\mathcal{H}$ as the space of self-adjoint trace class operators of trace one.

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**Lecture 2: Symplectic geometry**

This lecture is taken up with mathematical background on symplectic manifolds. A symplectic structure is a nondegenerate skew-symmetric bilinear form, and we contrast its basic theory with that of a positive definite symmetric form. We begin with linear algebra, where the salient fact is that whereas the space of inner products on a finite dimensional real vector space is a (contractible) convex cone, the space of symplectic forms has interesting topology. We move to affine geometry, where on both a Euclidean space and a symplectic affine space functions give rise to vector fields. In turn, vector fields generate motion, and it is that motion—in the symplectic case—which is needed for Axiom System 1.1(4).

New phenomena appear on smooth manifolds. Any smooth manifold admits a Riemannian metric, in fact a contractible space of them, whereas there are topological obstructions to the existence of a smoothly varying nondegenerate skew-symmetric bilinear form on the tangent spaces of an even dimensional manifold. Furthermore, there is an integrability condition in the symplectic case which is not present in the Riemannian case: we ask that a symplectic 2-form be closed. Functions on a symplectic manifold have a symplectic gradient, and there is a Lie algebra structure on the vector space of functions.

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\(^7\)The support is the complement of the union of open subsets of $\mathbb{R}$ with $\pi_A$-measure zero.
In the next lecture we apply these preliminaries to tell how a symplectic manifold equipped with a one-parameter family of symplectic diffeomorphisms gives rise to a mechanical system in the sense of Definition 1.11.

### 2.1 Bilinear forms in linear algebra

Let $V$ be a finite dimensional real vector space. A bilinear form $B: V \times V \to \mathbb{R}$ induces two maps $V \to V^*$. We say that $B$ is nondegenerate if these maps are isomorphisms.

**Definition 2.1.**

1. An inner product $\langle -, - \rangle$ is a positive definite symmetric bilinear form.
2. A symplectic form $\omega$ is a nondegenerate skew-symmetric bilinear form.

The positive definiteness is the condition that $\langle \xi, \xi \rangle > 0$ for all nonzero $\xi \in V$.

**Example 2.2.** Let $V = \mathbb{R}^n$ be the model vector space of dimension $n$. The model inner product is

$$\langle (\xi^i), (\eta^i) \rangle = \sum_i \xi^i \eta^i.$$

Let $n = 2m$ be even. Then the model symplectic form is

$$\omega((\xi^i), (\eta^i)) = (\xi^{m+1} \eta^1 - \xi^1 \eta^{m+1}) + \cdots.$$

**Remark 2.5.**

1. An inner product induces a norm—a length function on vectors—as well as a measurement of the angle between two nonzero vectors. From these one deduces a volume function on parallelepipeds of all dimensions. A symplectic form induces a signed area function on 2-dimensional parallelograms in $V$. Induced is a signed volume function on even dimensional parallelepipeds, in particular a signed volume function (on top dimensional parallelepipeds).

2. A skew-symmetric bilinear form on $V$ is an element of the exterior algebra of the dual space: $\omega \in \bigwedge^2 V^*$. The form is nondegenerate iff the top exterior product

$$\frac{\omega \wedge \cdots \wedge \omega}{m!} = \frac{\omega \wedge \cdots \wedge \omega}{m!} \in \bigwedge^{\text{top}} V^* = \bigwedge^{2m} V^* = \text{Det } V^*$$

is nonzero. In that case (2.6) is a signed volume form on $V$, so in particular determines an orientation of $V$.

The following basic result asserts the existence of normal forms.

**Theorem 2.7.** Let $V$ be a finite dimensional real vector space equipped with an inner product or symplectic form. Set $n = \dim V$. Then there exists an isomorphism $\mathbb{R}^n \to V$ that pulls back the inner product or symplectic form to the model (2.3) or (2.4).

In the symplectic case this implies that $n$ is even.
Corollary 2.8. The group $\text{GL}_n \mathbb{R}$ acts transitively on the space $\text{Met}(\mathbb{R}^n)$ of inner products and on the space $\text{Symp}(\mathbb{R}^n)$ of symplectic forms.

Sketch proof of Theorem 2.7. Consider first the case of an inner product space $(V, \langle - , - \rangle)$. Assuming $V \neq 0$, let $e_1 \in V$ be a unit length vector. Set $V_1 = (\mathbb{R} \cdot e_1) ^ \perp$ and apply the argument to the inner product space $(V_1, \langle - , - \rangle_{|V_1})$.

For $(V, \omega)$ a symplectic vector space, if $V \neq 0$ choose $e_1 \neq 0$ arbitrary and then, by nondegeneracy, choose $e_{m+1} \in V$ such that $\omega(e_{m+1}, e_1) = 1$. Set $V_1 = \mathbb{R} \cdot \{ e_1, e_{m+1} \} ^ \perp$ and apply the argument to the symplectic vector space $(V_1, \omega_{|V_1})$. □

The proof demonstrates that $n = \dim V = 2m$ is even in the symplectic case.

Corollary 2.8 asserts that the functions

\begin{align}
\text{GL}_n \mathbb{R} & \longrightarrow \text{Met}(\mathbb{R}^n) \\
\text{GL}_n \mathbb{R} & \longrightarrow \text{Symp}(\mathbb{R}^n)
\end{align}

that map an automorphism of $\mathbb{R}^n$ to the pullback of the model form are surjective, i.e., the Lie group $\text{GL}_n \mathbb{R}$ acts transitively on the respective spaces of forms. The stabilizer subgroups of the model forms are the orthogonal group $O_n$ and the symplectic group $\text{Sp}_{2m} \mathbb{R}$, respectively. The disparate natures of these stabilizer groups are behind the distinct natures of the associated linear and nonlinear geometries. For example, $O_n$ is compact and the inclusion $O_n \hookrightarrow \text{GL}_n \mathbb{R}$ is a homotopy equivalence: in fact, $\text{GL}_n \mathbb{R}$ deformation retracts onto the subgroup $O_n$. By contrast, $\text{Sp}_{2m} \mathbb{R}$ is not compact and the inclusion into $\text{GL}_n \mathbb{R}$ is not a homotopy equivalence. In the lowest dimensional case $m = 1$, we have $\text{Sp}_2 \mathbb{R} = \text{SL}_2 \mathbb{R}$. (The symplectic form is an area form and $\text{SL}_2 \mathbb{R} \subset \text{GL}_2 \mathbb{R}$ is the subgroup of invertible matrices that preserves the area form=determinant.)

The surjectivity of (2.9) and the identification of the stabilizer subgroups imply that

\begin{align}
\text{GL}_n \mathbb{R} / O_n & \overset{\approx}{\longrightarrow} \text{Met}(\mathbb{R}^n) \\
\text{GL}_n \mathbb{R} / \text{Sp}_{2m} \mathbb{R} & \overset{\approx}{\longrightarrow} \text{Symp}(\mathbb{R}^{2m})
\end{align}

are diffeomorphisms of homogeneous $\text{GL}_n \mathbb{R}$-manifolds.

Remark 2.11.

1. $\text{Met}(\mathbb{R}^n) \subset \text{SymBil}(\mathbb{R}^n; \mathbb{R})$ is a convex subset, in fact a convex cone: positivity is a convex condition. (Here $\text{SymBil}(\mathbb{R}^n; \mathbb{R})$ is the vector space of real-valued symmetric bilinear forms on the model vector space $\mathbb{R}^n$.) In particular, $\text{Met}(\mathbb{R}^n)$ is contractible.

2. By contrast, for $m = 1$ we have $\text{GL}_2 \mathbb{R} / \text{SL}_2 \mathbb{R} \approx \mathbb{R}^\times$ is not contractible. (The diffeomorphism is realized by the determinant homomorphism of Lie groups.) The space $\text{Symp}(\mathbb{R}^{2m})$ has more complicated topology in higher dimensions.

3. These assertions all transport from the model space $\mathbb{R}^n$ to arbitrary vector spaces. We discuss the details of that transportation in a future lecture, though we use it implicitly at the beginning of §2.3.

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8 The indices can be rearranged later: ‘$m$’ need not be defined.
2.2 Euclidean spaces and affine symplectic spaces

We pass now from linear geometry to affine geometry, which is the arena for flat geometry of global parallelism and the model for geometry on smooth manifolds. Let $V$ be a finite dimensional real vector space, and let $A$ be an affine space over $V$. We say $A$ is a Euclidean space if $V$ is equipped with an inner product, and $A$ is an affine symplectic space if $V$ is equipped with a symplectic form. A Euclidean space carries a constant/parallel/flat metric, and an affine symplectic space carries a constant/parallel/flat symplectic form. The normal form Theorem 2.7 implies the existence of distinguished affine coordinate systems. In the Euclidean case, write the coordinates as $x^1, \ldots, x^n$; the metric takes the form

$$g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n.$$  

(2.12)

In the symplectic case, write the coordinates as $q^1, \ldots, q^m, p_1, \ldots, p_m$; the symplectic form is

$$\omega = dp_i \wedge dq^i.$$  

(2.13)

Let $U \subset A$ be an open subset. A vector field on $U$ is a smooth function $U \to V$. A 1-form on $U$ is a smooth function $U \to V^*$. The metric or symplectic form induces an isomorphism between the space of vector fields and the space of 1-forms. In the symplectic case we write

$$\xi \leftrightarrow -\iota_\xi \omega,$$

where $\xi$ is a vector field and $\iota$ is the contraction of a vector field into a differential form; see (1.37).

Recall that vector fields induce local flows. The flows may patch to a global flow, which is an action of the Lie group $\mathbb{R}$ by diffeomorphisms $\varphi_t$ of $A$. In general, these diffeomorphisms are not affine, nor do they preserve the metric or symplectic form.

The differential of a function $f \in \Omega^0_U$ is a 1-form $df \in \Omega^1_U$. This is true on any affine space. In the presence of a Euclidean or affine structure we can convert the 1-form $df$ to a vector field $\xi_f \in \mathfrak{X}_U$ on $U$. In the Euclidean case, we use the notation $\nabla f = \xi_f$ and call this the gradient vector field of $f$. Observe that the gradient flow it induces is typically not an isometry.

---

9 Each coordinate $x^i : V \to \mathbb{R}$ is an affine function.

10 The summation convention tells that there is a sum over $i$ in (2.13).
Example 2.15. Consider the function $f = \frac{1}{2}x^2$ on the Euclidean line $\mathbb{R}^1_x$ with coordinate $x$. Then $\nabla f = x\partial/\partial x$, and distances grow exponentially under the gradient flow.

In the symplectic case, if $f \in \Omega^0_U$ is a smooth function, then its symplectic gradient $\xi_f$ satisfies:

\begin{equation}
\label{eq:2.16}
\text{df} = -\iota_{\xi_f}\omega.
\end{equation}

By contrast to the Euclidean case, the flow generated by a symplectic gradient vector field preserves the symplectic form.

Lemma 2.17. Let $A$ be an affine symplectic space, suppose $U \subset A$ is an open subset, and let $f: U \to \mathbb{R}$ be a smooth function with symplectic gradient $\xi_f \in \mathfrak{X}_U$. Then $\mathcal{L}_{\xi_f}\omega = 0$.

The conclusion is that the Lie derivative $\mathcal{L}_{\xi_f}\omega$ of $\omega$ along the vector field $\xi_f$ vanishes.

Proof. Apply Cartan’s formula for the Lie derivative of a differential form:

\begin{equation}
\label{eq:2.18}
\mathcal{L}_{\xi_f}\omega = (dt_{\xi_f} + t_{\xi_f}d)\omega = dt_{\xi_f}\omega = -d(df) = 0. \quad \square
\end{equation}

Definition 2.19. A vector field $\xi$ is **symplectic** if $\mathcal{L}_{\xi}\omega = 0$. It is **Hamiltonian** if it is the symplectic gradient of a function.

Note that a Hamiltonian vector field determines a function only up to locally constant functions.

2.3 Riemannian and symplectic manifolds

The vector space $V$ of translations of an affine space $A$ plays two roles. On the one hand, it acts as the Lie group of translations of $A$. On the other hand, it acts infinitesimally as the Lie algebra of infinitesimal translations of $A$: each element of $V$ gives rise to a constant/parallel/flat vector field on $A$. Equivalently, $V$ is the tangent space to $A$ at each point of $A$. It is this second, infinitesimal, role of $V$ that carries over to smooth manifolds in the form of the tangent bundle.

Let $M$ be a smooth manifold and let $TM \to M$ be its tangent bundle. There are two canonically associated fiber bundles:

\begin{equation}
\begin{array}{ccc}
\text{Met}(M) & \text{Symp}(M) \\
\downarrow \rho_1 & \downarrow \rho_2 \\
M & M
\end{array}
\end{equation}

At a point $m \in M$, the fiber of $\rho_1$ is the space Met$(T_mM)$ of inner products on the tangent space, and the fiber of $\rho_2$ is the space Symp$(T_mM)$ of symplectic forms on the tangent space.

Definition 2.21.

(1) A section of $\rho_1$ is a **Riemannian metric**.

(2) A section of $\rho_2$ is an **almost symplectic form**.
Recall from Remark 2.11(1) that the fibers of $\rho_1$ are convex subsets of a vector space.\footnote{Those vector spaces fit together to a vector bundle SymBil($T^*M; \mathbb{R}$) \to M.}

**Theorem 2.22.** There exist Riemannian metrics on $M$.

In fact, the space of Riemannian metrics (in an appropriate topology) is contractible.

**Proof.** A smooth manifold is locally modeled on affine space, and on affine space inner products exist. Use a partition of unity to average these local Riemannian metrics. Such averages are Riemannian metrics by the aforementioned convexity. \hfill \Box

By contrast, not every smooth manifold admits an almost symplectic form.

**Example 2.23.**

1. For $M = \mathbb{RP}^2$, the fiber bundle $\rho_2$ in (2.20) has typical fiber $\mathbb{R}^\times$; see Remark 2.11(2). As we traverse a circle $\mathbb{RP}^1 \subset \mathbb{RP}^2$, the two components of the fiber interchange, hence there is no section over that circle, much less over $\mathbb{RP}^2$.

2. Since the components of $\mathbb{R}^\times$ are contractible, this is the only obstruction to a section of $\rho_2$ in dimension two. Therefore, every orientable 2-manifold admits an almost symplectic structure.

3. The 4-sphere $S^4$ does not admit an almost symplectic structure; we do not give the proof here.

There is another crucial distinction between the Riemannian and symplectic cases. In the Riemannian case, there is no condition on a section of $\rho_1$. In the symplectic case, there is an **integrability condition**.

**Definition 2.24.** Let $N$ be a smooth manifold, and suppose $\omega \in \Omega^2_N$ is an almost symplectic form on $N$. Then $\omega$ is sympctic if $d\omega = 0$.

Thus an almost symplectic structure is a smoothly varying nondegenerate skew-symmetric bilinear form on the tangent spaces: an algebraic structure. **Definition 2.24** imposes a differential constraint.

**Remark 2.25.** There is a uniform theory of integrability conditions for structures on smooth manifolds. The triviality in the Riemannian case and the closure condition in the symplectic cases can be deduced from this general theory.

**Example 2.26.** The 4-sphere $S^4$ does not admit a symplectic structure. For if $\omega \in \Omega^2_{S^4}$ is a symplectic form, then its de Rham cohomology class

\begin{equation}
[\omega] \in H^2_{dR} \cong H^2(S^4; \mathbb{R}) = 0
\end{equation}

vanishes. But by **Remark 2.5**(2), the square of $\omega$ is a volume form and hence has a nonzero de Rham cohomology class

\begin{equation}
\left[\frac{\omega \wedge \omega}{2}\right] \in H^4_{dR}(S^4) \cong H^4(S^4; \mathbb{R}) \xrightarrow{\text{boundary}} \mathbb{R}
\end{equation}
The same argument gives a cohomological obstruction to a symplectic structure on any closed manifold.

In the positive direction, we have the following.

**Example 2.29.**

1. Any orientable 2-manifold admits a symplectic structure, which is a nonzero 2-form: all 2-forms are closed. In fact, as remarked above (2.10), a symplectic form in 2-dimensions is an area form, and in general the only obstruction to the existence of a volume form on a smooth manifold is orientability.

2. Let $M$ be a smooth manifold. We construct a symplectic form on the total space $T^*M$ of its cotangent bundle $\pi: T^*M \to M$. In fact, it is an *exact* symplectic form: the de Rham differential of a 1-form $\theta \in \Omega^1_{T^*M}$. To construct this tautological 1-form, suppose $m \in M$ and $\alpha \in T^*_m M$. Let $\eta \in T_\alpha(T^*M)$ be a tangent vector to $T^*M$ at $\alpha$. Then $\pi^*\eta \in T_m M$, and we define

\[(2.30) \quad \theta_\alpha(\eta) = \langle \alpha, \pi^*\eta \rangle.\]

The right hand side is the pairing between cotangent and tangent vectors at $m \in M$. Define the symplectic form $\omega = d\theta$. The nondegeneracy can be checked locally on $M$, where it reduces to a computation in affine space.

The construction of gradients carries over to Riemannian and symplectic manifolds. We only discuss the symplectic case. Let $(N, \omega)$ be a symplectic manifold. Define $\mathfrak{X}^\omega_N \subset \mathfrak{X}_N$ to be the subspace of vector fields $\xi$ that (infinitesimally) preserve the symplectic form in the sense that $L_\xi \omega = 0$. These vector fields generate *symplectic diffeomorphisms*, i.e., diffeomorphisms $\varphi$ that preserve the symplectic form: $\varphi^*\omega = \omega$. The proof of Lemma 2.17 shows that under the isomorphism $\mathfrak{X}_N \cong \Omega^1_N$ this subspace maps to the subspace of closed 1-forms. Then de Rham theory leads to the exact sequence

\[(2.31) \quad 0 \to H^0(N; \mathbb{R}) \to \Omega^0_N \xrightarrow{\text{symplectic gradient}} \mathfrak{X}^\omega_N \to H^1(N; \mathbb{R}) \to 0\]

In particular, there is an obstruction to expressing a symplectic vector field as the symplectic gradient of a function.
Example 2.32. Let $N = \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}/2\pi \mathbb{Z}$ be the 2-torus with coordinates $\theta^1, \theta^2$ and symplectic form $\theta^1 \wedge \theta^2$. The 1-parameter group $\varphi_t(\theta^1, \theta^2) = (\theta^1 + t, \theta^2)$ is generated by the symplectic vector field $\partial/\partial \theta^1$, but this vector field is not Hamiltonian, i.e., is not a symplectic gradient. (The corresponding 1-form $\pm d\theta^2$ is not exact, despite the notation.)

Recall that vector fields on a smooth manifold have a Lie algebra structure given by the Lie bracket. On a symplectic manifold, the subspace $X_N^\omega \subset X_N$ is closed under Lie bracket. Then the symplectic gradient map in (2.31) induces a Lie algebra structure on the vector space $\Omega^0_N$ of functions. This Poisson bracket is given by the formula

\begin{equation}
\{f_1, f_2\} = \xi_{f_1} \cdot f_2 = \omega(\xi_{f_1}, \xi_{f_2}), \quad f_1, f_2 \in \Omega^0_N.
\end{equation}

2.4 Symmetries and moment maps

Let $(N, \omega)$ be a symplectic manifold. A symmetry of $N$ is a diffeomorphism $\varphi: N \to N$ that preserves the symplectic form: $\varphi^*\omega = \omega$. Then $\varphi$ is called a symplectic diffeomorphism or symplectomorphism for short. An infinitesimal symmetry is a symplectic vector field $\xi$ (Definition 2.19): $\mathcal{L}_\xi \omega = 0$. If $\xi$ is Hamiltonian, so $\xi = \xi_f$ for some $f \in \Omega^0_N$, then we say that $f$ is a momentum of the infinitesimal symmetry $\xi$. Note that $f$ is determined up to additive shifts by a locally constant function. Suppose $H \in \Omega^0_N$ is a (Hamiltonian) function and $\varphi_t$ is the flow generated by $-\xi_H$. Assume that an infinitesimal symmetry $\xi$ preserves $H$: $\mathcal{L}_\xi H = 0$. Then if $\xi = \xi_f$ for some $f \in \Omega^0_N$,

\begin{equation}
-\xi_H \cdot f = -\{H, f\} = \{f, H\} = \xi_f \cdot H = 0.
\end{equation}

Therefore, $\varphi_t^* f = f$: the momentum $f$ is conserved by the Hamiltonian evolution. This relationship between infinitesimal symmetries and conserved momenta is due to Noether.

Now consider Lie groups of symmetries and Lie algebras of infinitesimal symmetries. For the latter, suppose $\mathfrak{g}$ is a Lie algebra and

\begin{equation}
\alpha: \mathfrak{g} \to X_N^\omega
\end{equation}

is an antihomomorphism of Lie algebras. (Antihomomorphisms are the infinitesimal version of left actions of Lie groups.) Recall the exact sequence (2.31).

**Definition 2.36.** Suppose the action $\alpha$ admits a lift to an antihomomorphism $\tilde{\alpha}$ in

\begin{equation}
0 \to H^0(N; \mathbb{R}) \to \Omega^0_N \to X_N^\omega \to H^1(N; \mathbb{R}) \to 0
\end{equation}

Then the transpose $\mu: N \to \mathfrak{g}^*$ of $\tilde{\alpha}$ is a moment map of $\alpha$. 

The composition $\alpha \to \chi_N^\omega \to H^1(N; \mathbb{R})$ is an obstruction to the existence of $\tilde{\alpha}$. If this composition vanishes, then there is an induced central extension of Lie algebras

\begin{equation}
0 \to H^0(N; \mathbb{R}) \to \Omega^0_N \to \chi_N^\omega \to H^1(N; \mathbb{R}) \to 0
\end{equation}

(2.38)

\begin{equation}
0 \to H^0(N; \mathbb{R}) \to \mathfrak{g} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0
\end{equation}

together with an antihomomorphism $\tilde{\mathfrak{g}} \to \Omega^0_N$. If the extension in the bottom row of (2.38) splits, then there exist lifts $\tilde{\alpha} : \mathfrak{g} \to \Omega^0_N$ and corresponding moment maps $\mu : N \to \mathfrak{g}^*$. A moment map is $\mathfrak{g}$-equivariant in the sense that

\begin{equation}
\mu_* [\alpha(\xi)_N] = \xi \cdot \mu(n), \quad \xi \in \mathfrak{g}, \quad n \in N,
\end{equation}

where on the right hand side the dot indicates the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.\(^\text{12}\) Unraveling the definitions, a moment map is characterized by the equations

\begin{equation}
d\mu_\xi = -\iota_{\alpha(\xi)} \omega, \quad \xi \in \mathfrak{g},
\end{equation}

where $\mu_\xi \in \Omega^0_N$ is the contraction of the $\mathfrak{g}^*$-valued function $\mu$ with $\xi \in \mathfrak{g}$.

If a Lie group $G$ acts on $N$ on the left by symplectic diffeomorphisms, then there is an induced infinitesimal action (2.35) of its Lie algebra $\mathfrak{g}$, and we ask that a moment map $\mathfrak{g} : N \to \mathfrak{g}^*$ be $G$-invariant.

We conclude with a geometric interpretation of the moment map. Suppose $\pi : T \to N$ is a principal $\mathbb{R}$-bundle with connection $\Theta \in \Omega^1_T$, whose curvature is

\begin{equation}
d\Theta = \pi^* \omega.
\end{equation}

(2.42)

If a Lie group $G$ acts on $N$ by symplectic diffeomorphisms, a lift to an action $G \circ T$ by automorphisms of the principal bundle with connection leads to a moment map as follows. For $\xi \in \mathfrak{g}$, let $\xi$ be the vector field on $T$ induced by the action $G \circ T$. Define the moment map $\mu_\xi \in \Omega^0_N$ by

\begin{equation}
\pi^* \mu(\xi) = \iota_\xi \Theta.
\end{equation}

(2.43)

(The map $\pi^*$ is injective, hence (2.43) determines the function $\mu_\xi \in \Omega^0_N$.) Then

\begin{equation}
0 = \mathcal{L}_\xi \Theta = (dt_\xi + \iota_\xi d) \Theta = \pi^* d\mu_\xi + \iota_\xi \pi^* \omega = \pi^* \left[ d\mu_\xi + \iota_{\alpha(\xi)} \omega \right],
\end{equation}

(2.44)

from which (2.41) follows.

\(^{12}\)There is a sign in the coadjoint action

\begin{equation}
(\xi \cdot \mu)(\eta) = -\mu([\xi, \eta]), \quad \xi, \eta \in \mathfrak{g}, \quad \mu \in \mathfrak{g}^*.
\end{equation}

(2.40)

that cancels the sign in the antihomomorphism property of $\alpha$ and leads to the $\mathfrak{g}$-equivariance of the moment map.
Lecture 3: Hamiltonian and Lagrangian mechanics

This lecture is a very condensed account of classical mechanics. The richness of the subject should be experienced through other sources, such as the classic text [Ar].

We begin with Hamiltonian systems, based on symplectic geometry. We illustrate with particle motion and also give more exotic examples, though without a particular physical model. A Hamiltonian system is the basic model of classical mechanics.

Often a Hamiltonian system can be derived from a Lagrangian system, and if so one has more tools and more control. We begin here with a discussion of variational principles in geometry, illustrating with the variational principle in Riemannian geometry that gives rise to the equation for geodesics. From here it is a short step to the Lagrangian system for particle motion. This is the principle of least action, for which I highly recommend the delightful lecture [Fey]. This lecture concludes with an extra structure one obtains in a Lagrangian system that is not present in a general Hamiltonian system: a connection whose curvature is the symplectic form. That connection plays a crucial role in quantization. We defer a discussion of Noether’s theorem, which tells how symmetries in a Lagrangian theory give rise to observables that are preserved by the Hamiltonian flow, to a future lecture. We refer to [DF2] for many details, elaborations, and examples of Lagrangian mechanics as well as Lagrangian field theories.

3.1 Hamiltonian systems

Definition 3.1.

(1) The data of a Hamiltonian system is a symplectic manifold \((N, \omega)\) and a 1-parameter group \(\varphi_t\) of symplectic automorphisms of \(N\).

(2) A Hamiltonian system is free if \((N, \omega)\) is an affine symplectic space and \(\varphi_t\) is a 1-parameter group of affine symplectic automorphisms.

The symplectic manifold \(N\) is often called the phase space.

A Hamiltonian system gives rise to a mechanical system in the sense of Definition 1.11. Given a symplectic manifold \((N, \omega)\) and a 1-parameter group \(\varphi_t\) of symplectic automorphisms, define (see also Section 1.2):

(1) The space of pure states is \(\mathcal{PS} = N\). The space of all states is the space \(\text{Prob}(N)\) of probability distributions on \(N\).

(2) The complex vector space \(\mathcal{O} = \text{Borel}(N; \mathbb{C})\) is the space of complex-valued Borel functions on \(N\). The involution of complex conjugation is the real structure. The dense smooth subspace \(\mathcal{O}^\infty = C^\infty(N; \mathbb{C})\) consists of smooth functions. It carries a Lie algebra structure given by Poisson bracket (2.33). Observables form the real vector space \(\mathcal{O}_\mathbb{R} = \text{Borel}(N; \mathbb{R})\) of Borel real-valued functions on \(N\), and there is a dense subspace \(\mathcal{O}^\infty_\mathbb{R} = C^\infty(N; \mathbb{R}) = \Omega^0_N\) of smooth observables.

(3) The measurement pairing \(\mathcal{O}_\mathbb{R} \times \mathcal{S} \to \text{Prob}(\mathbb{R})\) pairs an observable \(A : N \to \mathbb{R}\) and a state \(\sigma \in \text{Prob}(N)\) to the pushforward probability measure \(A_\ast \sigma \in \text{Prob}(\mathbb{R})\).

(4) A smooth observable \(f \in \Omega^0_N\) has a symplectic gradient vector field \(\xi_f \in \mathfrak{X}_N\). Under appropriate completeness assumptions it generates a 1-parameter flow \(\psi_t\) of symplectic
diffeomorphisms. There is an induced flow on states (measures push forward) and complex observables (functions pull back), and these induced flows are compatible (1.7).

(5) The chosen $\varphi_t$ generates the desired flows (1.12) on states and observables.

**Remark 3.2.** Note that not all smooth observables generate motion: vector fields always generate local flows, but a global flow may not exist. So Axiom System 1.1(4) is only aspirational in Hamiltonian systems.

**Example 3.3.** We revisit particle motion in Euclidean space (§1.2), and for simplicity consider only the Euclidean line $\mathbb{E}^1$ as target. Let $V: \mathbb{E}^1 \to \mathbb{R}$ be a smooth function (potential energy). Then Newton’s law for a motion $x: \mathbb{R} \to \mathbb{E}^1$ of a particle of mass $m \in \mathbb{R}^+ \setminus \{0\}$ is

\[
(3.4) \quad m \ddot{x} = -V'(x).
\]

Let $N$ be the space of classical trajectories, i.e., the space of functions $x: \mathbb{R} \to \mathbb{E}^1$ that satisfy (3.4). Given a position and velocity—a point of the Cartesian product $\mathbb{E}^1 \times \mathbb{R}$—there exists a local solution whose position and velocity at some specified time $t_0 \in \mathbb{R}$ is as given. (The solution is local in the sense that its domain is an open interval about $t_0$.) Assume the potential $V$ is sufficiently well-behaved that solutions exist for all time. Then for any $t_0 \in \mathbb{R}$, the evaluation map

\[
(3.5) \quad N \to \mathbb{E}^1 \times \mathbb{R}
\]

that maps a trajectory to its position and velocity at time $t_0$ is a diffeomorphism. Use ‘$x$’ for the coordinate on $\mathbb{E}^1$ and ‘$y$’ for the coordinate on $\mathbb{R}$. Then the symplectic form is

\[
(3.6) \quad \omega = m \, dy \wedge dx.
\]

The flow $\varphi_t: N \to N$ is generated by the Hamiltonian function

\[
(3.7) \quad H = \frac{my^2}{2} + V(x),
\]

which is the sum of kinetic and potential energy. The symplectic gradient (2.16) of $H$ is

\[
(3.8) \quad \xi_H = y \frac{\partial}{\partial x} - \frac{1}{m} V'(x) \frac{\partial}{\partial y},
\]

hence the equations for the flow $\varphi_t$ are

\[
(3.9) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -\frac{1}{m} V'(x) \end{cases}
\]

**Remark 3.10.**
The passage from Newton’s Law (3.4) to the Hamilton equations (3.9) is the traditional move that converts a second-order ODE to a system of first-order ODEs by introducing the first derivative as a new variable.

Observe that \( N \) is an affine space. For a free particle \( (V = 0) \) and for a harmonic oscillator \( (V = kx^2/2, \ k > 0) \), the flow \( \varphi_t \) is by affine symplectic diffeomorphisms, hence according to Definition 3.1(2) the system is free.

Example 3.3 generalizes to particle motion on a Riemannian manifold \((M, g)\) in the presence of a potential \( V: M \to \mathbb{R} \). In this case the acceleration \( \ddot{x} \) is computed using the Levi-Civita covariant derivative. Also, (3.5) is replaced by a diffeomorphism onto the total space of the tangent bundle \( TM \), again assuming solutions to the equations of motion are defined for all time. The Riemannian metric induces an isomorphism \( TM \cong T^*M \), and the symplectic form is \( m \) times the pullback of the tautological symplectic form defined in Example 2.29(2). The Hamiltonian function (3.7) generalizes to

\[
H(\xi) = \frac{m}{2}|\xi|^2 + V(\pi(\xi)), \quad \xi \in TM,
\]

where \( \pi: TM \to M \) is the tangent bundle.

The Riemannian manifold \( M \) is often called the configuration space. (Recall that \( N = TM \) is called the phase space.)

Later in the lecture we indicate how to derive (3.6) and (3.7) (and their generalizations for particle motion on a Riemannian manifold) from a more basic starting point: a Lagrangian.

Example 3.12. The phase space \( N \) of a Hamiltonian system need not be the tangent bundle to a manifold, and the 1-parameter group \( \varphi_t \) need not be generated by a Hamiltonian function. For example, as in Example 2.32 take

\[
N = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}
\]

and

\[
\varphi_t(\theta^1, \theta^2) = (\theta^1 + t, \theta^2)
\]

This is the quotient of a free system by the action of a discrete group \((2\pi\mathbb{Z} \times 2\pi\mathbb{Z})\) of automorphisms.

3.2 Variational principles

Variational principles are important in many parts of geometry, for example in global analysis on manifolds. Here we review the variational principle for free particle motion in Euclidean space. (The generalization to geodesic motion on a Riemannian manifold uses the Levi-Civita covariant derivative, but otherwise is the same.) We do not prove the assertions here, which can be proved using standard ideas in analysis; see [F1, Lecture 10], for example.
Let $V$ be a finite dimensional real inner product space, and suppose $A$ is an affine space over $V$ (a Euclidean space). Fix real numbers $a < b$, and consider smooth motions $x : [a, b] \to A$. The length of the image path is

\[(3.15) \quad \int_{[a,b]} |\dot{x}(t)| \, dt\]

and it does not depend on the parametrization of the image. In other terms, there is an infinite dimensional symmetry group of the function that maps $x$ to (3.15), and that symmetry makes (3.15) more difficult to use as the function(al) for a variation problem. Instead, consider the energy

\[(3.16) \quad E(x) = \int_{[a,b]} \frac{1}{2} |\dot{x}(t)|^2 \, dt = \int_{[a,b]} \frac{1}{2} \langle \dot{x}, \dot{x} \rangle \, dt.\]

To compute the variational formula, introduce a finite dimensional family of “variations”. That is, let $S$ be a smooth manifold and consider a smooth function

\[(3.17) \quad x : S \times [a,b] \to A.\]

For each $s \in S$ the function $x(s, -) : [a, b] \to A$ is a motion in $A$. Let $\delta$ be the de Rham differential along $S$, and write the de Rham differential on $[a, b]$ as $d = dt \frac{d}{dt}$. Then $E : S \to \mathbb{R}$ is a smooth function, and its differential is

\[(3.18) \quad \delta E = \int_{[a,b]} \langle \delta \dot{x}, \dot{x} \rangle \wedge dt \]

\[= \int_{[a,b]} \langle \delta x, \dot{x} \rangle \]

\[= -\langle d\delta x, \dot{x} \rangle \]

\[= -d\langle \delta x, \dot{x} \rangle - \langle \delta x \wedge d\dot{x} \rangle \]

\[= -\langle \delta x, \dot{x} \rangle \wedge dt - d\gamma, \]

where

\[(3.19) \quad \gamma = \langle \delta x, \dot{x} \rangle.\]

The important maneuver is the integration by parts that produces the penultimate line from its predecessor. (Observe that in the last line of (3.18) we exchanged the order of the terms from the previous line.) At this point one makes an argument that $\delta E$ vanishes for all variations iff

\[(3.20) \quad \ddot{x}(t) = 0\]

for all $t \in [a, b]$; see [F1, Lemma 10.18]. Equation (3.20) asserts that the motion has constant velocity, i.e., it is a geodesic motion in the Euclidean space $A$. 
Remark 3.21. The last term in (3.18) is the integral of an exact 1-form:

\[(3.22) \quad \int_{[a,b]} d\gamma = \gamma(b) - \gamma(a) = \langle \delta x, \dot{x} \rangle \bigg|_b - \langle \delta x, \dot{x} \rangle \bigg|_a.\]

One typically arranges that this vanish, for example by imposing that all paths begin and end at the same point of \(A\). Crucially, the geodesic equation (3.20) is local in time, and these boundary conditions are irrelevant for its derivation.

We extract from the computation (3.18) its essence, which is local in time and does not rely on integration. For convenience, we use the 1-form \(dt\) in place of the density \(|dt|\). Then the integrand of (3.16) is

\[(3.23) \quad L = \frac{1}{2} \langle \ddot{x}, \dot{x} \rangle dt \in \Omega^{0,1}_{S \times [a,b]},\]

and the integral is the energy function

\[(3.24) \quad E = (\text{pr}_1)_*(L) \in \Omega^0_S,\]

where \(\text{pr}_1 : S \times [a, b] \to S\) is projection onto the first factor. The geodesic equation lives on \(S \times [a, b]\), so should be computed from \(L\). Rather than restrict to a finite dimensional family of variations of \(\mathcal{F}\), we heuristically consider the space \(\mathcal{F}\) of all smooth paths \(x : [a, b] \to A\). Then we compute in the double complex

\[(3.25) \quad (\Omega^{\bullet, \bullet}_{\mathcal{F} \times [a,b]}, D = \delta + d)\]

of differential forms on the Cartesian product \(\mathcal{F} \times [a,b]\), assuming that calculus works in this infinite dimensional setting. (It does, with some care.) Thus we fit (3.23) and (3.19) into the diagram

\[(3.26) \quad \begin{array}{c}
\begin{array}{c|c|c}
0 & 1 \\
\hline
1 & L & \to \\
\hline
0 & \gamma & \\
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c|c|c}
0 & 1 \\
\hline
1 & \mathcal{F} & \to \\
\hline
0 & \gamma & \\
\end{array}
\end{array}\]

where the horizontal differential is \(\delta\) and the vertical differential is \(d\). The crucial point is that if we restrict to the finite dimensional manifold \(N\) of solutions to the equations of motion (3.20), then the solutions extend to all time and fit into the diagram

\[(3.27) \quad \begin{array}{c}
\begin{array}{c|c|c}
0 & 1 \\
\hline
1 & L & \to \\
\hline
0 & \gamma & \\
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c|c|c}
0 & 1 \\
\hline
1 & N & \to \\
\hline
0 & \gamma & \\
\end{array}
\end{array}\]

\[\mathbb{R}\]
In other words, on $N \times \mathbb{R}$ we have

\[(3.28)\quad D(L + \gamma) = \delta L + d\gamma = 0.\]

**Remark 3.29.**

1. If we begin with the density $L$ in (3.23), then we should ask: where does the *variational 1-form* $\gamma$ in (3.19) come from? In other words, can we codify the integration by parts in (3.18) without integration? Yes we can, but we do not do so here: see Definition 2.37 and §2.4 in [DF2].

2. Is there an *a priori* argument why the critical point equation of the total energy (3.16) is an equation (3.20) that is local in time? Yes there is. Namely, consider instead of the condition for a critical point the condition for a *minimizer*. Then for each time $t_0$, a minimizer must minimize the integral over the interval $[t_0, t_0 + \delta]$ for all $\delta > 0$. The limit $\lim_{\delta \to 0}$ is then a condition localized at $t_0$. It ends up being the *Euler-Lagrange equation* (3.20).

3. Variational principles are extremely useful in theory and in practice. If an ODE or a PDE is the Euler-Lagrange equation of some functional, then that functional can be used to set up and control a minimization procedure to find solutions. Specifically, the functional leads to estimates that are often crucial in the study of the PDE.

### 3.3 Lagrangian systems

The Lagrangian approach dates from the *principle of least action* or the *stationary action principle* is usually ascribed to Maupertuis, though perhaps Euler and Leibniz should also receive credit. Again, I defer to Feynman [Fey]. Here I simply focus on the formal aspects, which are developed in great detail in [DF2].

**Remark 3.30.**

1. Not every Hamiltonian system has a Lagrangian formulation.

2. A Lagrangian formulation of a Hamiltonian system leads to estimates and greater control, as indicated in Feynman’s lecture. Compare with Remark 3.29(3).

3. A Lagrangian formulation leads to more geometric structure, as we indicate in Section 3.4.

4. A Lagrangian formulation of a mechanical theory is the input into the *path integral*, which is an approach to quantum mechanics developed by Feynman.

We do not attempt any generality here, but simply consider a particle motion on a Riemannian manifold $(M, g)$ equipped with potential energy $V : M \to \mathbb{R}$. Then for $x : \mathbb{R} \to M$, set

\[(3.31)\quad S(x) = \int_{[a,b]} \left[ \frac{m}{2} |\dot{x}(t)|^2 - V(x(t)) \right] |dt|\]
Observe that both the integrand (Lagrangian or Lagrangian density) and the integral (action) have units/dimensions

\begin{equation}
\frac{ML^2}{T} = \frac{(\text{mass})(\text{length})^2}{(\text{time})},
\end{equation}

which we call the units of action.

Remark 3.33. The Lagrangian is a somewhat magical quantity that seemingly does not have as direct a physical interpretation as, say, quantities like energy, force, momentum, mass, etc.

As in (3.26) one does local calculus on \( \mathcal{F} \times \mathbb{R} \), where now \( \mathcal{F} \) is the space of smooth maps \( \mathbb{R} \to M \). One computes \( \gamma \), roughly using integration by parts as in (3.18), so that the equation of motion (Euler-Lagrange equation) is

\begin{equation}
D(L + \gamma) = \delta L + d\gamma = 0.
\end{equation}

For the particle on \( M \) we find

\begin{equation}
\gamma = m \langle \delta x, \dot{x} \rangle,
\end{equation}

and the equations of motion are Newton’s Law

\begin{equation}
\nabla \dot{x} + V'(x) = 0,
\end{equation}

where \( \nabla \) is the Levi-Civita covariant derivative. Let \( N \) be the space of solutions.

We briefly indicate how to derive a Hamiltonian system (Definition 3.1) from the Lagrangian. Set

\begin{equation}
\omega = \delta \gamma \in \Omega^{0,2}_{N \times \mathbb{R}},
\end{equation}

as depicted in the diagram

\begin{equation}
\begin{array}{c}
0 \downarrow 1 \\
L \rightarrow 0 \\
0 \uparrow \gamma \rightarrow \omega
\end{array}
\end{equation}

Lemma 3.39.

(1) \( D\omega = 0 \), from which \( \delta \omega = 0 \) and \( d\omega = 0 \).

(2) The restriction of \( \omega \) to \( N \times \{t\} \) is independent of \( t \in \mathbb{R} \).
(3) \( \omega \) is nondegenerate.

Proof. For (1), observe that \( \omega = D(\gamma - L) \); now apply \( D \). For (2), choose times \( t_0 < t_1 \) and apply Stokes’ theorem\(^{13}\) to the projection map \( N \times [t_0, t_1] \to N \) depicted in Figure 3:

\[
\delta \int_{[t_0, t_1]} \omega = \pm \int_{[t_0, t_1]} D\omega \pm \int_{\{t_1\} - \{t_0\}} \omega.
\]

The left hand side vanishes: the differential \( \delta \) and the integral \( \int_{[t_0, t_1]} \) commute. (This is the usual differentiation under the integral sign.) The first term on the right hand side vanishes by (1). The vanishing of the remaining term is the desired statement. For (3) compute that

\[
\omega = m \langle \delta \dot{x} \wedge \delta x \rangle,
\]

from which the description in Remark 3.10(3) follows. Now use the fact that the tautological 2-form on the cotangent bundle is nondegenerate. \( \square \)

**Corollary 3.42.** The restriction of \( \omega \) to \( N \times \{t\} \) is a symplectic form on \( N \), and it is independent of \( t \).

Next, we give a formula for a (Hamiltonian) function on \( N \) derived from the Lagrangian. The flow it generates is the 1-parameter group needed to complete the data in Definition 3.1(1). Let \( \xi \) be the vector field on \( \mathcal{F} \times \mathbb{R} \) that generates the negative of time translation. It is characterized by the contractions

\[
\iota_{\xi} dx = -1, \\
\iota_{\xi} \delta x = \dot{x}
\]

Then set

\[
H = \iota_{\xi}(L + \gamma) \in \Omega^{0,0}_{\mathcal{F} \times \mathbb{R}}.
\]

\(^{13}\)The version we need is Stokes’ theorem in families. Fortunately, we do not need to pin down signs.
Remark 3.45. The motivation for (3.44) comes from Noether’s theorem: the Hamiltonian is the conserved quantity derived from the negative of infinitesimal time translation. Compare (3.47) below to (2.41); see [DF2, §2.6] for further details.

Lemma 3.46. $H \big|_{N \times \{t\}}$ is independent of $t \in \mathbb{R}$ and

$$\delta H = -\iota_\xi \omega.$$  

Proof. On $N \times \mathbb{R}$ we use $D(L + \gamma) = \omega$ to compute

$$DH = D\iota_\xi (L + \gamma) = L_\xi (L + \gamma) - \iota_\xi D(L + \gamma) = -\iota_\xi \omega.$$  

The first assertion now follows from the conservation law $dH = 0$, which is the $(0, 1)$-component of (3.48).

Example 3.49. Consider particle motion on $\mathbb{E}^1$, as in Example 3.3. Then

$$L + \gamma = \left\{ \frac{m}{2} \dot{x}^2 - V(x) \right\} dt + m\dot{x}dx.$$  

Hence

$$H = \iota_\xi (L + \gamma) = -\left\{ \frac{m}{2} \dot{x}^2 - V(x) \right\} + mx^2$$  

$$= \frac{m}{2} \dot{x}^2 + V(x)$$  

which is the usual formula for the total kinetic energy. Notice the sign change on the potential from the Lagrangian (negative) to the Hamiltonian (positive).

3.4 Geometric meaning of $\gamma$

On the space $N$ of classical trajectories the equation $\delta L = -d\gamma$ holds, from which we deduce

$$\delta \int_{[t_0, t_1]} L = -\left\{ \gamma(t_1) - \gamma(t_0) \right\}$$  

for all times $t_0 < t_1$. We now interpret (3.52) geometrically; see [DF2, §1.1] for further discussion.

View $\mathbb{R}$ as a Lie group under addition. A connection on the trivial principal $\mathbb{R}$-bundle over $N$ is a (real-valued) 1-form on $N$. Then for each $t \in \mathbb{R}$, the 1-form $\gamma(t)$ is an $\mathbb{R}$-connection on $N \times \{t\}$. Equation (3.52) implies that addition by $\int_{[t_0, t_1]} L$ is an isomorphism of trivial principal $\mathbb{R}$-bundles with connection; it maps the connection $\gamma(0)$ to the connection $\gamma(1)$. Furthermore, these isomorphisms are compatible for three times $t_0 < t_1 < t_2$, as in Figure 4. Then taking invariant sections in the $\mathbb{R}$-direction, we obtain a (not trivialized) principal $\mathbb{R}$-bundle

$$T \longrightarrow N$$  

equipped with a connection whose curvature is the symplectic form. (This last assertion follows from $\delta \gamma = \omega$. This interpretation gives a new construction of the symplectic form on $N$ as the curvature of the connection on (3.53).)
Remark 3.54. The first step in the path integral formulation of the quantum theory is to exponentiate the bundle (3.53) with its connection. (Intuitively, quantization is an exponentiation process.) Observe that the units of $\gamma$ are the units $ML^2/T$ of action. Therefore, to carry out the exponentiation we need a numerical constant with units of action. Planck’s constant $\hbar$ is precisely such a number. It makes sense, then, to apply the function

\[(3.55) \quad \exp\left(\frac{\imath \hbar}{\hbar}\right) : \mathbb{R} \longrightarrow \mathbb{C}^\times\]

to form a principal $\mathbb{C}^\times$-bundle with connection associated to (3.53). The associated complex line bundle with covariant derivative is sometimes called the prequantum line bundle. Observe that it is constructed canonically from a Lagrangian formulation. This is the extra piece of structure alluded to earlier, and it is one of the powerful features of a Lagrangian formulation, when it is available.

Example 3.56. This illustrative example is called the particle on a ring. The ring is the Euclidean circle $\mathbb{E}^1/2\pi LZ$ of length $2\pi L$ for some $L \in \mathbb{R}^>0$. Let $m \in \mathbb{R}^>0$ be the mass of the particle. We define a 1-parameter family of theories parametrized by $\theta \in \mathbb{R}$. The Lagrangian is

\[(3.57) \quad L = \left(\frac{m}{2} \ddot{x}^2 + \frac{\theta}{2\pi} \dot{x}\right) dt.\]

The first term is the usual kinetic energy; the potential energy vanishes. The second term is locally exact but not globally exact, since there is no global function $x$ on the circle. We leave the reader to identify the space $N$ of classical solutions and show that

- the Hamiltonian system derived from (3.57) is independent of $\theta$; and
- the principal bundle (3.53) with connection changes with $\theta$.

The dependence on $\theta$ shows up in the corresponding quantum systems, as we will see in §7.3.

Remark 3.58. This example illustrates that different Lagrangian systems can give rise to the same Hamiltonian system. It also illustrates that Lagrangian systems can contain more information than Hamiltonian systems.
Lecture 4: Spectral theorems

We begin with a geometric take on diagonalization of linear operators on finite dimensional vector spaces. The general operator $A$ has a Jordan normal form, which we interpret in terms of a sheaf over the complex line. The support of the sheaf is the spectrum of the operator. If the operator is diagonalizable, then the sheaf can be re-expressed as a projection-valued measure. This is essentially the expression of $A$ as a multiplication operator. On inner product spaces there are two special classes of operators that are diagonalizable: self-adjoint operators and unitary operators.

Then we turn to bounded (or continuous) operators on Hilbert spaces. For self-adjoint operators the main result is von Neumann’s spectral theorem. We refer to §1.4.3 for background material.

In the next lecture we use these spectral ideas to introduce quantum mechanical systems.

4.1 Linear operators in finite dimensions

Let $W$ be a finite dimensional complex vector space of positive dimension. Suppose $A: W \to W$ is a linear operator. A subspace $W' \subset W$ is $A$-invariant if $A(W') \subset W'$. The operator $A$ is decomposable if there exists a direct sum decomposition $W = W' \oplus W''$ into proper $A$-invariant subspaces. An eigenspace of $A$ is a subspace on which $A$ acts as (multiplication by) a scalar; that scalar is the corresponding eigenvalue. The algebraic completeness of the complex numbers implies that every operator has an eigenline. For $\lambda \in \mathbb{C}$ define

$$W_\lambda = \{ \psi \in W : A\psi = \lambda \psi \}. \tag{4.1}$$

The previous assertion is that $W_\lambda \neq 0$ for some $\lambda \in \mathbb{C}$. There is an inclusion

$$\bigoplus_{\lambda \in \mathbb{C}} W_\lambda \longrightarrow W \tag{4.2}$$

The operator $A$ is diagonalizable if this inclusion is an isomorphism.

Example 4.3. Let $W = \mathbb{C}^3$, fix $\lambda \in \mathbb{C}$, and define $A: W \to W$ to be multiplication by the matrix

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \tag{4.4}$$

Then with respect to the standard basis $e_1, e_2, e_3$, the only eigenline is $\mathbb{C} \cdot \lambda$. This subspace has no $A$-invariant complement: the operator $A$ is indecomposable.

Definition 4.5. A Jordan block is an indecomposable linear operator.

It follows (not immediately!) that a Jordan block has the form (4.4) in some basis, where we allow the matrix to shrink or expand. The number $\lambda$ is called the generalized eigenvalue of the Jordan block.
block. In case \( \dim W = 1 \), a Jordan block is an eigenline. For a Jordan block, the domain of (4.2) is 1-dimensional.

Let \( A: W \to W \) be a Jordan block with generalized eigenvalue \( \lambda \). Then \( N = A - \lambda \text{id}_W \) is a nilpotent operator: \( A^{n-1} = 0 \), where \( n = \dim W \). The kernels of successive powers of \( N \) form a full flag in \( W \), i.e., a filtration

\[
0 \subset W^{(1)} \subset W^{(2)} \subset \cdots \subset W^{(n)} = W
\]

with \( \dim W^{(i)} = i \). The line \( W^{(1)} \) is a 1-dimensional eigenspace with eigenvalue \( \lambda \).

**Theorem 4.7.** Let \( W \) be a finite dimensional complex vector space, and suppose \( A: W \to W \) is a linear operator. Then there exists an \( A \)-invariant decomposition

\[
W = \bigoplus_{i=1}^{k} W_i
\]

into Jordan blocks. The decomposition is unique up to ordering.

The corresponding \( \lambda_1, \ldots, \lambda_k \) are the eigenvalues of \( A \). The \( \lambda_i \) may be repeated. The operator \( A \) is diagonalizable iff \( \dim W_i = 1 \) for all \( i \).

![Figure 5. The sheaf \( \mathcal{F}_A \to \mathbb{C} \) of the linear operator \( A: W \to W \)](image)

For a geometric view, let \( R = \mathbb{C}[\lambda] \) be the polynomial ring on one variable. Then a linear operator \( A: W \to W \) determines an \( R \)-module structure on \( W \): the indeterminate \( \lambda \) acts as \( A \). In algebraic geometry, to a commutative ring one attaches a space on which the ring is realized as a ring of functions. For the polynomial ring, that space is the affine line \( \text{Spec} \, R = \mathbb{C} \). A module over the commutative ring determines a sheaf\(^{15} \) over its spectrum. For the linear operator \( A \), denote the sheaf by \( \mathcal{F}_A \to \mathbb{C} \). The support of this sheaf—the spectrum of \( A \)—is the finite set of its eigenvalues; see Figure 5. For any open \( E \subset \mathbb{C} \), the sections of this sheaf over \( E \) form the vector space

\[
\Gamma_E(\mathcal{F}_A) = \bigoplus_{\lambda_i \subset E} W_i.
\]

\(^{15}\)We discuss sheaves in a later lecture; they are only used in passing here.
Remark 4.10. The decomposition (4.8) depicted in Figure 5 does not determine the operator $A$ in general. If there is a Jordan block of dimension $>1$, then the structure of the sheaf $\mathcal{F}_A \to \mathbb{C}$ in a formal neighborhood of the corresponding eigenvalue encodes the flag (4.6).

4.2 Projection-valued measures

Henceforth, we restrict to operators $A: W \to W$ that are diagonalizable. By definition, the eigenspace decomposition (4.2) is an isomorphism:

$$W = \bigoplus_{\lambda \in \mathbb{C}} W_\lambda.$$  

(4.11)

The operator $A$ is determined by this decomposition and by the eigenvalues: $A$ acts as multiplication by $\lambda$ on $W_\lambda$. This is nicely depicted in the skyscraper sheaf of Figure 5. The operator is multiplication by the coordinate $\lambda$ in the base $\mathbb{C}$, and it acts on the vector space $W$ which has been “spread out” or “sheafified” over the base.

There is an equivalent encoding of a multiplication operator in terms of projections. Recall from Definition 1.41(4) that a projection $P: W \to W$ is a linear operator that satisfies $P^2 = P$. A projection gives rise to a direct sum decomposition

$$W = P(W) \oplus (1 - P)(W)$$  

(4.12)

which in other terms is the direct sum of the image of $P$ and the kernel of $P$. Conversely, a decomposition of $W = W'' \oplus W'$ as the ordered direct sum of two subspaces determines a projection

$$\text{proj}[W'](W''): W \to W$$  

(4.13)

with image $W''$ and kernel $W'$. Let Projections($W$) denote the space of all projection operators on $W$.

Returning to a diagonalizable $A: W \to W$ with eigenspace decomposition (4.11), define

$$\pi_A: \text{Open}(\mathbb{C}) \to \text{Projections}(W)$$  

(4.14)

$$E \mapsto \text{proj} \left[ \bigoplus_{\lambda \notin E} W_\lambda \right] \left( \bigoplus_{\lambda \in E} W_\lambda \right)$$

Observe that although the domain is written as the set of open subsets of $\mathbb{C}$, in fact the map is well-defined on all subsets of $\mathbb{C}$. The map $\pi_A$ satisfies:

(i) $\pi_A(\emptyset) = 0$,
(ii) $\pi_A(\mathbb{C}) = \text{id}_W$,
(iii) If $E_1 \cap E_2 = \emptyset$, then $\pi_A(E_1 \cup E_2) = \pi_A(E_1) + \pi_A(E_2)$ and $\pi_A(E_1)$ commutes with $\pi_A(E_2)$.

These are similar to the properties of a probability measure; a projection has (eigen)values 0 and 1, and its value on a typical vector is a sort of average of 0 and 1. Property (i) says that the value of the empty set is the 0-map, and property (ii) says that the value of the entire space is the “1-map”. But there are the following differences from a probability measure:
• A probability measure is real-valued.
• A measure is defined on a $\sigma$-algebra of subsets, which for a topological space is the $\sigma$-algebra generated by the open subsets: the $\sigma$-algebra of Borel subsets.
• A measure satisfies countable additivity, not just the finite additivity in (iii).

The map (4.14) is called a projection-valued measure or spectral measure. The expression of $A$ as a multiplication operator—its diagonalization—is

$$A = \int_{\mathbb{C}} \lambda \, d\pi_A(\lambda).$$

As usual, once an operator $A$ is diagonalized we can define functions of the operator, such as its exponential $e^A$. If $f: \mathbb{C} \to \mathbb{C}$ is a function, set

$$f(A) = \int_{\mathbb{C}} f(\lambda) \, d\pi_A(\lambda).$$

This formula is the essence of functional calculus. When we come to Hilbert spaces, the function $f$ is required to be Borel measurable.

4.3 Generalizations and special cases

Later we encounter, in the infinite dimensional setting, some variations that are worth pointing out now in passing. We continue with a finite dimensional complex vector space $W$.

Suppose $A_1, \ldots, A^n: W \to W$ form a commuting set of diagonalizable operators: $A_i A_j = A_j A_i$ for all $1 \leq i, j \leq n$. Then these operators are simultaneously diagonalizable: there is a direct sum decomposition

$$W = \bigoplus_{j=1}^\ell W^{(j)}$$

and

$$A_i|_{W^{(j)}} = \lambda_{ij}$$

for some $\lambda_{ij} \in \mathbb{C}$.

A more geometric picture emerges if we consider all linear combinations of the operators $A_1, \ldots, A_n$. Let $V$ be a finite dimensional complex vector space, and suppose $A: V \to \text{End}(W)$ is a homomorphism of additive groups: it maps vector sums to operator sums. Then the eigenvalues of the operator $A(\xi)$, $\xi \in V$, are linear functions of $\xi$. There is one such linear function for each subspace $W^{(j)}$ in (4.17). This diagonalization can be expressed either as a skyscraper sheaf over $V^*$ supported at these linear functions, or, as in §4.2, as a projection-valued measure on $V^*$ with the same support. The previous paragraph is the model case $V = \mathbb{C}^n$.

Now suppose that $W$ is a finite dimensional complex inner product space. (Our convention for the inner product, which is a positive definite hermitian form, is (1.40).) There are two special
classes of diagonalizable linear operators $A : W \to W$, namely self-adjoint operators and unitary operators. The eigenvalues of a self-adjoint operator are real, and the associated projection-valued measure is supported on $\mathbb{R} \subset \mathbb{C}$. The eigenvalues of a unitary operator lie in the unit circle

$$\mathbb{T} = \{ \lambda \in \mathbb{C} : \lambda \bar{\lambda} = 1 \},$$

and the associated projection-valued measure is supported on $\mathbb{T} \subset \mathbb{C}$. Observe that $\mathbb{R} \subset \mathbb{C}$ is an (additive) subgroup, and $\mathbb{T} \subset \mathbb{C}^\times$ is a (multiplicative) subgroup. The exponential map

$$\exp(\sqrt{-1} \cdot) : \mathbb{R} \to \mathbb{T}$$

is a covering map, and using functional calculus we can exponentiate a self-adjoint operator to a unitary operator.

**Example 4.21.** Let $V$ be a finite dimensional real vector space, and suppose

$$A : V \to U(W)$$

is a unitary representation of the additive Lie group $V$. Then diagonalization produces a projection-valued measure on (the real dual space) $V^*$, but at a linear functional $\theta \in V^*$ the operator acts via the exponential of $\theta$:

$$A = \int_{V^*} e^{i\theta} d\pi_A(\theta).$$

**Remark 4.24.** Example 4.21 generalizes to unitary representations of abelian (Lie) groups; diagonalization occurs over the Pontrjagin dual group.

### 4.4 Spectral theorem on Hilbert space

We begin with a formal definition of a spectral measure. The books [Fo2, Nee, RS] have more detail and proofs of various spectral theorems.

**Definition 4.25.** Let $(X, \Sigma)$ be a measure space (so $\Sigma$ is a $\sigma$-algebra on $X$), and suppose $\mathcal{H}$ is a complex separable Hilbert space. Let $\text{Proj}^+(\mathcal{H})$ denote the space of self-adjoint projections in $\mathcal{H}$.

Then a map $\pi : \Sigma \to \text{Proj}^+(\mathcal{H})$ is a spectral measure if:

1. $\pi(\emptyset) = 0_{\mathcal{H}}$ and $\pi(X) = \text{id}_{\mathcal{H}}$;
2. if $E_1, E_2, \ldots \in \Sigma$ is a finite or countable sequence of disjoint sets, then

$$\pi\left( \bigcup_i E_i \right) \xi = \sum_i \pi(E_i)\xi, \quad \xi \in \mathcal{H};$$

3. $\pi(E_1 \cap E_2) = \pi(E_1)\pi(E_2)$ for all $E_1, E_2 \in \Sigma$. 


Recall now the material in §1.4.3. Since only bounded operators are discussed there, we supplement here with the following crucial result.

**Definition 4.27.** Let $\mathcal{H}$ be a separable Hilbert space. A strongly continuous 1-parameter unitary group is a homomorphism

$$U : \mathbb{R} \rightarrow U(\mathcal{H})$$

such that for every $\psi \in \mathcal{H}$, the map $t \mapsto U_t(\psi)$ is a continuous map $\mathbb{R} \rightarrow \mathcal{H}$.

A theorem of Stone [Stone] and von Neumann [vN2] tells that such 1-parameter groups can be differentiated at $t = 0$, and in fact the derivative generates the 1-parameter group.

**Theorem 4.29.** There is a one-to-one correspondence between self-adjoint operators and strongly continuous 1-parameter unitary groups.

Crucially, the self-adjoint operator may be unbounded. The key ingredient in the proof is the theory of unbounded self-adjoint operators in Hilbert space and the spectral theorem for such operators.

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**Lecture 5: Quantum mechanics; finite dimensional systems**

We introduced quantum mechanics (QM systems) quickly at the end of Lecture 1, and we quickly revisit that here. Then we discuss some general features of QM systems. First, there is an important composition law, which combines two QM systems without introducing any interaction between them. We introduce the crucial property of positive energy as well as the basic dichotomy of QM systems: gapped vs. gapless.

In the remainder of the lecture we illustrate with finite dimensional examples. We begin with a general discussion of systems with a 2-dimensional state space, now called *qubits*. Here we encounter an important piece of geometric structure in quantum mechanics: the transition probability function on pairs of pure states. Then we introduce the general class of *lattice models*, and describe one very accessible example in detail: the *toric code*.

### 5.1 QM systems

The following is parallel to Definition 3.1, which tells that the data of a classical Hamiltonian system is a pair $(N, \varphi)$ consisting of a symplectic manifold and a 1-parameter group $\varphi$ of symplectic diffeomorphisms, i.e., a homomorphism $\varphi : \mathbb{R} \rightarrow \text{SympDiff}(N)$.

**Definition 5.1.** The linear data of a quantum mechanical (QM) system is a pair $(\mathcal{H}, U)$ consisting of a complex separable Hilbert space $\mathcal{H}$ and a strongly continuous 1-parameter group $U_t : \mathcal{H} \rightarrow \mathcal{H}$, $t \in \mathbb{R}$, of unitary automorphisms.

Put differently, $U : \mathbb{R} \rightarrow U(\mathcal{H})$ is a strongly continuous unitary representation of the additive group $\mathbb{R}$ of time translation. By Stone’s Theorem 4.29 this unitary representation is equivalent to a (possibly unbounded) self-adjoint operator $H$, which is called the Hamiltonian.
Remark 5.2. We call this *linear* data since a quantum system is *projective* and can be specified by projective data instead, as we discuss in Lecture 8.

In §1.3 we explain how to obtain the data (Axiom System 1.1) of a mechanical system from the pair $(\mathcal{H}, U)$. We do not repeat here, but only emphasize the following points:

- Let $\mathcal{O} = \mathcal{O}^\infty = \text{End} \mathcal{H}$ be the algebra of *continuous* (bounded) operators on $\mathcal{H}$. That observables can be restricted to be bounded operators has been much discussed in the literature; for example, see [Seg, Ha]. This leads to $C^*$-algebras of observables.
- The formulas (1.30) for the flow on states and observables hold. They are defined directly in terms of the 1-parameter group of unitaries.

5.2 General remarks

5.2.1 Composition laws. There is a composition law on classical Hamiltonian systems and on QM systems: the composition of two systems is their conjunction with no interaction. In terms of the data in Definition 3.1 and Definition 5.1, the composition laws are

\[(N^{(1)}, \varphi^{(1)}) \times (N^{(2)}, \varphi^{(2)}) = (N^{(1)} \times N^{(2)}, \varphi^{(1)} \times \varphi^{(2)})\]

and

\[(\mathcal{H}^{(1)}, U^{(1)}) \otimes (\mathcal{H}^{(2)}, U^{(2)}) = (\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}, U^{(1)} \otimes U^{(2)}).\]

These composition laws are commutative and associative, but in a categorical sense: the structure is the categorification of a commutative monoid. There is a unit for the compositions: in the classical case it has $N = \text{pt}$ and in the quantum case $\mathcal{H} = \mathbb{C}$; for both units the 1-parameter groups are identity maps. These mechanical systems are the objects of a category (what are the morphisms?), and the composition law determines a symmetric monoidal structure.

Remark 5.5. The associative composition law and its unit determine the notion of an *invertible* quantum mechanical system: $(\mathcal{H}, U)$ is invertible if there exists $(\mathcal{H}', U')$ such that $(\mathcal{H}, U) \otimes (\mathcal{H}', U') \cong (\mathbb{C}, 1)$. If so, then dim $\mathcal{H} = 1$, and so the system only has a single state (but the evolution can be by any unitary character of $\mathbb{R}$). This is not a particularly interesting system. Its analogs in quantum field theory turn out to be surprisingly useful.

5.2.2 Positive energy; gapped vs. gapless. Recall the spectral decomposition of a unitary representation of a vector space (Example 4.21). Applied to the unitary time-evolution $U: \mathbb{R} \to U(\mathcal{H})$ of a QM system, we obtain a self-adjoint projection-valued measure $\pi_U$ such that

\[U_t = \int_{\mathbb{R}} e^{-itE/\hbar} \, d\pi_U(E).\]

This is the diagonalization of $U$ in terms of unitary characters $t \mapsto e^{-itE/\hbar}, E \in \mathbb{R}$, of the additive group $\mathbb{R}$. The support of the measure $\pi_U$ is the *spectrum* $\text{spec}(U)$ of the unitary representation $U$.

The following definition is critical.
Definition 5.7. Let \((\mathcal{H}, U)\) be the data of a quantum mechanical system.

1. \((\mathcal{H}, U)\) has **positive energy** if \(\text{spec}(U)\) is bounded below.
2. \((\mathcal{H}, U)\) is **gapped** if it has positive energy and the minimum of \(\text{spec}(U)\) is an isolated point of \(\text{spec}(U)\) which is an eigenvalue of finite multiplicity. Any eigenline for the minimum eigenvalue is a vacuum state. If \((\mathcal{H}, U)\) is not gapped, it is called gapless.

Many systems are normalized so that the minimum of the spectrum is 0, i.e., the Hamiltonian is a nonnegative self-adjoint operator. Also, in many systems there is a unique vacuum state; we will see an example below (toric code) with nonunique vacua. Finally, if \(\mathcal{H}\) is finite dimensional, then every Hamiltonian is gapped. For these discrete systems there is a more refined notion, at least heuristically; see Remark 5.28(2) below.

5.2.3 Unitarity. The evolution in quantum mechanics, as defined in (1.30), is unitary. This is a signature feature of quantum theory. The other pillar of quantum field theory is locality. In the general setup of quantum mechanics there is no notion of space—only time—so no notion of locality. Nonetheless, we will get a glimpse of locality in our description of spin systems and of the toric code, each of which takes place on a discrete space.

5.3 States and observables for two-dimensional Hilbert spaces

Let \(\mathcal{H} = W\) be a 2-dimensional complex inner product space. We can fix a basis, so in essence take \(\mathcal{H} = \mathbb{C}^2\), and we can give the basis elements descriptive names, such as ‘spin up’ \(\uparrow\) and ‘spin down’ \(\downarrow\), but to keep the geometry and all of the symmetry in the foreground we do not do so. Also, we need not fix a Hamiltonian to discuss states, observables, and the pairing to \(\text{Prob}(\mathbb{R})\). A QM system with a 2-dimensional state space is often called a qubit.

The pure states \(S_0\) comprise the complex projective line \(\mathbb{P}W\), the space of 1-dimensional subspaces in \(W\). The space of all states is

\[
(5.8) \quad S = \{ S \in \text{End}W : S^* = S, \ S \geq 0, \ \text{trace}(S) = 1 \}.
\]
Our goal is to make explicit the geometric structure of $S$. First, $\text{End } W$ is a 4-dimensional complex vector space. The real points ($S^0 = S$) lie in a 4-dimensional real subspace, depicted in Figure 6. For any self-adjoint $S$ there is an orthogonal decomposition $L \oplus L^\perp$ with respect to which the matrix of $S$ is diagonal. Either $S$ is a multiple of $\text{id}_W$ or there are distinct real numbers $\lambda > \lambda^\perp$ for which $S$ acts as multiplication by $\lambda$ on $L$ and multiplication by $\lambda^\perp$ on $L^\perp$. There is a closed convex cone of nonnegative self-adjoint operators, those for which $\lambda, \lambda^\perp \geq 0$; its interior is nonempty and consists of strictly positive operators ($\lambda, \lambda^\perp > 0$). The three-dimensional subspace of tracefree operators ($\lambda + \lambda^\perp = 0$) has an an affine translate of trace one operators ($\lambda + \lambda^\perp = 1$). The intersection with nonnegative operators is (5.8). It is a convex set whose extreme points satisfy $\lambda = 1$ and $\lambda^\perp = 0$, in which case $S$ is orthogonal projection onto the line $L \in \mathbb{P}W$. Identify $S$ as a 3-dimensional disk (closed ball) in the 3-dimensional affine space of self-adjoint $S$ with trace($S$) = 1; see Figure 7. The center of the disk is the map $\frac{1}{2} \text{id}_W$, and the boundary 2-sphere is identified with the projective line $\mathbb{P}W$: a point $L$ on the boundary 2-sphere is the self-adjoint projection onto the line $L \subset W$. Endow the affine space with a Euclidean metric so that the sphere has radius 1. Then the point at distance $0 < r \leq 1$ on the radius with endpoint $L$ represents the self-adjoint operator with eigenspaces $L, L^\perp$ and eigenvalues $(1 + r)/2, (1 - r)/2$.

![Figure 7. The space $S$ of states as a 3-disk](image)

**Remark** 5.9. It is easy to see in this example that the map $\text{Prob}(S_0) \to S$, which averages pure states to construct a state, while surjective is far from injective: any mixed state can be expressed in uncountably infinitely many ways as a convex combination of pure states, even of two pure states. Contrast with the case of classical/statistical mechanics in which that map is an isomorphism.

The space $\mathcal{O}_R$ of observables is the real 4-dimensional vector space of self-adjoint operators on $W$. There is a distinguished line: real multiples of $\text{id}_W$. For $\lambda \in \mathbb{R}$ the spectral measure of $\lambda \cdot \text{id}_W$ is depicted in Figure 8(i). It is supported at $\lambda \in \mathbb{R}$. The generic self-adjoint operator has distinct real eigenvalues $\lambda_1 < \lambda_2$ and there is an orthogonal decomposition $W = K_1 \oplus K_2$ as a sum of eigenlines; see Figure 8(ii) for its spectral measure.

Next, we work out the real-valued probability measure obtained by pairing an observable and a state, as defined in (1.27) and (1.28). Begin with the pure state $S$ that is projection onto a line $L \in \mathbb{P}W$. Consider first the observable $A = \lambda \cdot \text{id}_W$. Then the pairing works out to be the
Figure 8. The spectral measures of (i) $\lambda \cdot \text{id}_W$ and (ii) a self-adjoint operator with eigenvalues $\lambda_1 < \lambda_2$ point probability measure on $\mathbb{R}$ supported at $\lambda$. In other words, the result of the pairing is $\lambda$ with probability one: there is no uncertainty. For a self-adjoint operator $A$ with spectral measure as in Figure 8(ii), the pairing is a probability measure supported on the 2-element set $\{\lambda_1, \lambda_2\}$. Let’s work out the probabilities. Let $\psi \in L$ and $\xi_i \in K_i$ be unit norm vectors. Then orthogonal projection onto $K_i$ is the linear operator

(5.10) \[ P_i(\psi) = \langle \xi_i, \psi \rangle \xi_i. \]

Note that (5.10) is independent of the choice of unit vector $\xi_i$. The probability at $\lambda_i$ is

(5.11) \[ \langle \psi, P_i(\psi) \rangle = \langle \psi, \langle \xi_i, \psi \rangle \xi_i \rangle = \langle \xi_i, \psi \rangle \langle \psi, \xi_i \rangle = |\langle \psi, \xi_i \rangle|^2. \]

Observe that this is independent of the choice of unit vectors $\psi, \xi_i$. The expression for these probabilities is a fundamental piece of structure. Therefore, we introduce the function

(5.12) \[ p: \mathbb{P}W \times \mathbb{P}W \rightarrow [0, 1] \]

\[ L \rightarrow |\langle \psi, \xi \rangle|^2 \]

in which $\psi \in L$, $\xi \in K$ are unit vectors. The probability measures for the pairing of the operators in Figure 8 with the pure state $L \in \mathbb{P}W$ are depicted in Figure 9.

Remark 5.13.

(1) Note that in Figure 9(ii) the result of a measurement in a pure state is a probability measure with positive variance. In other words, in quantum mechanics, as opposed to in classical mechanics, even in a pure state measurement is probabilistic. (In both theories we expect measurement to be probabilistic in mixed states.)

(2) The function $p$ encodes the transition probability between pure states. It is a basic structure in quantum mechanics, which we explore in Lecture 8 in the general context of projective geometry. We will prove a generalization of the following formula for $p$ in terms of the geodesic distance $d$ on the 2-sphere of unit radius:

(5.14) \[ p = \cos^2\left(\frac{d}{2}\right). \]
Figure 9. The probability measures obtained by measuring the observables of Figure 8 in the pure state $L \in P_W$.

Try now to prove (5.14)—it is a fun problem in elementary geometry. You should also use it to verify that the sum of the weights in Figure 9(ii) is 1, a fact you can verify directly.

3. The entire discussion extends easily to mixed states: apply the first equation of (1.4).

4. There is a huge literature on measurement in quantum mechanics, and we simply make a brief comment. If the value of a measurement of an observable $A$ in a pure state $\sigma$ is a number in the point spectrum of $\sigma_A$, i.e., is an eigenvalue $\lambda$ of $A$, then after measurement the state of the system projects to the pure state obtained by orthogonal projection of $\sigma$ onto the $\lambda$-eigenspace. There is a similar rule when $\sigma$ is a mixed state. The story is more complicated for values in the continuous spectrum; see [DP] and the references therein.

5.4 Lattice models

We give only a brief introduction to spin systems; see [Sa] for a more extensive introduction. These systems introduce a discrete notion of space while retaining continuous time. Systems of this type are a focus of theoretical condensed matter physics. There are also systems with discrete space and discrete time, so-called “stat-mech systems”.

Figure 10. A spin system on a finite graph $\Gamma$

Let $\Gamma$ be a finite graph, as in Figure 10. For each vertex $v \in \text{Vert}(\Gamma)$ suppose given a finite dimensional complex Hilbert space $\mathcal{H}_v$. (Alternatively, one can attach Hilbert spaces to the edges.) Then define the state space of the model to be

\begin{equation}
\mathcal{H} = \bigotimes_{v \in \text{Vert}(\Gamma)} \mathcal{H}_v.
\end{equation}
The observables are the real points in the tensor product

$$\mathcal{O} = \bigotimes_{v \in \text{Vert}(\Gamma)} \text{End} \mathcal{H}_v.$$  

One can now speak of the support of an observable. For example, for a fixed vertex $v_0 \in \text{Vert}(\Gamma)$ observables of the form

$$A \otimes \bigotimes_{v \neq v_0} \text{id}_{\mathcal{H}_v}, \quad A \in \text{End} \mathcal{H}_{v_0},$$

are said to be supported at $v_0$. There is a similar definition of observables supported on any subset of $\text{Vert}(\Gamma)$. One imagines $\Gamma$ to be very large, and then one has local observables that are supported on a small number of neighboring vertices. The Hamiltonian of such a system is taken to be a sum of local observables with uniformly small support.

Remark 5.18. There is no corresponding notion of the support of a state: there is no distinguished nonzero vector in $\mathcal{H}_v$, whereas in (5.17) we use the distinguished identity operator in $\text{End} \mathcal{H}_v$.

This is only a small hint of this large field of inquiry. Next, we introduce a beautiful geometric example of a system of this type.

5.5 Toric code

The toric code was introduced by Alexei Kitaev [Ki]. It can be formulated on any finite CW complex, in which case the vertices, edges, and faces in our description below are the 0-cells, 1-cells, and 2-cells of the CW structure. The model also has a generalization for any finite group $G$; here $G$ is the cyclic group of order 2.

![Figure 11. A closed surface Y with an embedded graph \(\Lambda\)](image)

Let $Y$ be a closed smooth manifold, which you can assume is 2-dimensional. Let $\Lambda \subset Y$ be a “nice” finite graph that is “nicely embedded” in $Y$, as depicted in Figure 11. (Treating the model
with CW structures is a way to make this precise.) Let $\Lambda^0 \subset \Lambda$ denote the set of vertices. Consider the finite set

\[(5.19) \quad \mathcal{D}(\Lambda, \Lambda^0) = \left\{ (\tilde{\Lambda} \to \Lambda, s) : \tilde{\Lambda} \to \Lambda \text{ is a double cover}, s \text{ is a section of } \tilde{\Lambda}|_{\Lambda^0} \to \Lambda^0 \right\} / \cong\]

of isomorphism classes of double covers of $\Lambda$ equipped with a section over the vertices. Over each edge $e$ the double cover is trivializable, and so by parallel transport we can compare the trivializations at the two endpoints: they agree (+1) or disagree (−1). This leads to an isomorphism

\[(5.20) \quad \mathcal{D}(\Lambda, \Lambda^0) \cong \text{Map}(\text{Edge}(\Lambda), \{\pm 1\}).\]

The Hilbert space of the toric code is the complex vector space

\[(5.21) \quad \mathcal{H} = \text{Map}(\mathcal{D}(\Lambda, \Lambda^0), \mathbb{C})\]

of functions on the finite set (5.19). It carries the hermitian inner product which makes the basis of $\delta$-functions orthonormal.

Introduce observables $H_v, H_f$ associated to each vertex $v$ and each face $f$. A vertex $v \in \Lambda^0$ determines an automorphism

\[(5.22) \quad \varphi_v : \mathcal{D}(\Lambda, \Lambda^0) \to \mathcal{D}(\Lambda, \Lambda^0)\]

that flips the section at the vertex $v$. A face $f$ has a boundary which is homeomorphic to a circle, and a double cover of a circle is either trivial (0) or nontrivial (1). Let

\[(5.23) \quad h_f : \mathcal{D}(\Lambda, \Lambda^0) \to \{0, 1\}\]

be the function that returns the isomorphism type of the restriction of the double cover to $\partial f$. Set

\[(5.24) \quad H_v \psi = \frac{1}{2} (\psi - \varphi_v^* \psi) \quad \quad H_f \psi = h_f \psi\]

Each operator is self-adjoint and has spectrum $\{0, 1\}$, i.e. is a projection. The kernel of $H_v$ consists of functions invariant under flip of the section at $v$, and the kernel of $H_f$ consists of functions supported at double covers that extend over the face $f$. It is a nice exercise to verify that any two operators of the form (5.24) commute. Therefore, these operators are simultaneously diagonalizable.

The Hamiltonian of the toric code is the sum these of commuting projection operators:

\[(5.25) \quad H = \sum_v H_v + \sum_f H_f.\]
It follows that the spectrum of $H$ is contained in the set $\mathbb{Z}^{\geq 0} \subset \mathbb{R}$ of nonnegative integers. Furthermore, the kernel consists of functions supported on double covers that extend over $Y$. More precisely, in the diagram

$$
(5.26) \quad \mathcal{D}(\Lambda, \Lambda^0) \xrightarrow{\pi} \mathcal{D}(\Lambda) \xleftarrow{r} \mathcal{D}(Y)
$$

the map $\pi$ forgets the section $s$, and $r$ restricts a double cover over $Y$ to $\Lambda \in Y$. (These maps pass to sets $\mathcal{D}(-)$ of isomorphism classes of double covers.) The map $r$ is injective. Define an embedding

$$
(5.27) \quad \text{Map}(\mathcal{D}(Y), \mathbb{C}) \hookrightarrow \mathcal{H}
$$

that takes a function on $\mathcal{D}(Y)$, extends it by zero to a function on $\mathcal{D}(\Lambda)$, and then pulls back via $\pi$ to a function on $\mathcal{D}(\Lambda, \Lambda^0)$. The descriptions of $\ker H_v$ and $\ker H_f$ show that the image of (5.27) is the vacuum space $\ker H$.

**Remark 5.28.**

1. The vacuum space is independent of the graph $\Lambda \subset Y$. Also, it has *topological* significance: it can be identified with functions on the finite set $H^1(Y; \{\pm 1\})$ of isomorphism classes of double covers of $Y$. We will see other examples of systems—quantum mechanical and quantum field theoretic—whose vacuum “sector” can be described in topological terms.

2. To say a lattice system is gapped, one wants a local formulation of a family of systems over a family of $\Lambda \subset Y$, and the systems should have a *uniform* spectral gap independent of $\Lambda$. This data and property are present here: the toric code is defined for all suitable $\Lambda \subset Y$, and the uniform spectral gap is 1 since the spectrum for any $\Lambda$ is contained in $\mathbb{Z}^{\geq 0} \subset \mathbb{R}$.

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**Lecture 6: Infinite dimensional quantum mechanical systems**

In this lecture we discuss examples of quantum mechanical systems with an infinite dimensional state space $\mathcal{H}$. We begin with a particle on a Riemannian manifold with zero potential energy. We then move to the special case of a particle in Euclidean space with a quadratic potential energy: the *harmonic oscillator*. We do this in several forms, at one point emphasizing the essentially algebraic nature of the system, and then telling the crucial interpretation as a representation of the *Heisenberg group*. The corresponding Lie algebra representation is a realization of the *canonical commutation relations*. We state some general theorems about the one-dimensional Schrödinger equation, but we leave proofs to the homework. We conclude with some heuristics about the double well potential and tunneling.

**6.1 Particle on a Riemannian manifold**

Let $(M, g)$ be a Riemannian manifold. There is an associated quantum mechanical system $(\mathcal{H}, U_t)$ in which $\mathcal{H} = L^2(M; \mathbb{C})$ is the Hilbert space of $L^2$ complex functions on $M$ and $U_t$ is the strongly
continuous 1-parameter group of unitaries on $\mathcal{H}$ generated by the Laplace operator $H = \Delta$, where

$$\Delta f = d^* df \quad \text{(on the dense subspace of smooth functions $f \in \Omega^0_M(\mathbb{C})$).}$$

This is a system of positive energy (Definition 5.7(1)): $\text{spec } H \subset \mathbb{R}^\geq 0$. This follows since $\Delta$ is a nonnegative operator: for any smooth function $f$ on $M$ with compact support, integration by parts shows

$$\langle f, \Delta f \rangle = \langle f, d^* df \rangle = \langle df, df \rangle \geq 0,$$

where $\langle -, - \rangle$ is the $L^2$ hermitian inner product

$$\langle f, g \rangle = \int_M \bar{f} g \, d\mu_g.$$  \hfill (6.2)

(Here $d\mu_g$ is the Riemannian measure on $M$.) If $M$ has components of positive dimension, then the spectrum of $H$ is unbounded above: $\Delta$ is an unbounded self-adjoint operator.

The corresponding classical Hamiltonian system is described in Remark 3.10(3). In that case—assuming $M$ has components of positive dimension—the spectrum of the classical Hamiltonian function, which is the closure of its image in $\mathbb{R}$—equals $\mathbb{R}^\geq 0$. This is not always the case for the quantum theory.

Remark 6.3. One should not skip lightly over the problematic phrase ‘corresponding classical Hamiltonian system’. There is no easy correspondence between classical and quantum systems.

If $M$ is closed (compact without boundary) and positive dimensional, then the quantum Hamiltonian $H = \Delta$ has pure point spectrum: its spectrum is a countable sequence $0 < \lambda_1 < \lambda_2 < \cdots$ of nonnegative real eigenvalues, each with finite multiplicity. The discreteness of the spectrum and the finiteness of the multiplicities is a basic result in the theory of elliptic partial differential equations.

Hence this is a gapped system (Definition 5.7(2)). Furthermore, (6.1) proves that for $f \in \Omega^0_M$ we have $\Delta f = 0$ iff $df = 0$ iff $f$ is a locally constant function. Therefore, the vacuum space ker $H$ is finite dimensional and consists of locally constant functions, so it has topological significance: it is the space of functions on the finite set $\pi_0 M$ of (path) components of $M$. A generalization for $q \in \mathbb{Z}^\geq 0$: let $\mathcal{H}$ be the Hilbert space of complex differential $q$-forms on $M$, and let $H$ be the Hodge Laplace operator $\Delta = dd^* + d^* d$. The spectrum is pure point, as for $q = 0$, and the Hodge theorem tells that the vacuum space ker $H$ of complex harmonic $q$-forms is isomorphic to the complex cohomology group $H^q(M; \mathbb{C})$. Notice that this is independent of the Riemannian metric, and as in Remark 5.28(1) the vacuum space has topological significance. This is an important feature of many gapped systems.

Remark 6.4. On a compact manifold, then, the classical spectrum is continuous whereas the quantum spectrum is discrete. This is one meaning of ‘quantization’: the quantization of energy.

If $M$ is noncompact, then the Laplace operator may have continuous spectrum. For example, if $M = \mathbb{E}^d$ is standard Euclidean space then the spectrum is $\mathbb{R}^\geq 0$ and it is all continuous: there are no eigenvalues. There are certainly harmonic functions on $\mathbb{E}^d$—constant functions in any dimension $d$, polynomial functions such as $(x^1)^2 - (x^2)^2$, say, in dimension $d = 2$ on $\mathbb{E}^2_{(x^1, x^2)}$—but no nonzero harmonic function lies in $L^2$.

We can allow a potential energy function $V : M \to \mathbb{R}$ in each of these systems. Even on Euclidean space a potential function with suitable growth leads to a gapped quantum system with pure point spectrum, as we illustrate next.
6.2 Harmonic oscillator I: simple harmonic oscillator on the Euclidean line

We treat a unit mass particle moving on $E^1_x$ with potential function

$$V(x) = \frac{1}{2} x^2.$$  

Consider first the classical particle moving in this potential. The classical trajectories $x: \mathbb{R} \to E^1$ obey Newton’s law (1.16), which works out to be

$$\ddot{x} = -x.$$  

The classical trajectories are oscillatory and form a real affine space $N$, isomorphic to $E^1 \times \mathbb{R}$ by fixing a time $t_0$:

$$N \to \mathbb{A}^2$$

$$x \mapsto (x(t_0), \dot{x}(t_0))$$

The Hamiltonian on $N$ is the function

$$H = \frac{1}{2}(\dot{x}^2 + x^2),$$

which is evaluated at any time $t_0$; on a classical trajectory it is independent of $t_0$. The classical “spectrum” of the Hamiltonian function $H$, the closure of its range, is $\mathbb{R}^\geq 0$. The trajectories are illustrated in Figure 12.

![Figure 12. Classical trajectories of a harmonic oscillator](image)

We should, of course, include constants to get the units correct. Let $m \in \mathbb{R}^> 0$ have units of (mass) and $k \in \mathbb{R}^> 0$ have units of (mass)/(time)$^2$. Then the classical Hamiltonian

$$H = \frac{1}{2}(m\dot{x}^2 + kx^2),$$
has units of energy. We leave the reader to put these constants into the following treatment of the corresponding quantum system.

Let $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$ and define the self-adjoint Hamiltonian operator

$$H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 \right). \tag{6.10}$$

This is an unbounded operator, and it is nonnegative: the quantum harmonic oscillator has positive energy. Note that $H$ is the sum of two unbounded nonnegative operators, each of which has continuous spectrum $[0, \infty)$. We now show that the sum $H$ has pure point spectrum. In particular, the quantum harmonic oscillator is a gapped system.

Set

$$A = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right) \tag{6.11}$$

Then $A^*$ is the (formal) adjoint of $A$ and the commutator is $[A, A^*] = 1$ (meaning $\text{id}_{\mathcal{H}}$). Also,

$$H = A^*A + \frac{1}{2}, \tag{6.12}$$

which implies that $\text{spec}(H) \subset [\frac{1}{2}, \infty)$. We claim there exists a unique line $\mathbb{C} \cdot \Omega \subset \mathcal{H}$ such that $A\Omega = 0$. For if $\Omega$ is represented by a function $f: \mathbb{E}^1 \rightarrow \mathbb{C}$, then

$$\frac{df}{dx} + xf(x) = 0, \tag{6.13}$$

from which

$$f(x) = Ce^{-x^2/2} \tag{6.14}$$

for $C \in \mathbb{C}$. It follows from (6.12) that $\Omega$ is an eigenfunction of $H$ with eigenvalue $1/2$. Furthermore, any eigenfunction of $H$ with eigenvalue $1/2$ is annihilated by $A$; argue analogously to (6.1). Compute the Hamiltonian on $A^*\Omega$:

$$HA^*\Omega = \frac{1}{2}A^*AA^*\Omega + \frac{1}{2}A^*\Omega = \frac{1}{2}A^*A^*A\Omega + A^*\Omega + \frac{1}{2}A^*\Omega = \frac{3}{2}A^*\Omega, \tag{6.15}$$

so $A^*\Omega$ is an eigenfunction with eigenvalue $3/2$. Also

$$\langle A^*\Omega, \Omega \rangle = \langle \Omega, A\Omega \rangle = 0, \tag{6.16}$$

thus $\mathbb{C}(A^*\Omega)$ is orthogonal to $\mathbb{C}(\Omega)$, as it must be. Continue by induction to prove that for $n \geq 0$ the vector $(A^*)^n\Omega$ is an eigenfunction of $H$ with eigenvalue $\frac{1}{2} + n$. One can prove that $\{(A^*)^n\Omega\}_{n \in \mathbb{Z} \geq 0}$ is an orthonormal basis of $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$. Hence the spectrum of the Hamiltonian is $\text{spec}(H) = \frac{1}{2} + \mathbb{Z} \geq 0$ (Figure 13) and every eigenvalue is simple: the eigenspaces are one-dimensional.
6.3 Harmonic oscillator II: algebraic treatment

Now we consider what amounts to a finite number of simple harmonic oscillators, but we consider the collection as a whole—no decomposition into simple oscillators—which amounts to not choosing a basis for the vector space $W$ below. Our treatment focuses on operators rather than states, and we give a purely algebraic description, which applies quite generally, for example to vector spaces over arbitrary fields.

Let $W$ be a finite dimensional real vector space. Set

\begin{equation}
F(W) = \text{Sym}^* W_C^*,
\end{equation}

where $W_C = W \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $W$. This symmetric algebra, which is $\mathbb{Z}$-graded, is the Fock space of $W$. It is the algebra of complex-valued polynomials on $W$. For $w \in W$ and $\ell \in W^*$ define linear operators $A_w$ and $M_\ell$ of degrees $-1$ and $+1$, respectively, as follows:

\begin{align*}
A_w(\theta) &= \theta(w), & \theta & \in \text{Sym}^1 W_C^* \\
M_\ell(p) &= \ell \cdot p, & p & \in F(W),
\end{align*}

and extend $A_w$ to be a derivation: $A_w(p_1 \cdot p_2) = A_w(p_1) \cdot p_2 + p_1 \cdot A_w(p_2)$ for all $p_1, p_2 \in F(W)$. Then compute the commutator

\begin{equation}
[A_w, M_\ell] = \ell(w)
\end{equation}

to be scalar multiplication by $\ell(w)$. Other commutators vanish: for $w_1, w_2 \in W$, $\ell_1, \ell_2 \in W^*$,

\begin{equation}
[A_{w_1}, A_{w_2}] = [M_{\ell_1}, M_{\ell_2}] = 0.
\end{equation}

Equations (6.19) and (6.20) are the canonical commutation relations.

Remark 6.21.

(1) The real vector space of operators $A_w, M_\ell$ is the quotient in a central extension

\begin{equation}
0 \to \mathbb{R} \to \mathfrak{h} \to W \oplus W^* \to 0
\end{equation}

of real Lie algebras. $\mathfrak{h}$ is called a Heisenberg algebra. The commutator (6.19) induces a symplectic structure on $W \oplus W^*$. In the next section we replace $W \oplus W^*$ with finite dimensional real symplectic vector space. Here the symplectic vector space is polarized.
(2) There are operators $A_w, M_\ell$ for $w, \ell$ in the complexifications $W_C, W_C^*$, and they generate an
infinite dimensional complex algebra: the Weyl algebra.
(3) Observe that nonzero polynomials on $W$, which are the nonzero vectors in the Fock space $F(W)$,
cannot be $L^2$ with respect to any measure, since they do not tend to zero at infinity.

There is a canonical element $C \in W^* \otimes W$. It has the form $C = w_i \otimes w_i$ for any basis $w_1, \ldots, w_N$ of $W$ and dual basis $w^1, \ldots, w^N$ of $W^*$. It corresponds to the degree 0 linear operator

\begin{equation}
\hat{C} = \sum_i M_{w_i} \circ A_{w_i}
\end{equation}
on $F(W)$. The harmonic oscillator Hamiltonian is $H = \hat{C} + 1/2$.

Remark 6.24.

(1) The ordering in (6.23) is important since the commutator (6.19) is nonzero.
(2) If $W$ has a real inner product, then there is an induced hermitian inner product on $F(W)$
and one can complete it to a Hilbert space. We have not done so here, in part to emphasize
the algebraic nature of the harmonic oscillator.

6.4 Harmonic oscillator III: Heisenberg group

Suppose $(V, \omega)$ is a finite dimensional real symplectic vector space. The Heisenberg Lie algebra

\begin{equation}
\mathfrak{h}(V) = V \oplus \mathbb{R}
\end{equation}
is the direct sum in which $\mathbb{R}$ is central and $[v_1, v_2] = \omega(v_1, v_2) \in \mathbb{R}$ for $v_1, v_2 \in V$. This is a 2-step nilpotent Lie algebra:

\begin{equation}
[v_1, [v_2, v_3]] = 0, \quad v_1, v_2, v_3 \in V.
\end{equation}

A more geometric picture emerges for a general affine symplectic space $A$ over $V$. Then the Heisenberg Lie algebra $\mathfrak{h}(A)$ is defined from (2.31) by restriction to constant vector fields:

\begin{equation}
\begin{array}{cccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^0_A & \longrightarrow & \mathcal{X}_A & \longrightarrow & 0 \\
\longmapsto & & & & & & & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{h}(A) & \longrightarrow & V & \longrightarrow & 0
\end{array}
\end{equation}

In the first line we use $H^0(A; \mathbb{R}) = \mathbb{R}$ and $H^1(A; \mathbb{R}) = 0$. The subspace $\mathfrak{h}(A) \subset \Omega^0_A$ is the Lie algebra of affine functions on $A$: the Lie bracket is the Poisson bracket.

We can exponentiate $\mathfrak{h}(A)$ to obtain a Lie group $H(A)$, the Heisenberg group. In doing so replace the center $\mathbb{R}$ of (6.27) with $\sqrt{-1}\mathbb{R}$, and so upon exponentiation we obtain $T \subset \mathbb{C}$ in place of $\mathbb{R}^{>0} \subset \mathbb{R}$. The Heisenberg group is a group extension\textsuperscript{16}

\begin{equation}
1 \longrightarrow T \longrightarrow H(A) \longrightarrow \exp V \longrightarrow 1
\end{equation}
The main result about harmonic oscillators is the following.

\textsuperscript{16}We discuss group extensions more in §8.6.4.
Theorem 6.29 (Stone-von Neumann). There exists a unique isomorphism class of irreducible unitary representations of $H(A)$ on which $\mathbb{T} \subset H(A)$ acts as scalar multiplication.

In §6.3 we indicated how to construct an algebraic representation of a polarized Heisenberg Lie algebra. Recall that there is an increasing filtration on the underlying vector space. One can introduce inner products so that the exponentiated representation is unitary, and then one can complete to a Hilbert space representation. Then one must prove that the representation is independent of the polarization, up to isomorphism. There is a rich story associated to this Heisenberg representation; see [LV, Fo1] for example.

Remark 6.30.

(1) This unitary representation does not contain the 1-parameter group $U_t$ needed to define a QM system. In fact, the Heisenberg representation extends to a larger group that includes the metaplectic group, and it is in this larger unitary representation that one finds $U_t$. This too is a rich story recounted in many places, among them [GV, Se1].

(2) There is a generalization [Sh] of Theorem 6.29 for infinite dimensional $V$, but now a new geometric structure is required: a polarization (class) on $V$. See [Se1] for an exposition of this result.

6.5 One-dimensional Schrödinger operators

An important general class of QM systems is a single particle of mass $m \in \mathbb{R}_{>0}$ on $\mathbb{E}^1$ with some potential energy $V: \mathbb{E}^1 \rightarrow A$, which we assume is a smooth function. The Hamiltonian is

\begin{equation}
H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x).
\end{equation}

An eigenfunction of $H$ with eigenvalue $E \in \mathbb{R}$ is an $L^2$ function $\psi: \mathbb{E}^1 \rightarrow \mathbb{C}$ that satisfies the Schrödinger equation

\begin{equation}
-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + (V(x) - E) \psi(x) = 0.
\end{equation}

A nonzero eigenfunction represents a pure state called a bound state. Depending on $V$ there may be no bound states, the bound states may span (after closure) the Hilbert space $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$, or they may span a proper subspace. For $V = 0$ the first occurs; for $V = x^2/2$ the second occurs. Schrödinger equations have been the object of much mathematical study.

6.6 The double well potential

This is heuristic; rigorous and even exact solutions for particular potentials can be found in the literature. Here $V: \mathbb{E}^1 \rightarrow \mathbb{R}$ has two distinct minima and tends to $+\infty$ as the argument tends to $\pm \infty$, as depicted in Figure 14. For example, we can take the quartic $V(x) = (x^2 - a^2)^2$ for some $a \in \mathbb{R}_{>0}$. The minimal energy classical trajectories $x: \mathbb{R} \rightarrow \mathbb{E}^1$ are stationary sitting at a minima of $V$, so there are two: $x = \pm a$. We might therefore anticipate two quantum vacua, each
localized at a minimum. If we approximate $V$ near a minimum by a quadratic potential, then the resulting harmonic oscillator has a unique vacuum (6.14). These two smooth $L^2$ functions are only approximate vacua for the double well potential. In fact, the exact vacuum has energy less than the energy of each approximate vacuum, and there is a second eigenvalue that is slightly more. A detailed treatment of this problem leads to instantons, tunneling, and many more fun topics; see [C] for an entrée.

**Lecture 7: More QM systems; Klein’s Erlangen Program**

In the first part of this lecture we consider two more QM systems. Each of them comes in a parametrized family. The first depends on a real-valued function on the Euclidean line; the second depends on a number $\theta \in \mathbb{R}/2\pi \mathbb{Z}$. The first is a family of systems discussed by Hori [HKKPTVZ, Chapter 10]; it is also described in the lecture notes [To]. There are approximate vacua from quadratic (harmonic oscillator) approximations, and—at first glance surprisingly—one can solve explicitly for the exact vacua. This is due to a hidden symmetry—a supersymmetry—that we do not discuss in this lecture, though we do derive some consequences. We work on the Euclidean line; the entire discussion has a beautiful generalization to Riemannian manifolds, as explored in Witten’s influential paper [W6]. The second family of QM systems is the instructive particle on a ring; see [GKKS, Appendix D]. We focus on a symmetry group, which we show only acts projectively. This is motivation for Lecture 8, in which we explore the projective nature of quantum mechanics.

As preparation for that discussion, the second part of this lecture is a modern interpretation of Felix Klein’s Erlangen Program [K, BB]. We define a “type” of geometry as a model space $F$ together with the action of a group $G$. The pair $G \circ F$ is a specification; an instantiation is a right
A supersymmetric QM system

Although we use ‘supersymmetric’ in the section title, we do not elaborate on that term now. The Hilbert space of the system is a space of $L^2$ vector-valued functions on the Euclidean line:

\begin{equation}
\mathcal{H} = L^2(\mathbb{E}_1; \mathbb{C}^2) = \left\{ \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} : \psi_j \in L^2(\mathbb{E}_1; \mathbb{C}) \right\} = \mathcal{H}^0 \oplus \mathcal{H}^1
\end{equation}

Vectors in $\mathcal{H}^0$ have the form $\begin{pmatrix} \psi_0 \\ 0 \end{pmatrix}$ and are called even; vectors in $\mathcal{H}^1$ have the form $\begin{pmatrix} 0 \\ \psi_1 \end{pmatrix}$ and are called odd. Let $h: \mathbb{E}_1 \to \mathbb{R}$ be a smooth function. The (unbounded) Hamiltonian operator is

\begin{equation}
H = H_h = \frac{1}{2} \left( -\frac{d^2}{dx^2} + h'(x)^2 \right) \otimes \begin{pmatrix} +1 \\ 1 \end{pmatrix} - \frac{1}{2} h''(x) \otimes \begin{pmatrix} +1 \\ -1 \end{pmatrix}
\end{equation}

The first term is the standard Schrödinger operator (6.31), with $\hbar = 1$, $m = 1$, and potential energy

\begin{equation}
V = \frac{1}{2} (h')^2.
\end{equation}

Observe that $V \geq 0$ and $V(x) = 0$ iff $x \in \text{Crit}(h)$: the potential vanishes exactly at critical points of $h$. The key to the system is the second term $T$, which is also algebraic—no differentiation. We view (7.2) as a family $(\mathcal{H}, H_h)$ of QM systems parametrized by a function $h: \mathbb{E}_1 \to \mathbb{R}$.

Semiclassical vacua. We first ask about vacua of (7.2), that is, pure states of minimal energy. If $x = x_0$ is an isolated critical point of $h$, then we can approximate $h$ near $x = x_0$ by a quadratic function. Then $V$ is also approximated by a quadratic function. The first term of (7.2) is then approximated by a harmonic oscillator with minimal energy $|h''(x_0)|/2$, and there are approximate eigenfunctions in both $\mathcal{H}^0$ and $\mathcal{H}^1$. The second term $T$ now alters the minimal energy to 0, and the corresponding eigenfunction is even if $h''(x_0) > 0$ and is odd if $h''(x_0) < 0$. (Assume nondegeneracy of the critical point: $h''(x_0) \neq 0$.) Therefore, the prediction from a quadratic approximation is that, if each critical point is nondegenerate, then

(i) the set of (pure) vacuum states is isomorphic to $\text{Crit}(h)$, and
(ii) the vacuum at $x_0 \in \text{Crit}(h)$ lies in $\mathbb{P}\mathcal{H}^0$ if $h''(x_0) > 0$; it lies in $\mathbb{P}\mathcal{H}^1$ if $h''(x_0) < 0$.

We will see that these semiclassical predictions are usually false in the full theory.
7.1.2 Quantum vacua and first order operators. We analyze the quantum theory using the following observation. Introduce the (formally) adjoint\textsuperscript{17} first-order differential operators

\begin{equation}
Q = \left(-i \frac{d}{dx} - ih'(x)\right) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{equation}

\begin{equation}
Q^* = \left(-i \frac{d}{dx} + ih'(x)\right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{equation}

Observe that we can view these as differential operators

\begin{equation}
\mathcal{H}^0 \xrightarrow{Q} \mathcal{H}^1
\end{equation}

Then an easy computation shows that

\begin{equation}
H = \frac{1}{2} (QQ^* + Q^*Q).
\end{equation}

Lemma 7.7. $H\begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = 0$ iff $Q\psi^0 = 0$ and $Q^*\psi^1 = 0$.

Proof. The implication $\iff$ is clear from (7.6). In the converse direction, if $\Psi = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}$, then

\begin{equation}
\langle \Psi, H\Psi \rangle = \frac{1}{2} \|Q\psi^0\|^2 + \frac{1}{2} \|Q^*\psi^1\|^2,
\end{equation}

and the implication $\implies$ follows. (Compare (6.1).) \qed

Of course, we are assuming that $\Psi \in \mathcal{H}$ is $L^2$, which is used in the integration by parts implicit in (7.8).

The first-order ordinary differential equations $Q\psi^0 = 0$ and $Q^*\psi^1 = 0$ have solutions

\begin{equation}
\psi^0 = e^{-h} \\
\psi^1 = e^{h}
\end{equation}

and multiples thereof. So there are at most two vacua. However, these functions are not necessarily $L^2$, and in fact they cannot be simultaneously $L^2$. So there is at most one vacuum, and it lies in either $\mathbb{P}\mathcal{H}^0$ or in $\mathbb{P}\mathcal{H}^1$.

Remark 7.10. There is a variation of this system in which we replace $\mathbb{E}^1$ by the Euclidean circle $\mathbb{E}^1/2\pi\mathbb{Z}$. In this case the function $h$ is periodic, as are the functions (7.9). Then there are two vacua for every choice of $h$: one vacuum lies in $\mathcal{H}^0$ and the other lies in $\mathcal{H}^1$.

\textsuperscript{17}We have used usual conventions for adjoints. When we view these as odd in the world of $\mathbb{Z}/2\mathbb{Z}$-graded algebra, then there are additional factors of $i$ and signs to made the adjoints conform to the Koszul sign rule.
We make one more observation about the system (7.5) and the Hamiltonian (7.6). First, observe

\[
H = \begin{cases} \frac{1}{2}Q^*Q, & \text{on } \mathcal{H}^0; \\ \frac{1}{2}QQ^*, & \text{on } \mathcal{H}^1. \end{cases}
\]

**Lemma 7.12.** For \( \lambda \in \mathbb{R}^>0 \) let \( \mathcal{H}_\lambda^i, i = 0, 1, \) be the eigenspace of \( H \) with eigenvalue \( \lambda \). Then

\[
Q: \mathcal{H}_\lambda^0 \rightarrow \mathcal{H}_\lambda^1
\]

is an isomorphism.

**Proof.** The inverse is \( Q^*/2\lambda \). \( \square \)

Thus eigenspaces with positive eigenvalues are “paired up”. However, as we have already seen, the zero eigenspaces may have different dimensions.

We now illustrate various phenomena that occur for different \( h \).

![Figure 15. The harmonic oscillator (s > 0)](image)

### 7.1.3 Quadratic \( h \): the harmonic oscillator.

Consider the one parameter family

\[
h_s(x) = \frac{1}{2}sx^2, \quad s \in \mathbb{R}.
\]

Then

\[
V_s(x) = \frac{1}{2}s^2x^2,
\]

and the system is a harmonic oscillator if \( s \neq 0 \). The functions \( h_s, V_s \) are depicted in Figure 15 for \( s > 0 \). The second term in the Hamiltonian (7.2) is the constant matrix

\[
T = \begin{pmatrix} -s/2 \\ s/2 \end{pmatrix}
\]
From our analysis (§6.2) of the harmonic oscillator, one can easily deduce the spectrum of $H$. It is plotted in Figure 16 for $s > 0$, $s = 0$, and $s < 0$. Recall the fundamental bifurcation between gapped and gapless systems. Here $H_s$, $s \neq 0$ is gapped and $H_0$ is gapless. That is reflected in Figure 16: the spectrum is continuous down to 0 for $s = 0$, and there is a spectral gap at 0 for $s \neq 0$. (In fact, the entire spectrum is discrete for $s \neq 0$.) Finally, observe that in this case the semiclassical prediction of vacua (§7.1.1) is true, as expected since the semiclassical approximation is exact.

Figure 16. The spectrum of $H$ as $s$ varies

Remark 7.17. At first glance the following is strange: as $s \in \mathbb{R}$ increases through 0 from negative values to positive values, the unique vacuum for $s \neq 0$ jumps from $\mathcal{P}H^0$ to $\mathcal{P}H^1$. One might think it is due to the continuous spectrum down to 0 at $s = 0$. In fact, there are other examples in which it is only discrete spectrum that comes down to 0. These are examples of phase transitions at $s = 0$. One passes from a connected component of gapped systems to another connected component of gapped systems through a codimension one “wall” in the parameter space, and on that wall the gap in the spectrum is breached.

7.1.4 Cubic $h$. Next, consider a typical cubic function $h$, as depicted in Figure 17. Notice that neither function in (7.9) lies in $L^2$, so there are no zero energy states. The potential $V(x)$ still tends to $+\infty$ as $|x| \to \infty$; more precisely $V(x) = O(x^4)$ as $|x| \to \infty$. It follows (not by arguments we give here) that the spectrum of $H$ is discrete, and as we have argued all eigenvalues are positive. In this case $Q: \mathcal{H}^0 \to \mathcal{H}^1$ is an isomorphism. Observe that the semiclassical picture predicts two vacua, one in $\mathcal{P}\mathcal{H}^0$ and one in $\mathcal{P}\mathcal{H}^1$. This is in a sense correct, but the equal energies of these vacua is strictly positive. Also, we may have thought that the quartic potential $V$ would exhibit tunneling between the two minimal energy states, as described in §6.6. However, these states lie in different projective Hilbert spaces and there is no tunneling between them.

7.1.5 Quartic $h$. We use the quartic depicted in Figure 18: $h(x) \to \infty$ as $|x| \to \infty$. It follows that $\psi^0$ in (7.9) lies in $L^2$, and so there is a unique vacuum in $\mathcal{P}\mathcal{H}^0$. On the other hand, the semiclassical analysis predicts three vacua: two in $\mathcal{P}\mathcal{H}^0$ and one in $\mathcal{P}\mathcal{H}^1$. In this case there is tunneling between the two semiclassical vacua in $\mathcal{H}^0$ to produce a single true quantum vacuum; the energy of the
minimal energy state in $\mathcal{P}\mathcal{H}^1$ is strictly positive and matches the energy of the positive energy state from the splitting of the two semiclassical vacua.

It is illuminating to consider what happens in a family of quartics. Consider a family $h_s$, $s \in \mathbb{R}$, in which

- $h$ is a quartic polynomial for $x < 0$,
- $h$ is a quadratic polynomial for $x > 0$, and
- $h(x) \to +\infty$ as $|x| \to \infty$.

In this family there is a unique vacuum in $\mathcal{P}\mathcal{H}^0$ for all $s \in \mathbb{R}$. But think about semiclassical vacua and can convince yourself—at least heuristically—that as $s$ passes from negative to positive values, there are some eigenvalues that come close to 0. In fact, one can split $\mathcal{H}^0 = \mathcal{H}^+_1 \oplus \mathcal{H}^-_1$ into a direct sum of a finite dimensional space of “low lying eigenvectors” and an infinite dimensional space on which the Hamiltonian $H$ is bounded below. Such spectral splittings are useful in the geometric study of index theory, for example.
7.2 A quick word about symmetry

Suppose \((\mathcal{H}, U_t)\) is the data of a QM system. One might define a Lie group \(G\) of symmetries to be a unitary representation \(\rho: G \to \text{U}(\mathcal{H})\) such that

\[
\rho(g) U_t = U_t \rho(g), \quad g \in G, \quad t \in \mathbb{R}.
\]

These are symmetries that preserve the direction of time translation, and that is what we need for this lecture.

Remark 7.19. One should also allow time-reversing symmetries, as well as symmetries that act antilinearly. We will discuss these in Section 11.1.

![Figure 19. A symmetry group G acting on a family of QM systems](image)

More generally, consider a family \((\mathcal{H}^{(s)}, U_t^{(s)})\) of QM data parametrized by \(s \in S\) for a smooth manifold \(S\). Then a symmetry group \(G\) can also act nontrivially on the parameter space \(S\); it can permute the QM systems, as depicted in Figure 19. Alternatively, one can quotient by \(G\) and say that we have a family of systems parametrized by the groupoid or stack \(S//G\).

7.3 Particle on a ring

The classical version\(^{18}\) of this system was introduced in Example 3.56. Set \(M = \mathbb{E}^1/2\pi\mathbb{Z}\) and \(\mathcal{H} = L^2(M; \mathbb{C})\). Consider the family of Hamiltonian operators

\[
H_{\theta} = \frac{1}{2} \left( i \frac{d}{dx} + \frac{\theta}{2\pi} \right)^2, \quad \theta \in \mathbb{R}.
\]

Then \(H_{\theta} = T_{\theta}^2\) for the self-adjoint operator

\[
T_{\theta} = \frac{1}{\sqrt{2}} \left( -i \frac{d}{dx} - \frac{\theta}{2\pi} \right).
\]

\(^{18}\)The phrase ‘classical version’ is problematic: there is no isomorphism between classical and quantum systems. But in this simple system it is less controversial.
The spectrum is discrete, a basis of eigenfunctions is

\[ \psi_n(x) = e^{inx}, \quad x \in M, \quad n \in \mathbb{Z}, \]

and the corresponding eigenvalues of \( T_\theta \) and \( H_\theta \) are

\[ \mu_n(\theta) = \frac{1}{\sqrt{2}} \left( n - \frac{\theta}{2\pi} \right), \]
\[ E_n(\theta) = \frac{1}{2} \left( n - \frac{\theta}{2\pi} \right)^2 \]

respectively. Observe that the spectrum is periodic in \( \theta \); it is unchanged by \( \theta \to \theta + 2\pi \). In fact, the entire family of quantum systems is periodic: there is an automorphism of \( \mathcal{H} \) that sends \( H_\theta \) to \( H_{\theta+2\pi} \). So the parameter space of this family of systems is \( S = \mathbb{R}/2\pi\mathbb{Z} \).

**Remark 7.24.**

1. One can put in a parameter \( L \in \mathbb{R}^+ \) by setting \( M_L = \mathbb{E}^1/2\pi L \mathbb{Z} \) and modifying (7.20) to

\[ H_{\theta,L} = \frac{1}{2} \left( \frac{d}{dx} + \frac{\theta}{2\pi L} \right)^2. \]

For each \( \theta, L \) the QM system is gapped, but the gap is not uniform in \( L \); the gap tends to zero as \( L \to \infty \).

2. At \( \theta = 0 \) and \( \theta = \pi \) the eigenspaces are not all lines. In fact, other than the 0-eigenspace at \( \theta = 0 \) all eigenspaces are 2-dimensional. At \( \theta = \pi \) all eigenspaces are 2-dimensional.

![Figure 20. The flow of eigenvalues of \( \{H_\theta\} \) as \( \theta \) increases](image)

It is illuminating to visualize the flow of eigenvalues of \( H_\theta \) as \( \theta \) varies over \( S = \mathbb{R}/2\pi\mathbb{Z} \). In Figure 20 we indicate the direction of flow of the eigenvalues \( E_n(\theta) \) as \( \theta \) increases. We have depicted snapshots at \( \theta = 0 \) and \( \theta = \pi \). Note the doubling observed in Remark 7.24(2). One
can also contemplate the spectral flow of the eigenvalues of the first-order differential operators \( T_\theta \); this is depicted in Figure 21. The eigenvalues fit together into a single helix embedded in the cylinder \( S \times \mathbb{R} \).

Recall the classical Lagrangian (3.57) for a particle motion \( x: \mathbb{R} \rightarrow M \):

\[
(7.26) \quad \left( \frac{m}{2} \dot{x}^2 + \frac{\theta}{2\pi} \dot{x} \right) dt.
\]

There is an evident translation action by \( c \in G_0 = \mathbb{R}/2\pi\mathbb{Z} \) that acts trivially on \( \theta \in S \) and sends

\[
(7.27) \quad x \mapsto x + c.
\]

Then \( G_0 \) acts on \( \mathcal{H} \) and commutes with the Hamiltonian (as in (7.18)), so \( \mathcal{H} \) decomposes according to the irreducible representations of \( G_0 \). Since \( G_0 \) is abelian, these representations are 1-dimensional; they are characters of \( G_0 \). In fact, (7.22) shows that each character occurs with multiplicity one, independently of \( \theta \in S \).

There is also a reflection symmetry which has order 2 classically and acts nontrivially on the parameter space \( S \):

\[
(7.28) \quad x \mapsto -x \quad \theta \mapsto -\theta
\]

Note that the fixed points on the parameter space are at \( \theta = 0 \) and \( \theta = \pi \). Reflections reverse the direction of translation under conjugation, so altogether we have the classical symmetry group

\[
(7.29) \quad G = \mathfrak{g}_2 \rtimes \mathbb{R}/2\pi\mathbb{Z} = \mathfrak{g}_2 \rtimes G_0,
\]
where recall $\mu_2 = \{\pm 1\}$. The Lie group $G$ is isomorphic to the orthogonal group $O_2$.

In the quantum system, we can implement (7.27) and (7.28) by using the formula (7.22) as a guide. Restrict to $\theta = \pi$, where the energy eigenvalues (7.23) are $E_n = \frac{1}{2}(n - \frac{1}{2})^2$. (It is a good exercise to write the symmetry on the entire family of QM theories.) We can implement translation by $c$ and reflection by the unitary operators

$$T_c(\psi_n) = e^{inc}\psi_n$$
$$R(\psi_n) = \psi_{1-n}$$

Then $R^2 = id_\mathcal{H}$ and $\{T_c\}_{c \in \mathbb{R}/2\pi\mathbb{Z}}$ satisfies the group law of $G_0$: we obtain separate unitary representations of $\mu_2$ and $G_0$. However, we do not obtain a unitary representation of the semidirect product; rather,

$$RT_cR^{-1} = e^{-ic}T_{-c}.$$  

The homothety that is multiplication by $e^{-ic}$ ruins the group law of the semidirect product.

However, and this is the key point, the group $G$ does act (unitarily) on the projective space $\mathbb{P}\mathcal{H}$. In other words, it acts on the pure states of the QM system. Further analysis shows that $G$ acts on the convex space of all states, $G$ acts on the vector space of observables, and that these actions preserve the measurement pairing and all other data (Axiom System 1.1) of the QM system. There is nothing “sick” about this quantum symmetry; it is just as healthy as the symmetry of the corresponding classical system. This example is one indication of the projective nature of quantum mechanics—indeed, of quantum theory in general—as we pursue in Lecture 8.

**Remark 7.32.** One often hears that a classical symmetry has an anomaly when acting on the corresponding quantum system. This anomaly is not a sickness. It is simply a measure of the projectivity of the symmetry, an obstruction to its linearization.

As preparation for our discussion in the next lecture, we conclude this lecture with a rather different and general topic: geometric structures à la Felix Klein.

### 7.4 The Erlangen Program

Klein’s *Erlangen Program* [K, BB] states that “a geometry is determined by its symmetries”. We interpret this as follows, vaguely at first:

1. One specifies a geometric type as a group action $G \bowtie F$.
2. Then one gives data related to $G$ to construct an instance of this geometric type, or even better a parametrized family of these geometries.

Here $F$ is a model of the geometry, $G$ is the model symmetry group, and the pair $G \bowtie F$ completely determines the geometric type. We can carry this out in different contexts (categories). For our applications to physics, $F$ is a smooth manifold and $G$ is a Lie group.

**Example 7.33.** The geometric type ‘linear geometry of dimension $n$ over a field $\mathbb{F}$’ has model

$$GL_n\mathbb{F} \bowtie \mathbb{F}^n.$$
The geometric type ‘spherical geometry of dimension $n$’ has a model

$$O_{n+1} \circ S^n,$$

where $S^n \subset \mathbb{R}^{n+1}$ is the sphere of unit norm vectors.

To explain instantiation of geometric types, we begin with a digression on torsors. In the following definition the group can be discrete, topological, or Lie. The torsor has the corresponding structure of a set, topological space, or smooth manifold, and the group action is unconstrained, continuous, or smooth.

**Definition 7.36.** Let $G$ be a group. A right $G$-torsor is a set $T$ equipped with a simply transitive right $G$-action $T \times G \to T$. A left $G$-torsor is a set $T$ equipped with a simply transitive left $G$-action $G \times T \to T$.

Simple transitivity for a right $G$-torsor is the condition that the map

$$T \times G \longrightarrow T \times T$$

$$(t, g) \mapsto (t, t \cdot g)$$

be an isomorphism.

**Remark 7.38.** We use a strict convention for left vs. right: left group actions are geometric and right group actions are structural:

$$\text{left} = \text{geometric} \quad \text{right} = \text{structural}$$

There is a category of right $G$-torsors. Namely, a morphism $T' \to T$ of right $G$-torsors is a map that commutes with the right $G$-actions. Every morphism is invertible, so the category of right $G$-torsors is a groupoid. The automorphism group $\text{Aut}(T)$ of a right $G$-torsor $T$ acts on the left:

$$\text{Aut}(T) \circ T \circ G.$$
Each point of $T$ induces an isomorphism $F \xrightarrow{\cong} F_T$. The mixing construction (7.42) can also be called a sandwich: the group $G$ is sandwiched between right and left $G$-torsors to quotient out the $G$-action. The linear analog is the tensor product of right and left modules over a ring. Observe that $\text{Aut}(T)$ acts as symmetries of $F_T$:

$\text{Aut}(T) \circ F_T.$

The group action (7.43) is an instantiation of the model $G \circ F$.

As a special case of Definition 7.41, suppose $\phi: G \to H$ is a homomorphism of groups, and let $T$ be a right $G$-torsor. Then there is a left $G$-action on $H$ ($g \cdot h = \phi(g)h$), and

$\phi(T) = T \times_G H$

is a right $H$-torsor. This is analogous to a change of ring (base change) in the case of rings and modules.

In these terms we have the following.

**Definition 7.45** (Klein’s Erlangen Program).

1. A geometric type is a group action $G \circ F$.
2. A geometry of type $G \circ F$ is a right $G$-torsor $T$.
3. A family of geometries of type $G \circ F$ parametrized by $S$ is a fiber bundle of right $G$-torsors $P \to S$.

In (2) the corresponding geometric space is the mixing construction $F_T = T \times_G F$. A ‘fiber bundle of right $G$-torsors’ is also called a ‘principal $G$-bundle’.

**Example 7.46.** We usually specify a linear geometry (7.34) as an $n$-dimensional vector space $V$ over the field $\mathbb{F}$. Given $V$ define

$T(V) = \{b: \mathbb{F}^n \xrightarrow{\cong} V\}$
to be the set of bases of $V$. Then $T(V)$ is a right $\text{GL}_n\mathbb{F}$-torsor by precomposition, and there is a canonical isomorphism

$$T(V) \times_G \mathbb{F}^n \longrightarrow V$$

$$(b, v) \longmapsto b(v)$$

In other words, a vector space may be specified as an instantiation of linear geometry, i.e., via a right $\text{GL}_n\mathbb{F}$-torsor. In other geometries there is typically not a direct definition of an instantiation.

Lecture 8: Quantum mechanics and projective geometry

In Definition 5.1 we defined the data of a quantum mechanical system to be a pair $(\mathcal{H}, U)$ of a complex Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation of the additive group $\mathbb{R}$ of time translations. But the derived data of states and observables is projective rather than linear. For example, the space of pure states is the projective space $\mathbb{P}\mathcal{H}$, which comes equipped with the transition probability function (5.12). And in §7.3 we gave an example—the particle on a ring—of a symmetry group of a QM system that only acts projectively, not linearly.

In this lecture we examine the projective nature of quantum mechanics and determine the precise data needed to define a quantum mechanical system. We find that it is not a linear space $\mathcal{H}$ that we need, but rather it is a complex projective space. Recall (§7.4) that a model geometry is not simply a model space $F$, which for quantum mechanics is a complex projective space, but also needed is a model group action $G \acts F$ of a symmetry group $G$. It is the symmetry group that determines the precise nature of the geometry—that is Klein’s insight—and for quantum mechanics a fundamental result of Wigner determines $G$, as we explain in §8.2. A choice of linear space $\mathcal{H}$ is additional data, and we end the lecture (§8.5) by examining the nature of the choice of a linearization $\mathcal{H}$. In fact, if we consider a family of QM systems, or a QM system with symmetry, then there is an obstruction to the existence of a linearization.

The considerations in this lecture extend to any quantum system: quantum field theory and string theory included. The obstruction to linearization is called an anomaly, and it is a crucial and accessible piece of structure in a quantum theory.\(^{19}\) We like to summarize the basic idea underlying this lecture as a slogan

$$\text{(8.1) Quantum theory is projective. Quantization is linear.}$$

with a corollary

$$\text{(8.2) The anomaly of a quantum theory expresses its projectivity.}$$

\(^{19}\) Although the example in §7.3 is the anomaly of a symmetry, we emphasize that the projective nature of quantum theory is more fundamental and is not necessarily tied to symmetry.
8.1 What is a projective space?

Three of the classical geometries are: linear geometry, affine geometry, and projective geometry. As a prelude to the question posed above, let us briefly consider: What is a linear space? What is an affine space? Well, a linear space $V$ (also called a vector space) is defined by well-known data and axioms. It is an algebraic object, and it can be defined over any field $F$. For $F = \mathbb{R}, \mathbb{C}$ one often introduces a compatible topology, which is unique in finite dimensions. Definition 1.31(1) tells what is an affine space $A$ over a linear space $V$. Affine spaces are the arena of flat geometry with a global parallelism. They can be defined over any field $F$, and for $F = \mathbb{R}, \mathbb{C}$ one can introduce topologies as in linear geometry. The latter is the arena for calculus.

There are direct definitions of both linear spaces and affine spaces. The definition of a projective space, which can be given over any field $F$, is simplest if we begin with an auxiliary linear space $W$. Then there is a free action of the units $F^\times$ on $W \setminus \{0\}$ by scalar multiplication. Projective space is the quotient

$$\mathbb{P}W = (W \setminus \{0\}) / F^\times.$$  

The set $\mathbb{P}W$ parametrizes lines in $W$. If $F = \mathbb{R}, \mathbb{C}$ and $W$ has a topology, then $\mathbb{P}W$ has the quotient topology. In fact, it can be given a compatible structure of a smooth manifold—a complex manifold if $F = \mathbb{C}$. If $\dim W = 2$, then $\mathbb{P}W$ is a projective line. Over $\mathbb{R}$ a projective line is diffeomorphic to a circle $S^1$; over $\mathbb{C}$ a projective line is diffeomorphic to a 2-sphere $S^2$. We can work over the division ring of quaternions as well, and a quaternionic projective line is diffeomorphic to a 4-sphere $S^4$.

Can we construct a projective space $\mathbb{P}$ without first choosing a linear space $W$? Here we turn to the notion of geometric type, discussed in §7.4, which we illustrate in linear and affine geometry first. Fix a nonnegative integer $n$. The model $n$-dimensional linear geometry over a field $F$ is

$$\text{GL}_nF \subset F^n,$$

as in (7.34). The model $n$-dimensional affine geometry over $F$ is

$$\text{Aff}_nF \subset A^n_F,$$

where $\text{Aff}_nF$ is the group of affine transformations of the model $n$-dimensional affine space $A^n_F$; it fits into the group extension

$$1 \rightarrow F^n \rightarrow \text{Aff}_nF \rightarrow \text{GL}_nF \rightarrow 1.$$  

(As a set $A^n_F = F^n$, but it has a different structure than the model linear space.)

Remark 8.7. We give basic definitions about group extensions in §8.6.4. Group extensions appear often in subsequent lectures, as do variations of the particular group extension (8.6).
For $n$-dimensional projective geometry over $\mathbb{F}$ a natural model geometry is

\begin{equation}
\text{PGL}_{n+1}\mathbb{F} \circ \mathbb{F}^n,
\end{equation}

where the group of projective transformations $\text{PGL}_{n+1}\mathbb{F} = \text{GL}_{n+1}\mathbb{F} / \mathbb{F}^×$ is the quotient of the general linear group by the homotheties. It acts effectively on the set $\mathbb{F}^n = (\mathbb{F}^{n+1} \setminus \{0\}) / \mathbb{F}^×$. The following is the special case of Definition 7.45 for the model geometry (8.8).

**Definition 8.9.** Fix a field $\mathbb{F}$ and a nonnegative integer $n$. The data of an $n$-dimensional projective space over $\mathbb{F}$ is a right $\text{PGL}_{n+1}\mathbb{F}$-torsor $T$. The associated projective space is

\begin{equation}
P = T \times_{\text{PGL}_{n+1}\mathbb{F}} \mathbb{F}^n.
\end{equation}

A point of $T$ induces an isomorphism $\mathbb{F}^n \xrightarrow{\cong} P$ of the model projective space with the abstract projective space, a kind of “basis” in projective geometry. Such “bases” are determined up to the symmetry group of the geometry.

Matrix algebras are also “projective” in the following sense.

**Remark 8.11.** Let $M_{n+1}\mathbb{F}$ denote the algebra of $(n + 1) \times (n + 1)$ matrices over $\mathbb{F}$. The conjugation action $\text{GL}_{n+1}\mathbb{F} \circ M_{n+1}\mathbb{F}$ factors through the quotient group $\text{PGL}_{n+1}\mathbb{F}$. So associated to a projective space $P$ as in Definition 8.9 is an algebra

\begin{equation}
E = T \times_{\text{PGL}_{n+1}\mathbb{F}} M_{n+1}\mathbb{F}.
\end{equation}

An element $t \in T$ gives rise to an identification of $E$ as a matrix algebra.

The idea that “a matrix algebra is projective” can be expressed in the following terms. Let $W$ be a vector space (over any field), and let $K$ be a line. Then there is a canonical isomorphism

\begin{equation}
\text{End}(W) \rightarrow \text{End}(W \otimes K)
T \mapsto T \otimes \text{id}_K
\end{equation}

Similarly, projective space is projective: there is a canonical isomorphism

\begin{equation}
P W \rightarrow P(W \otimes K)
L \mapsto L \otimes K
\end{equation}

Specialize to $\mathbb{F} = \mathbb{C}$. The general model projective geometry (8.8) specializes to

\begin{equation}
\text{PGL}_{n+1}\mathbb{C} \circ \mathbb{C}^n,
\end{equation}

which is the model for holomorphic projective geometry. Restrict to the projective unitary group

\begin{equation}
\text{PU}_{n+1}\mathbb{C} \circ \mathbb{C}^n,
\end{equation}

to obtain the model for Fubini-Study geometry. The smaller symmetry group in (8.16) has more invariants, in this case the Fubini-Study Riemannian metric on $\mathbb{C}^n$. (See §8.6.3 for a brief introduction to this geometry.) It turns out that neither (8.15) nor (8.16) is the correct model geometry for quantum mechanics.
8.2 The model geometry in finite dimensional quantum mechanics

In quantum mechanics the base field is $\mathbb{F} = \mathbb{C}$: complex numbers encode interference phenomena. We might at first think the appropriate symmetry group for a quantum mechanical system with an $n$-dimensional space of pure states is the complex Lie group $\text{PGL}_{n+1}\mathbb{C}$. Then the model for the space $\mathcal{S}_0$ of pure states would be $\text{PGL}_{n+1}\mathbb{C} \otimes \mathbb{CP}^n$, and the model for the complex vector space $\mathcal{O}$ would be $\text{PGL}_{n+1}\mathbb{C} \otimes M_{n+1}\mathbb{C}$. But there is more data in mechanics (Axiom System 1.1): there is an entire convex space $\mathcal{S}$ of states, the vector space $\mathcal{O}$ has a real structure and a Lie algebra structure, and there is a pairing of states and observables to probability measures on $\mathbb{R}$. This additional data depends on the adjoint map, which is the conjugate transpose map $A \mapsto A^*$ on complex matrices. If $U \in \text{GL}_{n+1}\mathbb{C}$ is to preserve this map, then for all $A \in M_{n+1}\mathbb{C}$ we must have

$$UA^*U^{-1} = (UAU^{-1})^* = (U^{-1})^*A^*U^* = (U^*)^{-1}A^*U^*,$$

from which it follows that $(U^*U)A^*(U^*U)^{-1} = A^*$ for all $A$. Hence $U^*U$ is a scalar multiple of the identity. Therefore, the image of $U$ in the projective linear group $\text{PGL}_{n+1}\mathbb{C}$ equals the image of a unitary matrix. In other words, the projective unitary subgroup

$$\text{PU}_{n+1} \subset \text{PGL}_{n+1}\mathbb{C}$$

preserves the map $A \mapsto A^*$. It also preserves the subspace of positive matrices and the trace one condition on states. Hence Fubini-Study geometry

$$\text{PU}_{n+1} \otimes \mathbb{CP}^n$$

seems to be a reasonable model geometry for quantum mechanics.

Remark 8.20. For $n = 1$ the model geometry is $\mathbb{CP}^1 \approx S^2$ with the round metric. The Lie group $\text{PU}_2 \cong \text{SO}_3$ is not the full group of isometries—that group is $\text{O}_3$. It has $\text{SO}_3$ as its identity component, and the second component $\text{O}_3 \setminus \text{SO}_3$ acts as antiholomorphic transformations. We will argue now that elements in this second component are symmetries in quantum mechanics.

Recall from (5.12) that in quantum mechanics the space of pure states comes equipped with a function

$$p: \mathbb{CP}^n \times \mathbb{CP}^n \longrightarrow [0, 1]$$

that expresses transition probabilities. The pairing of states and observables is expressed in terms of $p$; see (5.11).

Hypothesis 8.22. The symmetry group of the model pure state space $\mathbb{CP}^n$ in quantum mechanics is the group of self-maps of $\mathbb{CP}^n$ that preserves the function $p$.

The following theorem is the first step toward determining this group.
Theorem 8.23. Let

\[(8.24) \quad d: \mathbb{CP}^n \times \mathbb{CP}^n \longrightarrow \mathbb{R}^\geq 0\]

be the distance function of the Fubini-Study Riemannian metric. Then

\[(8.25) \quad \cos(d) = 2p - 1.\]

In Lecture 5 we stated (8.25) in the equivalent form

\[(8.26) \quad p = \cos^2(d/2).\]

We prove (8.25) in the next section.

Next, we apply the Myers-Steenrod theorem in Riemannian geometry [MySt, Pa]. The first part of the theorem is the following. A Riemannian manifold \((M, g)\) has an underlying metric space \((M, d)\) with distance function \(d\). If \(\varphi: M \rightarrow M\) is a metric space isometry, then \(\varphi\) is smooth and is a Riemannian isometry. The second part of the theorem asserts that the group of Riemannian isometries of \(M\) has a natural Lie group structure. Putting this together with Theorem 8.23 we are led to the following consequence of Hypothesis 8.22.

Definition 8.27. Fix a nonnegative integer \(n\), and let \(\text{PQ}_{n+1}\) be the Lie group of isometries of \(\mathbb{CP}^n\). Then the model geometry for quantum mechanics with an \(n\)-dimensional space of pure states is

\[(8.28) \quad \text{PQ}_{n+1} \triangleleft \mathbb{CP}^n.\]

Therefore, the data of a quantum mechanical system is a right \(\text{PQ}_{n+1}\)-torsor and a Hamiltonian.

A right \(\text{PQ}_{n+1}\)-torsor \(T\) has an associated projective space

\[(8.29) \quad \mathbb{P} = T \times_{\text{PQ}_{n+1}} \mathbb{CP}^n,\]

and the claim is that the rest of the data that defines a quantum mechanical system is also associated to \(T\). We leave that verification to a homework problem.

Remark 8.30. We emphasize the logic at work and the power of the Myers-Steenrod theorem. The model of pure states in quantum mechanics is the set \(\mathbb{CP}^n\) equipped with the function \(p\) in \((8.21)\), and so what comes to us from physics is the group of permutations of \(\mathbb{CP}^n\) that preserve \(p\). Note that such permutations are simply maps of an uncountable set to itself; there is no \textit{a priori} topology, so no continuity. We apply Theorem 8.23 and the Myers-Steenrod theorem to prove that this group has a natural \textit{Lie group} structure, and that it is precisely the group of isometries of \(\mathbb{CP}^n\) with the Riemannian metric defined by Fubini-Study.
It remains to determine the group $\text{PQ}_{n+1}$. Note that the projective unitary group $\text{PU}_{n+1}$ is a subgroup, since it acts by isometries on $\mathbb{C}P^n$. It turns out that $\text{PQ}_{n+1}$ has two components, if $n > 0$, and $\text{PU}_{n+1}$ is the identity component. As an example, recall from Remark 8.20 that $\text{PQ}_2 \cong \text{O}_3$; the identity component is $\text{SO}_3$, which fits into the group extension

\begin{equation}
1 \to \mathbb{T} \to \text{U}_2 \to \text{SO}_3 \to 1.
\end{equation}

The Kähler manifold $\mathbb{C}P^n$ is a symmetric space, in fact, it is a symmetric space of a special type: a Hermitian symmetric space. Groups of isometries of symmetric spaces have been determined by general theory [Loo]. Here we outline an argument for $\mathbb{C}P^n$.

The key result is called Wigner’s theorem, though it seems [Bo] that the statement of the theorem first appeared in a joint paper [vNW] of von Neumann and Wigner. To state it we need to introduce antilinear or conjugate linear transformations (see §8.6.2 for definitions) of a Hilbert space that preserve the underlying real inner product.

**Definition 8.32.**

1. Let $W$ be a complex Hilbert space. A real linear map $A: W \to W$ is **antiunitary** if it is conjugate linear and

\begin{equation}
\langle A\psi_1, A\psi_2 \rangle = \langle \overline{\psi_1}, \overline{\psi_2} \rangle, \quad \psi_1, \psi_2 \in W.
\end{equation}

2. For $W = \mathbb{C}^{n+1}$, let $Q_{n+1}$ be the 2-component Lie group of unitary and antiunitary transformations of $\mathbb{C}^{n+1}$.

Note that the composition of two antiunitary transformations is a unitary transformation. The structure of $\text{PQ}_{n+1}$ is encapsulated in the following. (See §8.6.4 for the definition of a group extension.) Let $\mathbb{T} \subset \mathbb{C}$ denote the group complex numbers of modulus one.

**Theorem 8.34.** There is a group extension

\begin{equation}
1 \to \mathbb{T} \to Q_{n+1} \to \text{PQ}_{n+1} \to 1.
\end{equation}

It is straightforward that elements of $Q_{n+1}$ act on $\mathbb{C}P^n$ by isometries, and that homotheties in $\mathbb{T}$ act as the identity. The import of Theorem 8.34 is the surjectivity of the map $Q_{n+1} \to \text{PQ}_{n+1}$: every isometry of $\mathbb{C}P^n$ lifts to a unitary or antiunitary transformation of $\mathbb{C}^{n+1}$. Wigner’s theorem holds for infinite dimensional Hilbert spaces as well.

**Remark 8.36.** For $n = 1$ the Lie group $Q_2$ sits inside a Clifford algebra: $Q_2 \cong \text{Pin}_3$. (Spin and pin groups and their position inside Clifford algebras are discussed in [ABS]. We give an introduction in the next lecture.)

**Remark 8.37.** The preceding discussion leads to the model geometry for quantum mechanics with a choice of state space:

\begin{equation}
Q_{n+1} \subset \mathbb{C}^{n+1}.
\end{equation}

Elements in the identity component $U_{n+1} \subset Q_{n+1}$ act unitarily; elements in the off-component act antiunitarily.
Remark 8.39. Symmetries in quantum theory which reverse the direction of time are typically antiunitary; see Lemma 11.3.

8.3 Sketch proofs

Here we only give some ideas toward proofs of Theorem 8.23 and Theorem 8.34. Complete proofs, including the infinite dimensional case, can be found in [F3].

Sketch proof of Theorem 8.23. The first step is to reduce to the case of $\mathbb{C}P^1 \approx S^2$. Observe that (8.25) holds on the diagonal of $\mathbb{C}P^n \times \mathbb{C}P^n$, so fix $(L_1, L_2) \in \mathbb{C}P^n \times \mathbb{C}P^n$ with $L_1 \neq L_2$ and let $V = L_1 + L_2 \subset \mathbb{C}^{n+1}$ be the 2-dimensional subspace they span. Then the unitary automorphism $\varphi$ of $\mathbb{C}P^n$ induced from $\text{id}_V \oplus -\text{id}_{V^\perp}$ on $\mathbb{C}^{n+1} = V \oplus V^\perp$ has $PV \subset \text{Fix}(\varphi)$ as a component of its fixed point set. It follows that $PV \subset \mathbb{C}P^n$ is totally geodesic: a geodesic from $L_1$ to $L_2$ in $PV$ is also a geodesic in $\mathbb{C}P^n$. This means that $d(L_1, L_2)$ can be computed in the complex projective line $PV$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{stereographic_projection.png}
\caption{Stereographic projection}
\end{figure}

At this point the proof is elementary geometry in the 2-sphere. Let $e_1 \in L_1$ be a unit norm vector, and choose $e_2$ so that $\{e_1, e_2\}$ is an orthonormal basis of $V$. Then $\lambda e_1 + e_2 \in L_2$ for a unique $\lambda \in \mathbb{C}$. After normalizing to obtain a unit vector, we compute

\begin{equation}
(8.40) \quad p(L_1, L_2) = \frac{|\lambda|^2}{|\lambda|^2 + 1}.
\end{equation}

In case $\lambda = 0$ the lines $L_1, L_2$ are orthogonal, and so $d = \pi$ and $p = 0$: (8.25) holds. It suffices to assume $\lambda \neq 0$, and then identify

\begin{equation}
(8.41) \quad PV \setminus \mathbb{C}\langle e_2 \rangle \leftrightarrow \mathbb{C} \cong \{0\} \times \mathbb{C} \leftrightarrow S^2 \setminus \{(1, 0)\} \subset \mathbb{R} \times \mathbb{C},
\end{equation}

where the first identification is $\mathbb{C}\langle e_1 + \mu e_2 \rangle \leftrightarrow \mu, \mu \in \mathbb{C}$, and the second is stereographic projection as depicted in Figure 23. Compute

\begin{equation}
(8.42) \quad L_1 \leftrightarrow (-1, 0), \quad L_2 \leftrightarrow \left(\frac{|\lambda|^2 - 1}{|\lambda|^2 + 1}, \frac{2|\lambda|^2 + 1}{|\lambda|^2 + 1} \frac{1}{\lambda}\right).
\end{equation}
Finally, the cosine of the distance between two points on the unit sphere in a real inner product space is the inner product of the unit vectors from the origin to the points. \[ \square \]

Two proofs of Theorem 8.34 are given in [F3], one leaning on Riemannian geometry and the other on complex geometry. For the latter, the first step is the following.

**Lemma 8.43.** An isometry of $\mathbb{C}P^n$ is either holomorphic or antiholomorphic.

The argument is as follows. Let $I: T\mathbb{C}P^n \to T\mathbb{C}P^n$ be the complex structure. Then $I$ is parallel with respect to the Levi-Civita covariant derivative. So too is its pullback by an isometry. We use the fact that a parallel almost complex structure commutes with the Riemann curvature tensor to prove that the only parallel complex structures are $\pm I$.

The second step is standard.

**Lemma 8.44.** Let $W$ be a finite dimensional complex vector space. Then any holomorphic map $\phi: \mathbb{P}W \to \mathbb{P}W$ lifts to a linear map $W \to W$.

**Proof.** To prove this let $\mathcal{L} \to \mathbb{P}W$ be the tautological holomorphic line bundle whose fiber at $L \subset W$ is the line $L$. Then $\phi^* \mathcal{L} \to \mathbb{P}W$ is isomorphic to $\mathcal{L} \to W$, since a holomorphic line bundle over $\mathbb{P}W$ is determined by its topology (its Chern class) and the group of holomorphic automorphisms of $\mathbb{P}W$ is connected. Choose an isomorphism. The pullback on global sections of the dual hyperplane bundle

\[
\phi^*: H^0(\mathbb{P}W; \mathcal{L}^*) \longrightarrow H^0(\mathbb{P}W; \phi^* \mathcal{L}^*) \cong H^0(\mathbb{P}W; \mathcal{L}^*)
\]

is the automorphism of $W^*$ dual to a linear automorphism of $W$ that lifts $\phi$. \[ \square \]

### 8.4 The model geometry in infinite dimensions

The main issue here is to use appropriate topologies on spaces of operators and on projective spaces. I hope to return to that issue in a future version of these notes. For now I refer to [F3], where proofs are given for infinite dimensional versions of the theorems in the previous section, and to [AS, Appendix 1], [FM1, Appendix D], where operator topologies are discussed.

As for a model geometry, fix a separable complex Hilbert space, say $\ell^2$ the $L^2$ sequence space. Let $Q$ be the group of unitary and antiunitary transformations of $\ell^2$, topologized with the compact-open topology as in the references. There is a projective quotient group $PQ$ which fits into the group extension

\[
1 \longrightarrow \mathbb{T} \longrightarrow Q \overset{\pi}{\longrightarrow} PQ \longrightarrow 1
\]

as in (8.35), and the model geometry for an infinite dimensional quantum system is

\[
PQ \subset \mathbb{P}\ell^2.
\]
8.5 Linearization of projective space

The considerations here apply to any of the three projective geometries we have discussed, both in finite and infinite dimensions, but the situation for quantum projective geometry has an extra twist. We begin with a generic definition, which ultimately is too vague (why?).

**Definition 8.48.** Let \( P \) be a projective space.

1. A *linearization* of \( P \) is a pair \((W, \theta)\) consisting of a vector space \( W \) and an isomorphism

\[
\theta: P \xrightarrow{\cong} PW.
\]

2. An *isomorphism* \((W', \theta') \to (W, \theta)\) of linearizations is a linear isomorphism \( W' \to W \) whose projectivization fits into the commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\theta'} & PW' \\
\downarrow{\theta} & & \downarrow{\theta} \\
PW & \xrightarrow{\pi} & \pi(PW)
\end{array}
\]

Linearizations and their isomorphisms comprise a groupoid \( \mathcal{L}(P) \).

A better version of Definition 8.48 uses the symmetry/structure groups to pin down the precise projective geometry in question. Recall that these groups fit into group extensions, written here for the infinite dimensional case:

\[
1 \to T \to GL \xrightarrow{\pi} PGL \to 1
\]

(8.51)

\[
1 \to T \to U \xrightarrow{\pi} PU \to 1
\]

\[
1 \to T \to Q \xrightarrow{\pi} PQ \to 1
\]

Crucially, in the first two the kernel \( T \) is *central*; in the third it is not: scalars anticommute with antiunitary transformations. For the following definition, we choose one of the three geometries and write the corresponding group extension (8.51) as

\[
1 \to T \to G \xrightarrow{\pi} PG \to 1
\]

(8.52)

Recall the mixing construction, and especially (7.44).

**Definition 8.53.** Let \( T \) be a right PG-torsor.

1. A *linearization* of \( T \) is a pair \((\widetilde{T}, \theta)\) consisting of a right \( G \)-torsor \( \widetilde{T} \) and an isomorphism

\[
\theta: T \xrightarrow{\cong} \pi(\widetilde{T})
\]

of PG-torsors, where the associated torsor \( \pi(\widetilde{T}) \) is defined in (7.44) below.
(2) An isomorphism \((\tilde{T}', \theta') \rightarrow (\tilde{T}, \theta)\) of linearizations is an isomorphism \(\tilde{T}' \rightarrow \tilde{T}\) of G-torsors whose induced isomorphism of PG-torsors fits into the commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\theta'} & \pi(\tilde{T}') \\
\downarrow{\theta} & & \downarrow{\pi(\tilde{T})} \\
& & \\
\end{array}
\]

Linearizations and their isomorphisms comprise a groupoid \(\mathcal{L}(\mathbb{P})\), where here \(\mathbb{P} = T \times_{PG} \mathbb{P} \ell^2\) is the projective space associated to \(T\) (with its induced “quantum projective geometry”).

For the two geometries in which the kernel \(T\) is central, the groupoid \(\mathcal{L}(\mathbb{P})\) can be given the structure of a gerbe. (See Definition 8.58 in §8.6.1.) To see this, observe that automorphisms of an object \((T, \theta)\) are given by the action of some \(\lambda \in \mathbb{T}\) on \(T\): since \(\mathbb{T} \subset G\) is central, the action of \(\lambda\) commutes with the G-action, so it is an automorphism of the G-torsor. It follows that every morphism space is naturally a \(T\)-torsor. Observe too that given a \(T\)-torsor \(S\) and an object \((\tilde{T}, \theta)\) of \(\mathcal{L}(\mathbb{P})\), then we can construct a new object by tensoring \(\tilde{T}\) with \(S\).

In the third geometry—quantum projective geometry, the case of most interest in these lectures—the groupoid \(\mathcal{L}(\mathbb{P})\) is a twisted version of a gerbe. We do not spell out that structure now, but encourage the reader to do so.

A single projective space \(\mathbb{P}\) always admits linearizations: simply choose an object of \(\mathcal{L}(\mathbb{P})\). However, there are two closely related situations in which linearizations may not exist: parametrized families of projective spaces and projective spaces equipped with the action of a Lie group of symmetries. A motivating example is the particle on a ring with its symmetry group, as explained in §7.3.

Remark 8.56. In algebraic geometry one can express linearization in the terms of the proof of Lemma 8.44. Namely, a linearization of a projective space \(\mathbb{P}\) is a choice of tautological holomorphic (or algebraic) line bundle \(\mathcal{L} \rightarrow \mathbb{P}\). For a single projective space it exists, but in a family there is an obstruction, as there is to lifting a group action on \(\mathbb{P}\) to \(\mathcal{L} \rightarrow \mathbb{P}\).

Remark 8.57. In quantum theory, the twisted gerbe \(\mathcal{L}(\mathbb{P})\) of linearizations is sometimes called the anomaly. It is an accessible invariant of a theory, and it has strong implications in quantum field theory and string theory. When one wants to quantize, the anomaly is an obstruction—see the guiding slogans (8.1) and (8.2).

8.6 Mathematical background

8.6.1 Gerbes. For an abelian group \(A\), the category \(\text{Tor}_A\) of right \(A\)-torsors is a Picard groupoid: there is a multiplication operation \(\text{Tor}_A \times \text{Tor}_A \rightarrow \text{Tor}_A\) which tensors \(A\)-torsors. The following is probably nonstandard, but sufficient.
**Definition 8.58.** Let $A$ be an abelian group. An $A$-gerbe is a groupoid $\mathcal{C}$ and, for each pair of objects $x, y \in \mathcal{C}$, the structure of a right $A$-torsor $T_{x,y}$ on the morphism set $\mathcal{C}(x, y)$. Furthermore, for $x, y, z \in \mathcal{C}$ there are isomorphisms

$$T_{x,y} \otimes T_{y,z} \xrightarrow{\cong} T_{x,z}$$

which fit into a commutative diagram with composition:

$$\eta(x, y) \times \eta(y, z) \xrightarrow{\circ} \eta(x, z)$$

A trivialization of $\mathcal{C}$ is a choice of object.

The isomorphisms (8.59) are required to be coherent; I leave you to construct the appropriate commutative diagram for four objects in $\mathcal{C}$.

**8.6.2 Conjugate linear vector spaces and transformations.**

**Definition 8.61.** Let $W$ be a complex vector space.

1. The complex conjugate vector space $\overline{W}$ is the same underlying real vector space with the complex conjugate scalar multiplication:

$$\bar{\lambda} \cdot \bar{w} = \overline{\lambda w}, \quad w \in W, \quad \lambda \in \mathbb{C}.$$ (8.62)

Here $\bar{w} \in \overline{W}$ corresponds to $w \in W$ under the identification of underlying real vector spaces.

2. The complex conjugate of a linear map $T : W' \to W$ between complex vector spaces is the linear map

$$\overline{T} : \overline{W'} \to \overline{W}$$

$$(\bar{w'}) \mapsto \overline{T\bar{w'}}$$

(8.63)

3. An antilinear transformation of $W$ is a linear map $A : W \to \overline{W}$.

Alternatively, an antilinear transformation of $W$ is a real linear transformation of the underlying real vector space $W_\mathbb{R}$ such that $A(\lambda w) = \bar{\lambda} A(w)$ for all $w \in W, \lambda \in \mathbb{C}$.

**8.6.3 The Fubini-Study metric on complex projective space.** Let $W$ be a finite dimensional complex inner product space and let $\mathbb{P}W$ be the associated complex projective space, which is a complex manifold. Then at $L \in \mathbb{P}W$, the tangent space to $\mathbb{P}W$ is canonically the complex vector space

$$T_L \mathbb{P}W = \text{Hom}(L, L^\perp).$$ (8.64)
(The graph of a linear map \( L \rightarrow L^\perp \) is a point of \( \mathbb{P}W \). So a smooth motion in \( \mathbb{P}W \) with initial position \( L \) can be expressed, for small times, as a motion in \( \text{Hom}(L, L^\perp) \), hence the initial velocity lies in that vector space.) The Fubini-Study metric is a hermitian metric on each tangent space \( T_L \mathbb{P}W \).

In terms of the identification (8.64) it is

\[
\langle f_1, f_2 \rangle \mapsto -\text{trace}(f_1^* \circ f_2)
\]

The composition \( f_1^* \circ f_2 \) is an endomorphism of the line \( L \), so is scalar multiplication by a complex number; the trace extracts the complex number. The formula makes evident that the projectivization of unitary and antiunitary transformations of \( W \) are isometries of the Fubini-Study metric.

### 8.6.4 Group extensions

We do not comment further on topology, but say once and for all that for topological groups all group homomorphisms are assumed continuous, and for Lie groups all group homomorphisms are assumed smooth.

**Definition 8.66.** A *group extension* is a sequence of group homomorphisms

\[
1 \rightarrow G' \xrightarrow{i} G \xrightarrow{\pi} G'' \rightarrow 1
\]

that is exact in the sense that the kernel of any homomorphism equals the image of the preceding homomorphism. We call \( G' \) the *kernel* and \( G'' \) the *quotient*.

Exactness at \( G' \) implies that \( i \) is injective. The inclusion \( i \) identifies \( G' \) with its image, which is a subgroup of \( G \). At \( G \) the exactness implies that \( \pi \) factors through an injective map of the quotient group \( G/G' \) into \( G'' \), and exactness at \( G'' \) implies that this is an isomorphism. We use it to identify \( G'' \) as this quotient.

There is a category of group extensions with fixed kernel and quotient.

**Definition 8.68.** Let \( G', G'' \) be groups and \( G'^{\tau_1}, G'^{\tau_2} \) group extensions with kernel \( G' \) and quotient \( G'' \). A *morphism of group extensions* \( G'^{\tau_1}, G'^{\tau_2} \) is a group homomorphism \( \varphi: G'^{\tau_1} \rightarrow G'^{\tau_2} \) which fits into the commutative diagram

\[
\begin{CD}
1 @>>> G' @>>> G @>>> G'' @>>> 1
\end{CD}
\]

As usual, \( \varphi \) is an *isomorphism* if there exists a homomorphism \( \psi: G'^{\tau_2} \rightarrow G'^{\tau_1} \) of group extensions so that the compositions \( \psi \circ \varphi \) and \( \varphi \circ \psi \) are identity maps, which is simply equivalent to the condition that \( \varphi \) be an isomorphism of groups.

There is a notion of a trivialization of a group extension.

**Definition 8.70.** A *splitting* of the group extension (8.67) is a homomorphism \( j: G'' \rightarrow G \) such that \( \pi \circ j = \text{id}_{G''} \).

Not every group extension is split: for example, the cyclic group of order 4 is a nonsplit extension of the cyclic group of order 2 by the cyclic group of order 2:

\[
1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1
\]
Lecture 9: Introduction to superalgebra

So far our discussion has been for mechanical systems which are *bosonic*. But in the real world there are two species of particles: *bosons* and *fermions*. This statement depends on features of space, and so far our axioms only have incorporated time. Furthermore, in two spatial dimensions more exotic possibilities abound: *anyons* [Wil]. Hence in our current context we cannot give a convincing argument for a boson/fermion dichotomy. Nonetheless, we begin the lecture with a heuristic discussion.

The main part of the lecture is an introduction to superalgebra, the algebraic formalism which neatly encodes the boson/fermion dichotomy, and which appears in many other mathematical contexts as well. The word ‘super’ is synonymous with \( \mathbb{Z}/2\mathbb{Z} \)-graded. The key point is the Koszul sign rule (9.11): there is a minus sign when commuting two odd elements. For example, this sign dictates the proper definition of commutativity in super associative algebras, of skew-symmetry and the Jacobi identity in super Lie algebras, and of the hermitian structure and adjoints in the context of a super Hilbert space.

In the last part of the lecture we give an introduction to Clifford algebras and the spin group. The venerable paper [ABS] has a more comprehensive treatment, and there are many other accounts.

9.1 Quantum statistics

Suppose \((\mathcal{H}, U)\) is a complex Hilbert space and unitary strongly continuous 1-parameter group, data that defines a QM system. We use language as if the system models a single particle: \(\mathcal{H}\) is the Hilbert space of *1-particle states*. Consider the conjunction of two copies of this system with no interaction. The data for this composition follows from (5.4):

\[
(\mathcal{H} \otimes \mathcal{H}, U \otimes U).
\]

Then if \(\psi_1, \psi_2 \in \mathcal{H}\) are unit norm vectors, \(\psi_1 \otimes \psi_2\) represents a decomposable pure state in the combined system. To encode that (9.1) is a QM system of two *identical* particles, we ask that \(\psi_1 \otimes \psi_2\) and \(\psi_2 \otimes \psi_1\) represent the same state, that is, they are scalar multiples of each other. In other words, the exchange map

\[
\mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}
\]

\[
\psi_1 \otimes \psi_2 \longmapsto \psi_2 \otimes \psi_1
\]

should reduce to scalar multiplication:

\[
\psi_2 \otimes \psi_1 = \lambda \psi_1 \otimes \psi_2.
\]

Furthermore—this is the unmotivated step—we ask that applying (9.3) twice we get back to \(\psi_1 \otimes \psi_2\) on the nose. Therefore, we posit \(\lambda = \pm 1\):

\[
\psi_2 \otimes \psi_1 = \begin{cases} 
\psi_1 \otimes \psi_2, & \text{(boson)}; \\
-\psi_1 \otimes \psi_2, & \text{(fermion)},
\end{cases}
\]
and so factor the putative 2-particle state space $\mathcal{H} \otimes \mathcal{H}$ to the quotient

\begin{equation}
\mathcal{H} \otimes \mathcal{H} \rightarrow \begin{cases} 
\text{Sym}^2 \mathcal{H}, & \text{(boson)}; \\
\bigwedge^2 \mathcal{H}, & \text{(fermion)}. 
\end{cases}
\end{equation}

**Remark 9.6.**

1. Exterior algebra encodes the *Pauli exclusion principle*.
2. A typical QM system has both bosonic and fermionic states, so we want a formalism that incorporates both simultaneously. That is provided by superalgebra. There are also states that are mixtures; they are neither bosonic nor fermionic.
3. We also use superalgebra to define bosonic and fermionic observables, though ‘observable’ is inappropriate for fermionic observables: only bosonic observables can be measured.
4. The presence of both bosonic and fermionic states and observables does not imply that the system is *supersymmetric*. Supersymmetries exchange bosonic states and fermionic states, and those may or may not be present.

We expound on these points in the next lecture.

### 9.2 Superalgebra

The word ‘super’ is synonymous with ‘$\mathbb{Z}/2\mathbb{Z}$-graded’. The following discussion applies over any field $F$ not of characteristic 2, but for our application we only use $F = \mathbb{R}$ and $F = \mathbb{C}$.

**Definition 9.7.**

1. A *super vector space* $V = V^0 \oplus V^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. A homogeneous element $v \in V^i$ has *parity* $i \in \mathbb{Z}/2\mathbb{Z}$. A *morphism* of super vector spaces is a parity-preserving linear map. The *grading automorphism* of $V$ is

\begin{equation}
\epsilon_V = \text{id}_{V^0} \oplus -\text{id}_{V^1}.
\end{equation}

If $V$ is finite dimensional, then its *dimension* is $\dim V = n^0|n^1$, where $n^i = \dim V^i, i \in \mathbb{Z}/2\mathbb{Z}$.
2. Let $V, W$ be super vector spaces. The *tensor product* $V \otimes W$ is the usual tensor product with $\mathbb{Z}/2\mathbb{Z}$-grading

\begin{equation}
(V \otimes W)^k = \bigoplus_{i+j=k} (V^i \otimes W^j), \quad k \in \mathbb{Z}/2\mathbb{Z}.
\end{equation}

The associativity isomorphism for the tensor product of super vector spaces $U, V, W$ is

\begin{equation}
(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)
\end{equation}

\begin{equation}
(u \otimes v) \otimes w \rightarrow u \otimes (v \otimes w)
\end{equation}

The commutativity isomorphism for the tensor product of super vector spaces $V, W$ is

\begin{equation}
\sigma_{V,W} : V \otimes W \rightarrow W \otimes V
\end{equation}

\begin{equation}
v \otimes w \rightarrow (-1)^{|v||w|} w \otimes v
\end{equation}

The symmetry (9.11) is the *Koszul sign rule*; it satisfies $\sigma_{W,V} \circ \sigma_{V,W} = \text{id}_{V \otimes W}$.
(3) A \textit{superalgebra} is a super vector space $A$ equipped with a multiplication

\begin{equation}
A \otimes A \rightarrow A
\end{equation}

which is associative and unital. The unit $1 \in A$ is necessarily even. The superalgebra $A$ is \textit{commutative} if

\begin{equation}
ab = (-1)^{|a||b|}ba
\end{equation}

for all homogeneous elements $a, b$ of $A$.

The standard super vector space of dimension $n^0|n^1$ is denoted $F_{n^0|n^1}$. The linear maps (9.10), (9.11), and (9.12) are assumed to be morphisms of super vector spaces, that is, they are parity-preserving. Note that an ordinary vector space is an even super vector space; similarly for ordinary algebras. I leave the reader to define modules over a superalgebra as well as super Lie algebras. If $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ is a super Lie algebra, then the Lie bracket restricts to a \textit{symmetric} bilinear pairing

\begin{equation}
\mathfrak{g}^1 \times \mathfrak{g}^1 \rightarrow \mathfrak{g}^0.
\end{equation}

In fact, a super Lie algebra gives rise to

- a Lie algebra $\mathfrak{g}^0$,
- a $\mathfrak{g}^0$-module $\mathfrak{g}^1$,
- a map $\text{Sym}^2 \mathfrak{g}^1 \rightarrow \mathfrak{g}^0$ of $\mathfrak{g}^0$-modules,

and these data satisfy

\begin{equation}
[x, [x, x]] = 0, \quad x \in \mathfrak{g}^1.
\end{equation}

Conversely, one can prove that this data subject to (9.15) determines a super Lie algebra. We refer to [DM] for much more on superalgebra, as well as for an exposition of the theory of supermanifolds.

\textit{Remark 9.16.} \textit{Definition 9.7(1)} defines a linear category of super vector spaces. \textit{Definition 9.7(2)} is the main data required to endow it with a symmetric monoidal structure. Its most salient feature is the Koszul sign rule (9.11). Then we can rephrase \textit{Definition 9.7(3)} as follows: a superalgebra is an algebra object in the category of super vector spaces. See [DM] for this point of view.

\textit{Example 9.17.}

(1) Let $U$ be a vector space. Then the exterior algebra $A = \bigwedge U$ has a $\mathbb{Z}$-grading, so too a quotient $\mathbb{Z}/2\mathbb{Z}$-grading. The resulting superalgebra is commutative.

(2) Let $V = V^0 \oplus V^1$ be a super vector space. Then the algebra $\text{End} V$ of all endomorphisms of the underlying ungraded vector space is a superalgebra:

\begin{equation}
\text{End} V = \text{End}^0 V \oplus \text{End}^1 V
\end{equation}

where the even elements preserve parity and the odd elements reverse parity.
(3) If \( x_1, \ldots, x_N \) are even indeterminates, the polynomial algebra \( \mathbb{R}[x_1, \ldots, x_N] \) is commutative as an ungraded algebra, but is not commutative as a superalgebra (with \( \mathbb{Z}/2\mathbb{Z} \)-grading the parity of the total degree).

(4) If \( \eta_1, \ldots, \eta_N \) are odd indeterminates, then \( \mathbb{R}[\eta_1, \ldots, \eta_N] \) is the free commutative real superalgebra they generate. It is canonically isomorphic to the exterior algebra on the free real vector space generated by \( \{\eta_1, \ldots, \eta_N\} \).

Remark 9.19. For even indeterminates \( x_1, \ldots, x_N \), the free commutative algebra \( \mathbb{R}[x_1, \ldots, x_N] \) is the algebra of polynomial functions on an affine space. In the odd case, we can also view \( \mathbb{R}[\eta_1, \ldots, \eta_N] \) as functions on a space, but now an odd affine space; see [F5, Lecture 1] for a heuristic introduction to odd affine spaces and more generally to supermanifolds. The text [DM] is a much more thorough and rigorous treatment.

Let \( \mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \) be a super vector space. Then

\[
\text{Sym}^2 \mathcal{H} = \mathcal{H} \otimes \mathcal{H} / S,
\]

where \( S \subset \mathcal{H} \otimes \mathcal{H} \) is the subspace generated by

\[
\psi_1 \otimes \psi_2 - (-1)^{|\psi_1||\psi_2|} \psi_2 \otimes \psi_1
\]

over all homogeneous \( \psi_1, \psi_2 \in \mathcal{H} \). The reader is encouraged to prove the following.

Lemma 9.22. \( \text{Sym}^2 \mathcal{H} \cong \left[ \text{Sym}^2 \mathcal{H}^0 \oplus \wedge^2 \mathcal{H} \right] \oplus \left[ \mathcal{H}^0 \otimes \mathcal{H}^1 \right] \).

If \( \mathcal{H} \) is the 1-particle state space of a QM system, then \( \text{Sym}^2 \mathcal{H} \) is the 2-particle state space: bosonic states are products of two bosonic states or of two fermionic states, and fermionic states are the product of one bosonic state and one fermionic state. This shows how the symmetric algebra of a super vector space captures the statistics of identical particles.

Remark 9.23. Here \( \mathcal{H} \) is an algebraic super vector with no topology. If it has a topology, say a Hilbert space structure, then there is are completions of the algebraic symmetric tensor powers which are also Hilbert spaces. The algebraic symmetric algebra \( \text{Sym}^* \mathcal{H} \) is called the \textit{Fock space}.

9.3 The Koszul sign rule and antilinearity

Specialize to the ground field \( \mathbb{C} \).

Definition 9.24.

(1) Let \( A \) be an algebra over \( \mathbb{C} \). A \( \ast \)-operation on \( A \) is a real linear map

\[
A \longrightarrow A
\]

\[
a \longmapsto a^\ast
\]

(9.25)
such that for all \( \lambda \in \mathbb{C} \), \( a, a_1, a_2 \in A \) we have

\begin{align}
(9.26) & \quad a^{**} = a \\
(9.27) & \quad (\lambda a)^* = \bar{\lambda} a^* \\
(9.28) & \quad (a_1a_2)^* = a_2^*a_1^*
\end{align}

In brief, \( * \) is an antilinear antilinear involution of \( A \). The pair \((A,*)\) is a \( * \)-algebra.

(2) Let \( A \) be a superalgebra over \( \mathbb{C} \). Then a \( * \)-structure is as in (1) with (9.28) replaced by

\[ (9.29) \quad (a_1a_2)^* = (-1)^{|a_1||a_2|}a_2^*a_1^* \]

where \( a_1, a_2 \) are homogeneous.

The transition from (1) to (2) is dictated by the Koszul sign rule. Notice that if \( A \) is a commutative \( * \)-superalgebra, then

\[ (9.30) \quad (a_1a_2)^* = a_1^*a_2^* \]

for all \( a_1, a_2 \in A \).

We turn now to hermitian inner products.

**Definition 9.31.** Let \( \mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \) be a complex super vector space. A hermitian inner product is a bilinear pairing

\[ (9.32) \quad \langle -,- \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C} \]

such that for all homogeneous \( \psi_1, \psi_2 \in \mathcal{H} \) we have

\[ (9.33) \quad \langle \psi_1, \psi_2 \rangle = (-1)^{|\psi_1||\psi_2|}\langle \overline{\psi_2}, \psi_1 \rangle, \]

and for all nonzero homogeneous \( \psi \in \mathcal{H} \) we have

\[ (9.34) \quad \begin{cases} 
\langle \psi, \psi \rangle > 0, & \psi \in \mathcal{H}^0; \\
-i\langle \psi, \psi \rangle > 0, & \psi \in \mathcal{H}^1.
\end{cases} \]

Note that (9.33) implies that for \( \psi \) odd the inner product \( \langle \psi, \psi \rangle \) is purely imaginary. Observe too that \( \mathcal{H}^0 \perp \mathcal{H}^1 \) since (9.32) is even. The restriction of \( \langle -,- \rangle \) to \( \mathcal{H}^0 \) is an ordinary hermitian inner product, as is the restriction of \(-i\langle -,- \rangle\) to \( \mathcal{H}^1 \). Hence there are notions of completeness and so of a super Hilbert space.

The adjoint of a (bounded) homogeneous endomorphism \( T : \mathcal{H} \rightarrow \mathcal{H} \) is characterized by

\[ (9.35) \quad \langle T^*\psi_1, \psi_2 \rangle = (-1)^{|T||\psi_1|}\langle \psi_1, T\psi_2 \rangle \]

for all homogeneous \( \psi_1 \) and all \( \psi_2 \in \mathcal{H} \). The map \( T \mapsto T^* \) is a \( * \)-structure, so this makes \( \text{End} \mathcal{H} \) into a \( * \)-superalgebra.

**Remark 9.36.** If \( T \) is odd, then \( T \) has no nonzero homogeneous eigenvectors.

**Remark 9.37.** There is a way around the awkward sign in (9.35) for the adjoint. That leads to a more standard convention; see [DM, §4.4], [De2, §4].
9.4 Clifford algebras

In the remainder of the lecture we introduce a particular noncommutative superalgebra: the Clifford algebra. The orthogonal group $O_n$ is a subset of an algebra: the algebra $M_n\mathbb{R}$ of $n \times n$ matrices. The Clifford algebra plays a similar role for a double cover group of the orthogonal group, called the pin group. Its identity component, the double cover of the special orthogonal group, is the spin group.

We begin with a heuristic motivation for Clifford algebras. Orthogonal transformations are products of reflections. For a unit norm vector $\xi \in \mathbb{R}^n$ define

$$\rho_\xi(\eta) = \eta - 2\langle \eta, \xi \rangle \xi,$$

where $\langle - , -\rangle$ is the standard inner product.

**Theorem 9.39** (Cartan-Dieudonné). Any $g \in O_n$ is the composition of $\leq n$ reflections.

**Proof.** The statement is trivial for $n = 1$. Proceed by induction: if $g \in O_n$ fixes a unit norm vector $\zeta$ then it fixes the orthogonal complement $(\mathbb{R} \cdot \zeta)^\perp$, and we are reduced to the theorem for $O_{n-1}$. If there are no fixed unit norm vectors, then for any unit norm vector $\zeta$ set $\xi = \frac{g(\zeta) - \zeta}{|g(\zeta) - \zeta|}$. The composition $\rho_\xi \circ g$ fixes $\zeta$ and again we reduce to the $(n-1)$-dimensional orthogonal complement. $\square$

Now generate an algebra from the unit norm vectors, with relations inspired by those of reflections. Note immediately that the vectors $\pm \xi$ both correspond to the same reflection $\rho_\xi = \rho_{-\xi}$. Therefore, we expect from the beginning that the Clifford algebra “double counts” orthogonal transformations. Now since the square of a reflection is the identity, impose the relation

$$\xi^2 = \pm 1, \quad ||\xi|| = 1.$$  \hspace{1cm} (9.40)

The sign ambiguity is that described above, and we choose a sign independent of $\xi$. It follows that

$$\xi^2 = \pm ||\xi||^2$$  \hspace{1cm} (9.41)

for any $\xi \in \mathbb{R}^n$. Now if $\langle \xi_1, \xi_2 \rangle = 0$, then $(\xi_1 + \xi_2)/\sqrt{2}$ has unit norm and from

$$\pm 1 = \left(\frac{\xi_1 + \xi_2}{\sqrt{2}}\right)^2 = \frac{\xi_1^2 + \xi_2^2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2} = \frac{\pm 2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2}$$  \hspace{1cm} (9.42)

we deduce

$$\xi_1\xi_2 + \xi_2\xi_1 = 0, \quad \langle \xi_1, \xi_2 \rangle = 0.$$  \hspace{1cm} (9.43)

Equations (9.40) and (9.43) are the defining relations for the Clifford algebra. Check that the reflection (9.38) is given by

$$\rho_\xi(\eta) = -\xi \eta \xi^{-1}$$  \hspace{1cm} (9.44)

in the Clifford algebra. By composition using Theorem 9.39 we obtain the action of any orthogonal transformation on $\eta \in \mathbb{R}^n$. 
**Definition 9.45.** For $n \in \mathbb{Z}$ define the real Clifford algebra $\mathcal{C}l_{n}$ as the unital associative real algebra generated by $e_1, \ldots, e_{|n|}$ subject to the relations

$$
e_i^2 = \pm 1, \quad i = 1, \ldots, |n|$$

$$e_i e_j + e_j e_i = 0, \quad i \neq j.$$ 

The complex Clifford algebra $\mathcal{C}l^C_{n}$ is the complex algebra with the same generators and same relations. These Clifford algebras are $\mathbb{Z}/2\mathbb{Z}$-graded by declaring that $e_1, \ldots, e_{|n|}$ are odd.

Note $\mathcal{C}l_0 = \mathbb{R}$ and $\mathcal{C}l_0^C = \mathbb{C}$.

**Example 9.47.** There is an isomorphism of superalgebras $\mathcal{C}l_{-n}^C \cong \mathcal{C}l_{n}^C$ obtained by multiplying each generator $e_i$ by $\sqrt{-1}$.

**Example 9.48.** $\mathcal{C}l_{-1}$ can be embedded in the matrix super algebra $M_{1|1} \mathbb{R} = \text{End}(\mathbb{R}^{1|1})$—see Example 9.17(2)—by setting

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

The same equation embeds $\mathcal{C}l_{-1}^C$ in $M_{1|1} \mathbb{C}$.

**Example 9.50.** Construct an isomorphism of superalgebras $\mathcal{C}l_{-2}^C \cong \text{End}(\mathbb{C}^{1|1})$ by

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$. There is no such isomorphism over the reals. The product

$$e_1 e_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

is $-i$ times a grading operator on $\mathbb{C}^2$.

**Example 9.53.** The real Clifford algebras $\mathcal{C}l_{1}$ and $\mathcal{C}l_{-1}$ are not isomorphic. In particular, the doubled orthogonal group $\{ \pm 1, \pm e_1 \}$ is different in the two cases: in $\mathcal{C}l_{1}$ it is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, whereas in $\mathcal{C}l_{-1}$ it is cyclic of order four.

We may in the future discuss basic Morita theory, in which we will see that super matrix algebras are in some sense trivial. That is the spirit of the following algebraic form of Bott periodicity.

**Theorem 9.54.** There are isomorphisms of superalgebras

$$\mathcal{C}l_{-2}^C \xrightarrow{\cong} \text{End}(S), \quad \dim S = 1|1,$$

$$\mathcal{C}l_{-8}^C \xrightarrow{\cong} \text{End}(S_{\mathbb{R}}), \quad \dim S_{\mathbb{R}} = 8|8.$$
Proof. The complex case is Example 9.50. For the real case, let \( C_\ell_{-2} \) act on \( W = \mathbb{C}^{1|1} \) via the formulas in (9.51). This action commutes (in the graded sense) with the odd real structure

\[
J(z^0, z^1) = (\overline{z^1}, \overline{z^0}).
\]

That is, \( J : W \to W \) is antilinear, odd, and squares to \(-\text{id}_W\). Set \( S = W^\otimes 4 \). It carries an action of \( C_\ell_{-2} \cong C_\ell_{-8} \) which commutes with \( J^\otimes 4 \). The latter is antilinear, even, and squares to \( \text{id}_S \), so is a real structure. \( \square \)

Observe that \( W^\otimes 2 \) carries a quaternionic structure \( J^\otimes 2 \): the Koszul sign rule (9.11) implies that \( J^\otimes 2 \) squares to minus the identity. (Check that sign!)

Next, we indicate the definitions of the pin and spin groups, as they sit in the Clifford algebras. For \( n > 0 \) let \( S(\mathbb{R}^n) \subset \mathbb{R}^n \) denote the sphere of unit norm vectors. Since \( \mathbb{R}^n \) embeds in \( C_\ell_{\pm n} \), so too does \( S(\mathbb{R}^n) \). We assert without proof that the group it generates is a Lie group \( \text{Pin}_{\pm n} \subset C_\ell_{\pm n} \).

It follows from Theorem 9.39 that there is a surjection \( \text{Pin}_{\pm n} \to \text{O}_n \) defined by composing the reflections (9.44). The inverse image \( \text{Spin}_{\pm n} \) of the special orthogonal group \( \text{SO}_n \) consists of products of an even number of elements in \( S(\mathbb{R}^n) \). There is an isomorphism \( \text{Spin}_n \cong \text{Spin}_{-n} \), but as we saw in Example 9.53 this is not true in general for Pin.

The Clifford algebra in Definition 9.45 arises from the following question, posed by Dirac: Find a square root of the Laplace operator. We work on flat Euclidean space \( E^n \), which is the affine space \( \mathbb{A}^n \) endowed with the translation-invariant metric constructed from the standard inner product on the underlying vector space \( \mathbb{R}^n \) of translations. Let \( x^1, \ldots, x^n \) be the standard affine coordinates on \( E^n \). The Laplace operator is

\[
\Delta = -\sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2}.
\]

A first-order operator

\[
D = \gamma^i \frac{\partial}{\partial x^i}
\]

satisfies \( D^2 = \Delta \) if and only if \( \gamma^i \) satisfy the Clifford relation

\[
\gamma^i \gamma^j + \gamma^j \gamma^i = -2\delta^{ij}, \quad 1 \leq i, j \leq n,
\]

as in (9.46). If we let (9.57), (9.58) act on the space \( C^\infty(\mathbb{E}^n; S) \) of functions with values in a vector space \( S \), then we conclude that \( S \) is a \( C_\ell_{-n} \)-module.
9.5 Abstract Clifford algebras

For more details see [ABS, Part I], [De1, §2].

A quadratic form on a vector space $V$ is a function $Q: V \to k$ such that

\begin{equation}
B(\xi_1, \xi_2) = Q(\xi_1 + \xi_2) - Q(\xi_1) - Q(\xi_2), \quad \xi_1, \xi_2 \in V,
\end{equation}

is bilinear and $Q(n\xi) = n^2 Q(\xi)$.

**Definition 9.61.** The Clifford algebra $\mathcal{C}\ell(V, Q) = \mathcal{C}\ell(V)$ of a quadratic vector space is an algebra equipped with a linear map $i: V \to \mathcal{C}\ell(V, Q)$ which satisfies the following universal property: If $\varphi: V \to A$ is a linear map to an algebra $A$ such that

\begin{equation}
\varphi(\xi)^2 = Q(\xi) \cdot 1_A, \quad \xi \in V,
\end{equation}

then there exists a unique algebra homomorphism $\tilde{\varphi}: \mathcal{C}\ell(V, Q) \to A$ such that $\varphi = \tilde{\varphi} \circ i$.

We leave the reader to prove that $i$ is injective and that $\mathcal{C}\ell(V, Q)$ is unique up to unique isomorphism. Furthermore, there is a surjection

\begin{equation}
\otimes V \longrightarrow \mathcal{C}\ell(V, Q)
\end{equation}

from the tensor algebra, as follows from its universal property. This gives an explicit construction of $\mathcal{C}\ell(V, Q)$ as the quotient of $\otimes V$ by the 2-sided ideal generated by $\xi^2 - Q(\xi) \cdot 1_\otimes V$, $\xi \in V$. The tensor algebra is $\mathbb{Z}$-graded, and since the ideal sits in even degree the quotient Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$-graded. The increasing filtration $\otimes^0 V \subset \otimes^{\leq 1} V \subset \otimes^{\leq 2} V \subset \cdots$ induces an increasing filtration on $\mathcal{C}\ell(V, Q)$ whose associated graded is isomorphic to the ($\mathbb{Z}$-graded) exterior algebra $\wedge^\bullet V$. There is a canonical isomorphism

\begin{equation}
\mathcal{C}\ell(V' \oplus V'', Q' \oplus Q'') \cong \mathcal{C}\ell(V', Q') \otimes \mathcal{C}\ell(V'', Q''),
\end{equation}

deduced from the universal property. The standard Clifford algebras in Definition 9.45 have the form $\mathcal{C}\ell(V, Q)$ for $V = \mathbb{R}^n, \mathbb{C}^n$ and $Q$ the positive or negative definite standard quadratic form on $V$.

The Clifford algebras are *central simple* as $\mathbb{Z}/2\mathbb{Z}$-graded algebras. I will leave the simplicity (there are no nontrivial 2-sided homogeneous ideals) as an exercise and here prove the centrality.

**Proposition 9.65.** $\mathcal{C}\ell(V, Q)$ has center $k$.

**Proof.** Suppose $x = x^0 + x^1$ is a central element. Fix an orthonormal basis $e_1, \ldots, e_n$ of $V$. Then for every $i = 1, \ldots, n$ we have

\begin{equation}
x^0 e_i = e_i x^0
\end{equation}

\begin{equation}
x^1 e_i = -e_i x^1
\end{equation}
There is a unique decomposition $x^0 = a^0 + e_i b^1$ where $a^0, b^1$ belong to the Clifford algebra generated by the basis elements excluding $e_i$. Then

\begin{align}
(9.67) \quad x^0 e_i &= a^0 e_i + e_i b^1 e_i = e_i a^0 - (e_i)^2 b^1 \\
&= e_i a^0 + (e_i)^2 b^1.
\end{align}

Since $x^0$ is central we have $x^0 e_i = e_i x^0$, and so (9.67) implies that $b^1 = 0$. Since this holds for every $i$, we conclude that $x^0$ is a scalar. Similarly, write $x^1 = a^1 + e_i b^0$ so that

\begin{align}
(9.68) \quad x^1 e_i &= a^1 e_i + e_i b^0 e_i = -e_i a^1 + (e_i)^2 b^0 \\
&= -e_i a^1 - (e_i)^2 b^0.
\end{align}

from which $x^1 = 0$. \hfill \Box

For a vector space $L$ and $\theta \in L^*$ let $\epsilon_\theta$ denote exterior multiplication by $\theta$, which is an endomorphism of the exterior algebra $\bigwedge^\bullet L^*$. For $\ell \in L$ the adjoint of exterior multiplication by $\ell$ is contraction $\iota_\ell$, an endomorphism of $\bigwedge^\bullet L^*$ of degree $-1$.

**Proposition 9.69.** Suppose $V = L \oplus L^*$ with the split quadratic form $Q(\ell + \theta) = \theta(\ell)$, $\ell \in L$, $\theta \in L^*$. Set $S = \bigwedge^\bullet L^*$ with its $\mathbb{Z}/2\mathbb{Z}$-grading by the parity of the degree. Then the map $V \to \text{End} S$

\begin{align}
(9.70) \quad \ell &\mapsto \iota_\ell \\
\theta &\mapsto \epsilon_\theta
\end{align}

extends to an isomorphism $\text{Cf}(V) \xrightarrow{\cong} \text{End} S$ of the Clifford algebra with a super matrix algebra.

**Proof.** Using (9.64) we reduce to the case $\dim L = 1$ which can be checked by hand; it is essentially Example 9.50. \hfill \Box

Sitting inside the Clifford algebra $\text{Cf}(V, Q)$ is the pin group $\text{Pin}(V, Q)$ generated by $S(V)$ and its even subgroup $\text{Spin}(V, Q) = \text{Pin}(V, Q) \cap \text{Cf}(V, Q)^0$. When $V$ is real and $Q$ is definite these are compact Lie groups. In that case we can average a metric over a real or complex Clifford module $S = S^0 \oplus S^1$ so that $\text{Pin}(V, Q)$ acts orthogonally (unitarily in the complex case). It follows that $e \in S(V)$ is self- or skew-adjoint, according as $Q$ is positive or negative definite. (Here we use a sign-adjust notion of adjoint, alluded to in Remark 9.37.)

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**Lecture 10: Bosons and fermions**

We revisit Axiom System 1.1, which is the axiom system for states and observables. Now we incorporate two types of states: bosonic states and fermionic states as well as mixtures. The vector space of observables is now $\mathbb{Z}/2\mathbb{Z}$-graded, and the homogeneous elements are called bosonic observables and fermionic observables. Measurement can be carried out only for bosonic observables.
Then we turn to QM systems, where the basic data is linear (Definition 5.1) or, better, projective (Definition 8.27). Here we incorporate fermionic states and observables by postulating that the (projective) Hilbert space be \( \mathbb{Z}/2\mathbb{Z} \)-graded.

The bulk of the lecture gives examples of QM systems with fermions. The first is a fermionic analog of the harmonic oscillator. The harmonic oscillator—bosonic or fermionic—has a canonical projective incarnation as an irreducible representation of a Heisenberg group. We develop this idea in the bosonic case, including the intertwining projective representation of the symplectic group. We conclude by reinterpreting the QM system in §7.1 as a system with fermions, and we exhibit its supersymmetry at the quantum level, at least in special cases.

10.1 General mechanical systems with fermions

For the generalization of Axiom System 1.1 to incorporate fermions we only indicate the modifications in the data, as listed there.

1. There are two real convex spaces \( S^0, S^1 \) whose elements are called **bosonic states** and **fermionic states**, respectively. The labels ‘0’ and ‘1’ lie in \( \mathbb{Z}/2\mathbb{Z} \). A general state is a convex combination of a bosonic and fermionic state, i.e., it is a point in the join

\[
S = S^0 \ast S^1.
\]

A pure state in \( S \) is either a pure state in \( S^0 \) or a pure state in \( S^1 \); it is either bosonic or fermionic.

2. The vector space \( \mathcal{O} \) is \( \mathbb{Z}/2\mathbb{Z} \)-graded, the real structure is even, there is a super Lie algebra structure on \( \mathcal{O}^\infty \), and the map (1.2) applies only to even elements of \( \mathcal{O}_\mathbb{R} \):

\[
\text{Borel}(\mathbb{R}; \mathbb{R}) \times \mathcal{O}_{\mathbb{R}}^0 \longrightarrow \mathcal{O}_{\mathbb{R}}^0
\]

\[
f , A \mapsto f(A)
\]

Elements of \( \mathcal{O}_{\mathbb{R}}^0 \) are **even** observables and elements of \( \mathcal{O}_{\mathbb{R}}^1 \) are **odd** observables.

3. Only bosonic observables can be measured:

\[
\mathcal{O}_{\mathbb{R}}^0 \times S \longrightarrow \text{Prob}(\mathbb{R})
\]

\[
A , \sigma \mapsto \sigma_A
\]

4. Only bosonic observables—those in \( (\mathcal{O}_{\mathbb{R}}^\infty)^0 \)—generate one-parameter groups, and the action on states preserves the decomposition (10.1).

**Remark 10.4.** One should relax (1) as follows: the labeling set of the spaces \( S^0, S^1 \) can be any 2-element set. In other words, rather than labeling by the cyclic group \( \mathbb{Z}/2\mathbb{Z} \), which has a definite identity element and a definite non-identity element, we allow the labeling to be a \( \mathbb{Z}/2\mathbb{Z} \)-torsor. This allows the two sets of states to be exchanged by symmetries, and also allows such an exchange in a family of systems. Symmetries that exchange the two spaces of states are called **supersymmetries**.

---

21 The join of topological spaces \( X, Y \) is the quotient of the Cartesian product \([0, 1] \times X \times Y \) by the subspaces \( \{0\} \times \{x\} \times Y \) and \( \{1\} \times X \times \{y\} \) for each \( x \in X, y \in Y \); see [H, p. 9].

22 For QM systems with fermions, **Remark 9.36** indicates that measurement of fermionic observables is absurd.

23 If a family is parametrized by a space or stack \( S \), then the labels form the total space of a double cover of \( S \).
For a motion (Definition 1.11), the constituent one-parameter groups of automorphisms of $S$ and of $O$ are required to preserve the $\mathbb{Z}/2\mathbb{Z}$-gradings.

### 10.2 Abstract QM systems with fermions

In Lecture 1 we deduced the data (1)–(4) above from a separable complex Hilbert space. In Lecture 8 we explored the projective nature of quantum mechanics and told how to specify a projective space rather than a Hilbert space. The analog of the latter discussion that incorporates fermionic states is interesting, and we will make comments below, but we begin with linear QM data (Definition 5.1).

Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a super Hilbert space. Recall from the comments following (9.34) that each ungraded vector space $\mathcal{H}^0, \mathcal{H}^1$ inherits a Hilbert space structure. Define a mechanical system with fermions as follows:

1. The space $S^0$ of bosonic states is the convex space of self-adjoint endomorphisms $S$ of $\mathcal{H}^0$ that are nonnegative, trace class, and trace $S = 1$. Fermionic states are self-adjoint endomorphisms of $\mathcal{H}^1$ that satisfy the same conditions. Pure states are the disjoint union of two projective spaces:

\[
\mathcal{P}S = \mathcal{P}\mathcal{H}^0 \amalg \mathcal{P}\mathcal{H}^1.
\]

2. Set $O = \text{End} \mathcal{H} = \text{End}^0 \mathcal{H} \oplus \text{End}^1 \mathcal{H}$. (Recall Example 9.17(2).) The real structure is the adjoint $A \mapsto A^\ast$, and the super Lie algebra structure is (1.26) modified by the Koszul sign rule (9.11):

\[
[A_1, A_2]_\hbar = -\frac{i}{\hbar} (A_1 \circ A_2 - (-1)^{|A_1||A_2|} A_2 \circ A_1),
\]

An even self-adjoint operator has spectral measures—self-adjoint projection-valued “probability” measures on $\mathbb{R}$ given by the spectral theorem—but now two of them: one for the operator on $\mathcal{H}^0$ and one for the operator on $\mathcal{H}^1$. So there are two spectral decompositions, one bosonic and the other fermionic, so to speak. As mentioned in footnote 22, it does not make sense to measure fermionic observables.

3. Measurement is as in (1.27) and (1.28), though one must account for the bosonic and fermionic spectra and apply the one appropriate to the state, which is either bosonic or fermionic.

4. The 1-parameter unitary group $U_t$ in Definition 5.1 is parity-preserving on $\mathcal{H}$, and it determines 1-parameter groups (1.30) of states and observables that preserve parity.

### 10.3 The fermionic harmonic oscillator

This is an analog of the abstract simple harmonic oscillator discussed in §6.2. Let $A$ be odd, and posit that $[A, A^\ast] = 1$. More formally, let $\mathcal{A}$ be the complex $*$-superalgebra generated by the odd

\[24\text{As in Remark 10.4, in place of a } \mathbb{Z}/2\mathbb{Z}\text{-graded Hilbert space we can grade by a } \mathbb{Z}/2\mathbb{Z}\text{-torsor.}\]
element $A$ subject to the relation $[A, A^*] = 1$. Note the Koszul sign rule for the commutator:

$$[A, A^*] = AA^* + A^*A.$$  

The Koszul sign rule also implies $A^2 = 0$, and so $\mathcal{A}$ is 4-dimensional, with basis $1, A, A^*, A^*A$. Construct a module $M$ over $\mathcal{A}$ of dimension $1|1$ as follows. Assume $\Omega \in M$ satisfies $A\Omega = 0$. Then

$$M = \mathbb{C}\langle \Omega \rangle \oplus \mathbb{C}\langle A^*\Omega \rangle$$

and the action of $\mathcal{A}$ is determined by the commutator $[A, A^*] = 1$. There is a choice of $\mathbb{Z}/2\mathbb{Z}$-grading of $M$. If $\Omega$ is assumed even, then $A^*\Omega$ is odd; if $\Omega$ is assumed odd, then $A^*\Omega$ is even. More naturally, we do not make a choice and grade $M$ by a $\mathbb{Z}/2\mathbb{Z}$-torsor. For the fermionic harmonic oscillator with Hamiltonian

$$H = A^*A + 1/2$$

the lines in (10.8) are eigenlines with eigenvalues $1/2, 3/2$. In terms of the axioms for a mechanical system, the spaces $\mathcal{S}^0, \mathcal{S}^1$ of bosonic and fermionic states each consist of a single point. The $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathcal{O}$ is the underlying vector space of the superalgebra $\mathcal{A}$.

There is a generalization parallel to our description (§6.3) of the polarized bosonic harmonic oscillator. Let $W$ be a finite dimensional real vector space.²⁵ Set

$$F(W) = \bigwedge^\bullet W_\mathbb{C}^*.$$  

This exterior algebra, which is finite dimensional and $\mathbb{Z}$-graded, is the fermionic Fock space of $W$. (Note that we only use the quotient $\mathbb{Z}/2\mathbb{Z}$-grading of the $\mathbb{Z}$-grading.) For $w \in W$ and $\ell \in W^*$ define linear operators $A_w$ and $M_\ell$ of degrees $-1$ and $+1$, respectively:

$$A_w(\theta) = \theta(w), \quad \theta \in \bigwedge^1 W_\mathbb{C}^*$$
$$M_\ell(\omega) = \ell \wedge \omega, \quad \omega \in F(W),$$

and extend $A_w$ to be a derivation: $A_w(\omega_1 \wedge \omega_2) = A_w(\omega_1) \wedge \omega_2 + \omega_1 \wedge A_w(\omega_2)$ for all $\omega_1, \omega_2 \in F(W)$. Then

$$[A_w, M_\ell] = \ell(w)$$

is scalar multiplication by $\ell(w)$.

Remark 10.13.

²⁵As above, there is an important variation for infinite dimensional topological vector spaces with a polarization; see [PS, §12].
(1) The odd\(^{26}\) real vector space of operators \(A_w, M_\ell\) is the quotient in a central extension

\[
(10.14) \quad 0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{h} \longrightarrow \Pi(W \oplus W^*) \longrightarrow 0
\]

of real super Lie algebras. The commutator (10.12) induces an orthogonal structure on \(W \oplus W^*\) which is split: the signature\(^{27}\) of the nondegenerate symmetric bilinear form is \((N, N)\) if \(\dim W = N\). There is a generalization of this model in which we replace \(W \oplus W^*\) by a finite dimensional real orthogonal vector space, often taken to be positive definite.

(2) There are operators \(A_w, M_\ell\) for \(w, \ell\) in the complexifications \(W_\mathbb{C}, W^*_\mathbb{C}\), and they generate a finite dimensional complex algebra: the Clifford algebra.

One can define a Hamiltonian as in the bosonic case; see (6.23) and the text that follows, and also note the special case (10.9) above.

10.4 Schur’s lemma

As a preliminary, we review a basic result about irreducible representations.

Let \(G\) be a (Lie) group. We work over \(\mathbb{C}\) though any algebraically closed field will do. Recall that a representation on a vector space \(W\) is irreducible if \(W\) has no proper \(G\)-invariant subspaces. (If \(W\) has a topology, then we ask for no closed \(G\)-invariant subspaces.) Also, recall that a morphism of \(G\)-representations \(\pi': G \rightarrow \text{Aut} W'\) and \(\pi: G \rightarrow \text{Aut} W\) is a linear map \(T: W' \rightarrow W\) such that \(T \circ \pi'(g) = \pi(g) \circ T\) for all \(g \in G\).

**Theorem 10.15.** Suppose \(\pi: G \rightarrow \text{Aut} W\) is an irreducible finite dimensional linear representation of \(G\) on a complex vector space \(W\). Then if \(T: W \rightarrow W\) is a linear automorphism of \(\pi\), there exists \(\lambda \in \mathbb{C}\) such that \(T = \lambda \text{id}_W\).

**Proof.** Let \(\lambda \in \mathbb{C}\) be an eigenvalue of \(T\). The \(\lambda\)-eigenspace \(\ker(\lambda \text{id}_W - T)\) is a nonzero \(G\)-invariant subspace of \(W\), hence equals \(W\) by the assumption that \(\pi\) is irreducible. \(\square\)

**Corollary 10.16.** Let \(G\) be a (Lie) group. Then to each isomorphism class \(\rho\) of irreducible linear representation of \(G\) is canonically attached a projective space \(\mathbb{P}_\rho\) with a \(G\)-action.

In other words, the associated projective representation is canonical (whereas the linear representation is only canonical up to tensoring by a line).

Next is Schur’s lemma for unitary representations, which may be infinite dimensional.

**Theorem 10.17.** Suppose \(\pi: G \rightarrow U(\mathcal{H})\) is an irreducible unitary representation of \(G\) on a complex Hilbert space \(\mathcal{H}\). Then if \(T: \mathcal{H} \rightarrow \mathcal{H}\) is a unitary automorphism of \(\pi\), there exists \(\lambda \in \mathbb{T}\) such that \(T = \lambda \text{id}_\mathcal{H}\).

\(^{26}\)Let \(\Pi\) denote the ground field \(F\), viewed as a \(\mathbb{Z}/2\mathbb{Z}\)-graded vector space which is odd. Then for any super vector space \(V\), let \(\Pi V = \Pi \otimes V\). We call \(\Pi V\) the parity-reversal of \(V\).

\(^{27}\)Here I use ‘signature’ for the pair \((p, n)\) of dimensions of maximal positive and negative definite subspaces. Usually ‘signature’ is the integer \(p - n\), which is zero here.
Proof. There is a spectral theorem for unitary operators, which produces a self-adjoint projection-valued measure on \( T \subseteq \mathbb{C} \). If that spectral measure of \( T \) is not concentrated at a point, then there is a self-adjoint projection whose kernel is a closed proper \( G \)-invariant subspace. (Alternative proof: Apply the spectral theorem for self-adjoint operators to \( T + T^* \) and \( i(T - T^*) \).)

\[ \square \]

**Corollary 10.18.** Let \( \pi' : G \to U(\mathcal{H}') \) and \( \pi : G \to U(\mathcal{H}) \) be isomorphic unitary irreducible representations. Then

1. the collection of unitary morphisms \( \mathcal{H}' \to \mathcal{H} \) of \( G \)-representations is a \( \mathbb{T} \)-torsor; and
2. there is a unique isomorphism \( \mathbb{P}\mathcal{H}' \to \mathbb{P}\mathcal{H} \) which preserves the Fubini-Study structures and commutes with the \( G \)-actions.

See (8.19) for the definition of Fubini-Study geometry in finite dimensions. Also, there is an analog of Corollary 10.16 for unitary representations.

**Corollary 10.19.** Let \( \pi : G \to U(\mathcal{H}) \) be an irreducible unitary representation of \( G \) on a complex Hilbert space \( \mathcal{H} \), and assume given an automorphism \( \alpha : G \to G \) of \( G \) such that the composition \( G \xrightarrow{\alpha} G \xrightarrow{\pi} U(\mathcal{H}) \) is isomorphic to \( \pi \) as a \( G \)-representation. Then there is an \( \mathbb{T} \)-torsor of unitary operators on \( \mathcal{H} \) which implement \( \alpha \) as an intertwiner. As a consequence, there is a unique projective unitary automorphism of \( \mathbb{P}\mathcal{H} \) which implements \( \alpha \) as an intertwiner.

Recall the projective space \( \mathbb{P}_\rho \) in Corollary 10.16.

**Corollary 10.20.** Let \( G \) be a (Lie) group, and let \( \rho \) be an isomorphism class of irreducible unitary representations of \( G \). Suppose \( H \) is a group of automorphisms of \( G \) such that each \( h \in H \) preserves \( \omega \). Then there exists a canonical extension of the \( G \)-action on \( \mathbb{P}_\rho \) to an action of the semidirect product \( H \ltimes G \).

10.5 The bosonic harmonic oscillator revisited

We build on the material in §6.4.

Let \( A \) be a finite dimensional real symplectic affine space. Thus \( A \) is affine over a finite dimensional real vector space \( V \) which is equipped with a nondegenerate skew-symmetric pairing \( \omega : V \times V \to \mathbb{R} \). The pairing is then a translation-invariant symplectic form on \( A \). The symmetry groups of \( A \) and \( V \) fit into a group extension

\[
1 \longrightarrow V \longrightarrow \text{Sp}(A) \longrightarrow \text{Sp}(V) \longrightarrow 1
\]

in which \( \text{Sp}(V) \) is the group of linear automorphisms of \( V \) that preserve \( \omega \), and \( \text{Sp}(A) \) is the group of affine automorphisms of \( A \) that preserve \( \omega \). The kernel \( V \) consists of translations in \( A \). Define

\[
\hat{\text{Sp}}(V) = \{ T : V \to V \text{ real linear} : T^*\omega = \pm \omega \},
\]

\footnote{This means that if \( \pi : G \to \text{Aut}W \) is a linear representation in the isomorphism class \( \rho \), then the composition \( G \xrightarrow{h} G \xrightarrow{\pi} \text{Aut}W \) is also in the isomorphism class \( \rho \).}
and let $\widetilde{\text{Sp}}(A)$ be the corresponding 2-component Lie group of affine symmetries. These fit into group extensions

$$
1 \rightarrow \text{Sp}(V) \rightarrow \widetilde{\text{Sp}}(V) \rightarrow \mu_2 \rightarrow 1
$$

(10.23)

$$
1 \rightarrow \text{Sp}(A) \rightarrow \widetilde{\text{Sp}}(A) \rightarrow \mu_2 \rightarrow 1
$$

$$
1 \rightarrow V \rightarrow \widetilde{\text{Sp}}(A) \rightarrow \widetilde{\text{Sp}}(V) \rightarrow 1
$$

Remark 10.24. A particular case is the model geometry in which $V = \mathbb{R}^{2m}$ acts on $A = \mathbb{A}^{2m}$ and $\omega$ is a standard symplectic form on $\mathbb{R}^{2m}$. It is this model geometry that is used to make the canonical constructions.

Recall the Heisenberg Lie algebra $\mathfrak{h}(A)$, defined in (6.27) as the Lie algebra of affine functions $A \rightarrow \mathbb{R}$ with its Poisson bracket. The Heisenberg group $H(A)$ is the corresponding Lie group; it fits into the group extension

$$
1 \rightarrow \mathbb{T} \rightarrow H(A) \rightarrow \exp V \rightarrow 1
$$

(10.25)

The Stone-von Neumann Theorem 6.29 asserts that there is a canonical isomorphism class of irreducible unitary representations of $H(A)$ on which the center $\mathbb{T} \subset H(A)$ acts by scalar multiplication. Note that since the scalars $\mathbb{T} \subset H(A)$ act by homotheties on the unitary representation of Theorem 6.29, they act trivially on the projectivization of any unitary representation in the isomorphism class. Now our discussion of Schur’s lemma in §10.4—in particular, the infinite dimensional unitary analog of Corollary 10.16—immediately implies the following.

**Corollary 10.26.** There is a canonical Fubini-Study projective space $\mathbb{P}_A$ attached to the symplectic vector space $A$, and it carries a projective unitary action of $\exp V$.

The connected Lie group $\text{Sp}(A)$ acts by conjugation on the normal translation subgroup $V$, the action drops to the quotient group $\text{Sp}(V)$, and by the Stone-von Neumann theorem this action preserves the isomorphism class of the Heisenberg representation. Therefore, by Corollary 10.19 there exists a canonical projective unitary action of $\text{Sp}(A)$ on $\mathbb{P}_A$, and furthermore it extends the action of $V$. In a linearization the symplectic group $\text{Sp}(V)$ acts through a central extension, the metaplectic group.\(^{29}\) Altogether (Corollary 10.20) we obtain a projective action of $\text{Sp}(V) \times V$ on $\mathbb{P}_A$.

By our discussion of Schur’s lemma we can also implement the larger group $\widetilde{\text{Sp}}(A)$ as projective transformations of $\mathbb{P}_A$, but the off-component acts by projective antunitaries. This follows since elements in the off-component reverse the sign of the symplectic form $\omega$, hence map the Heisenberg Lie algebra $\mathfrak{h}(A)$ to the Lie algebra obtained from the negative symplectic form. On the group level (10.25) this action inverts the kernel $\mathbb{T}$, which acts by scalar multiplication in the Heisenberg representation: this inversion is the antilinearity. The projective space $\mathbb{P}_A$ has a projective quantum geometric structure, as in Definition 8.27, but now in infinite dimensions.

This discussion is summarized as follows. Recall the model group PQ of projective unitary and antunitary transformations in (8.46).

\(^{29}\)The metaplectic group has kernel $/\mu_2$ whereas this central extension has kernel $\mathbb{T}$, so is called the “metaplectic-c” group by analogy with the spin-c group.
**Theorem 10.27.** There is a canonical projective representation

\[(10.28) \quad \widehat{\text{Sp}(A^{2m})} \rightarrow \text{PQ}.\]

**Remark 10.29.**

1. The projective representation (10.28) does not lift to a linear representation into Q. The obstruction is not topological, though it can be measured in a suitable generalization of standard topological cohomology theories.

2. As explained above, the representation (10.28) associates a family of projective spaces to a fiber bundle of affine spaces via the associated principal $\text{Sp}(A^{2m})$-bundle. For example, any symplectic manifold gives rise to such a family via its affine tangent bundle.

3. There is an infinite dimensional variant of (10.28); see [Sh, Se1, PS].

One can construct spaces of states and observables from the projective space, as explained in Lecture 8. However, there is no canonical harmonic oscillator Hamiltonian that is invariant under $\widehat{\text{Sp}(A^{2m})}$. Rather, there is a class of Hamiltonians: elements in the (closure of the) second filtration of the dense algebraic subspace of the Heisenberg representation.

### 10.6 The fermionic harmonic oscillator revisited

There is a parallel discussion for the fermionic harmonic oscillator, but with some major differences: the vector space $V$ is odd in the sense of superalgebra, the Heisenberg Lie algebra encodes canonical anticommutation relations, and the quantum system is $\mathbb{Z}/2\mathbb{Z}$-graded and finite dimensional. Our discussion is abbreviated and only highlights these differences.

Let $V$ be an odd finite dimensional real vector space, so $V = \Pi U$ for an even vector space $U$. Then a skew-symmetric bilinear form on $V$ satisfies

\[(10.30) \quad \omega(v_1, v_2) = -(-1)^{|v_1||v_2|}\omega(v_2, v_1) = \omega(v_2, v_1),\]

so it comes from a symmetric bilinear form on $U$. We assume that this form is positive definite, i.e., is an inner product on $U$. For the odd vector space $V$, the symplectic group $\text{Sp}(V)$ in (10.21) is the orthogonal group $O(U)$, and for an odd affine space $A$ over $V$ the affine symplectic group is the Euclidean group of an associated even affine space. Observe that these Lie groups have two components: elements in the identity component preserve orientation and elements in the off-component reverse orientation. Now the analogs of $\widehat{\text{Sp}}(V)$ and $\widehat{\text{Sp}}(A)$ have 4 components: they are simply direct products of $\mu_2$ with the unhatted groups. The Heisenberg group $H(V)$ is a super Lie group. For the analog of Theorem 6.29 we need to assume that $\dim V$ is even.$^{31}$ Then the representation is on a $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional complex Hilbert space $S = S^0 \oplus S^1$, and Theorem 9.54 states that the complex Clifford algebra on an even dimensional vector space is a super matrix algebra. The odd symplectic group $\text{Sp}(V)$ (so the orthogonal group $O(U)$) is

---

$^{30}$ We have not defined this notion, so the reader may want to contemplate the definition.

$^{31}$ If $\dim V$ is odd, then there is a representation on a module over the complex Clifford algebra $Cl_2^c$; see [FHT3] for a detailed discussion.
implemented projectively; it acts via the pin-c covering group. The analog of Corollary 10.26 is a pair of projective spaces $\mathbb{P}^0_V, \mathbb{P}^1_V$ with an action of $\widetilde{Sp}(A)$. The numbering ‘0,1’ is not canonical: in a family whose base is not simply connected the two projective spaces can exchange as one traverses a closed loop.

**Remark 10.31.**

(1) As opposed to Remark 10.29(1) in the even case, here the central extension of the orthogonal group is detected in topology: the class of the pin-c central extension is the Bockstein of the second Stiefel-Whitney class.

(2) There is a family of odd harmonic oscillators (without a particular Hamiltonian) associated to a fiber bundle of Euclidean spaces, so in particular to a real vector bundle with an inner product. The obstruction to linearizing the family is the Bockstein of the second Stiefel-Whitney class of the vector bundle: a linearization is induced from a spin-c structure on the vector bundle. The family parametrized by a Riemannian manifold (using its tangent bundle) occurs in the simplest supersymmetric system: the superparticle.

(3) There is an infinite dimensional variant [PS, §12].

### 10.7 Addendum to Section 7.1

We revisit the QM system analyzed in §7.1, now with the modification

\[(10.32) \quad \mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C}^{1|1}) = \left\{ \left( \begin{array}{c} \psi^0 \\ \psi^1 \end{array} \right) : \psi^i \in L^2(\mathbb{E}^1; \mathbb{C}) \right\} = H^0 \oplus H^1.\]

In other words, we take the state space to be $\mathbb{Z}/2\mathbb{Z}$-graded with the indicated grading. Recall the operator $Q_h$ in (7.4). The adjoint is $Q_h^*$, but the formulas must be modified slightly due to the signs and factors of $\sqrt{-1}$ that occur in adjoints of operators on $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert spaces; see (9.35).

Set $R_h = Q_h + Q_h^*$. Then the Hamiltonian is

\[(10.33) \quad H_h = \frac{1}{2} R^2_h.\]

Assume that $R_h$ has no kernel, so that $R_h$ is an isomorphism. For example, this occurs if $h$ is a cubic polynomial (§7.1.4). Then $R_h$ permutes the pure states (compare (7.5)):

\[(10.34) \quad \mathbb{P}\mathcal{H}^0 \xrightarrow{R_h} \mathbb{P}\mathcal{H}^1.\]

A general bosonic or fermionic state $S$ lies in $\text{End} \mathcal{H}^0$ or in $\mathcal{H}^1$, and an observable $A$ lies in $\text{End} \mathcal{H}$. The action of $R_h$ on pure states extends to an action

\[(10.35) \quad S \mapsto R_h S R_h^{-1}, \quad A \mapsto R_h^{-1} A R_h\]

on all homogeneous states and all observables. This symmetry commutes with the Hamiltonian (10.33), so it also commutes with the unitary flows the Hamiltonian generates. This is an example of a *supersymmetry.*
Lecture 11: Wick rotation, algebras of observables

This lecture, the last on quantum mechanics per se, treats a few miscellaneous topics.

We begin with a brief discussion of affine time, emphasizing as always (§7.4) the symmetries. We always assume a time-orientation, which we sometimes call an arrow of time. But we also consider the larger symmetry group that reverses the arrow of time. We prove an easy, but basic, Lemma 11.3 for symmetries of QM systems and its corollary: If the Hamiltonian \( H \) is unbounded above and bounded below, then a time-preserving symmetry acts unitarily and a time-reversing symmetry acts antiunitarily. (If \( H \) is similar to \(-H\), then this need not hold.)

Next, we give the definition of a correlation function in quantum mechanics. These are basic quantities that encode physical information.

Wick rotation applies to QM systems of positive energy (Definition 5.7). It expresses time evolution and correlation functions as boundary values of holomorphic expressions. In particular, the unitary time evolution is the boundary value of a holomorphic 1-parameter semigroup of contractive operators. Wick rotation [Wk] is the restriction to a real 1-parameter semigroup, an imaginary time evolution. The original context for Wick rotation is relativistic quantum field theory. In that context it leads to modern geometric axiom systems, as we shall see.

Our axiomatization of mechanical systems in general (Axiom System 1.1), and QM systems in particular, treats states and observables on an equal footing. With some additional hypotheses one can put observables in the primary position and treat states as a derived concept. This has a long history in quantum mechanics and quantum field theory, dating back to papers of Jordan–von Neumann–Wigner [JvNW] and Irving Segal [Seg]. Observables are assumed to form an algebra—in fact, a \( C^* \)-algebra (or perhaps another form of topological \( \ast \)-algebra)—and states are special linear functionals on the algebra. We give a brief introduction to this viewpoint.

You may want to review the material on group extensions (§8.6.4).

11.1 Time, time-orientation, and symmetries

In mechanics, time is an oriented 1-dimensional Euclidean space \( M \). Thus \( M \) is an affine space over a 1-dimensional oriented real inner product space \( U \). The vector line \( U \) is the group of time translations. Note that \( U \) has a distinguished basis: a positively oriented vector of unit norm. The inner product on \( U \) can be regarded as a “clock”: it is used to measure elapsed time. The orientation of \( U \) allows to distinguish in which direction the clock is running. The symmetry group \( O^\uparrow(M) \) of this structure sits in a group extension, and it is a subgroup of the group \( O(M) \) of all isometries of \( M \), time-orientation preserving or not:

\[
\begin{array}{cccccc}
1 & \longrightarrow & U & \longrightarrow & O^\uparrow(M) & \longrightarrow & O^\uparrow(U) & \longrightarrow & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & \longrightarrow & U & \longrightarrow & O(M) & \longrightarrow & O(U) & \longrightarrow & 1 \\
\end{array}
\]

Note that \( O^\uparrow(U) \) consists only of the identity element, hence \( O^\uparrow(M) \) consists only of (time) translations. The larger group \( O(M) \) also contains (time) reflections.
Remark 11.2. The model geometry $G \triangleleft F$ for time is the affine line $F = \mathbb{A}^1$ with symmetry group $G = \text{Euc}_1 \cong \mathbb{R}$ translations of the affine line. Observe that the inner product on a real line is determined up to multiplication by a positive real number, and the symmetry groups (11.1) are independent of the inner product. Hence we need not fix an inner product on $U$. Indeed, an inner product fixes a clock to measure in seconds, or minutes, or hours, or light-years, depending on the scale of the inner product. It is sometimes convenient not to fix an inner product.

Let $(\mathcal{H}, U_t)$ be linear data for a QM system, and suppose $H$ is the Hamiltonian: $U_t = e^{-itH}/\hbar$. (In terms of our previous discussion, $t \in U \cong \mathbb{R}$ is an interval of time.) Recall our discussion of quantum symmetries, and in particular Wigner’s Theorem 8.34. It states that the group of quantum symmetries sits in a group extension (8.46): each quantum symmetry $g$ lifts to a real linear symmetry $\tilde{g}$ which is either unitary or antiunitary. Write $\phi(g) = \pm 1$ to track the two possibilities. Now we also consider how the symmetry acts on time, and we allow a quantum symmetry $g: \mathbb{R} \mathcal{H} \to \mathbb{R} \mathcal{H}$ to act by preserving or reversing time. Let $t(g) = \pm 1$ track this dichotomy.

**Lemma 11.3.**

1. 

   \begin{equation}
   \tilde{g}^{-1}H\tilde{g} = \phi(g)t(g)H.
   \end{equation}

2. If $H$ is bounded below and unbounded above, then $\phi(g) = t(g)$.

**Proof.** Symmetries act on time evolution by conjugation, and they may preserve or reverse the direction of that evolution. Hence for all $t \in \mathbb{R}$ we have

   \begin{equation}
   \tilde{g}^{-1}e^{-itH}/\hbar \tilde{g} = e^{t(g)(-itH)}/\hbar,
   \end{equation}

   from which we deduce

   \begin{equation}
   \tilde{g}^{-1}(iH)\tilde{g} = t(g)(iH),
   \end{equation}

   and then (11.4). For (2) observe that under the assumptions given, $H$ is not conjugate to $-H$. 

**Remark 11.7.** In many quantum systems, for example most quantum field theories, the Hamiltonian satisfies the conditions in (2) and so time-preserving symmetries are unitary and time-reversing systems are antiunitary. There are exceptions, notably in finite dimensional systems. For a more elaborate discussion of symmetry in QM systems, see [FM1].

### 11.2 Correlation functions

Suppose given a mechanical system as in **Axiom System 1.1**. Let $M$ be a time line. It is natural in this context to assume that the observables form a complex vector bundle $\mathcal{O} \to M$, and the smooth observables form a dense subbundle $\mathcal{O}^\infty \to M$. Time evolution lifts time translation $U \triangleleft M$ to $U \triangleleft \mathcal{O}$. This is the time evolution of observables: the Heisenberg picture in the case of QM systems.
Suppose \((\mathcal{H}, U_t)\) is the linear data of a QM system. Fix \(t_0 < t_1 < \cdots < t_\ell < t_f\) a finite set of ordered points in \(M\). Let \(A^{(t_i)}_i \in \mathcal{O}_{t_i}, i \in \{1, \ldots, \ell\}\) be observables. Fix a state \(\sigma \in \mathcal{S}\). We illustrate this data in Figure 24. For simplicity assume that \(\sigma\) is a pure state—a line \(L \subset \mathcal{H}\)—and choose a unit norm vector \(\psi \in L\). Then the correlation function is the complex number

\[
\langle A^{(t_\ell)}_\ell \cdots A^{(t_1)}_1 \rangle_{[t_0, t_f]} = \langle U_{t_0-t_f} \psi, A^{(t_\ell)}_\ell \circ \cdots \circ A^{(t_1)}_1 \psi \rangle
\]  

(11.8)

Note we have translated the (final) state \(\sigma\) back to the initial time \(t_0\). (We can have different initial and final states.) Let us rewrite (11.8) in the Schrödinger picture. Assume that each observable is the time translate of an observable at initial time:

\[
A^{(t_i)}_i(t) = U_{t_0-t_i} A_i U_{t_i-t_0}, \quad i \in \{1, \ldots, \ell\},
\]

for some observable \(A_i \in \mathcal{O}_{t_0}\). Then the correlation function is

\[
\langle A^{(t_\ell)}_\ell \cdots A^{(t_1)}_1 \rangle_{[t_0, t_f]} = \langle \psi, U_{t_f-t_\ell} A_\ell \cdots A_2 U_{t_2-t_1} A_1 U_{t_1-t_0} \psi \rangle
\]

(11.9)

The last expression reads: begin in the pure state \(\mathbb{C} \langle \psi \rangle\), evolve for time \(t_1 - t_0\), apply observable \(A_1\), evolve for time \(t_2 - t_1\), apply observable \(A_2\), \ldots, apply observable \(A_\ell\), evolve for time \(t_f - t_\ell\). Often one takes \(\sigma\) to be a vacuum state, in which case (11.10) is called a vacuum correlation function.

Remark 11.11. The correlation function (11.10) only involves differences of time, which lie in \(U\). That reflects the time translation symmetry.

### 11.3 Wick rotation

Let \(\mathcal{H}\) be a finite dimensional hermitian vector space, and suppose \(H \in \text{End} \mathcal{H}\) is self-adjoint. The real 1-parameter group \(t \mapsto e^{-i t H}, t \in \mathbb{R}\) of unitary operators has an extension to a complex 1-parameter group of endomorphisms: simply take the parameter \(t\) to be a complex number. For each \(t \in \mathbb{C}\) the operator \(e^{-i t H}\) is of course bounded, since \(\mathcal{H}\) is finite dimensional, though the family of operators may not be uniformly bounded. But if \(\mathcal{H}\) is infinite dimensional, then the operator \(e^{-i t H}\) may be unbounded for fixed \(t\). Note that for \(\lambda \in \mathbb{R}\),

\[
|e^{-i t \lambda}| = e^{\text{Im}(t) \lambda}.
\]

(11.12)

If \(\lambda \geq 0\) and \(\text{Im}(t) < 0\), then \(|e^{-i t \lambda}| \leq 1\). Hence, if \(H\) is a nonnegative operator, then for \(\text{Im}(t) < 0\) we have \(\|e^{-i t H}\| \leq 1\) in operator norm.

\[32\text{In the application to quantum mechanics we have set } \hbar = 1.\]
**Definition 11.13.** Let $\mathcal{H}$ be a Hilbert space. A linear operator $C : \mathcal{H} \to \mathcal{H}$ is **contractive** if its operator norm satisfies $\|C\| \leq 1$.

Let $(\mathcal{H}, U_t)$ be the linear data of a QM system that has **positive energy** (Definition 5.7): the Hamiltonian is nonnegative. Apply the preceding to the time evolution, as illustrated in Figure 25. Real time evolution in quantum mechanics is a 1-parameter real group of unitary operators. Under the positive energy hypothesis, this unitary evolution is the boundary value of a **holomorphic** 1-parameter semigroup\(^{33}\) of contractive operators parametrized by

$$C_- = \{ t \in \mathbb{C} : \text{Im}(t) < 0 \}.$$

Now restrict to the real semigroup $-i\mathbb{R}^> \subset \mathbb{C}_-$. Thus for $\tau > 0$ set $t = -i\tau$ to obtain

$$\tau \mapsto C_\tau = e^{-\tau H}, \quad \tau \in \mathbb{R}^>.$$

This real semigroup of contractive operators is usually better behaved analytically than the group of unitary operators. The analytic difference is more stark when Wick rotating correlation functions, especially in quantum field theory.

**Example 11.16.** For the quantum mechanical particle on a closed Riemannian manifold $M$, as in Lecture 5, the Hilbert space is $\mathcal{H} = L^2(M; \mathbb{C})$ and the Hamiltonian $H = \Delta$ is the Laplace operator. The Wick-rotated “evolution” is via the heat operator $e^{-\tau H}$, $\tau > 0$. This operator is not only contractive, but it is smoothing: the image of an $L^2$ function is a smooth function. In fact, it extends to distributions, which then also map to smooth functions. Of course, the unitary evolution operator does not have that property.

There is also a Wick rotation of correlation functions. For example, the correlation function (11.10) with $\ell = 1$ (for convenience) rotates to a function of imaginary time differences\(^{34}\)

$$t_1 - t_0 = -i\tau_1$$

$$t_2 - t_1 = -i\tau_2$$

$$t_f - t_2 = -i\tau_f$$

\(^{33}\)A **semigroup** is a set with an associative composition law. If it has a unit, then it is a monoid.

\(^{34}\)The times $t_0, t_1, t_2, t_f$ live on an affine time line $M$; evolution is a function of time differences in a vector line $U$, and these are what appear in (11.10), as noted in Remark 11.11. Their imaginary counterparts appear in the Wick-rotated correlation function (11.18).
using the contractive evolution (11.15):

\[ \langle \psi, C_{\tau_1} A_2 C_{\tau_2} A_1 C_{\tau_1} \psi \rangle \]

11.4 Algebras of observables

We already listed a few of the original references [JvNW, Seg] at the beginning of the lecture; see also [Str, §1.3] and [Ha, §1.1] for further discussions. For example, there you will find elaborations of an argument justifying restriction to bounded operators; the argument is based on the fact that real-world measurements are bounded.

In the general Axiom System 1.1 for mechanics, the vector space \( \mathcal{O} \), whose real points are observables, is not assumed to be an algebra. In the main examples—classical or statistical mechanics on a Riemannian manifold, quantum mechanics in which we take \( \mathcal{O} \) to consist of bounded operators—it does have the structure of an associative unital algebra. Also, the convex space of states is separate data, not derived from \( \mathcal{O} \). Observe that a state \( \sigma \in \mathcal{S} \) determines a real linear functional on observables:

\[ \mathcal{O}_\mathbb{R} \rightarrow \text{Prob}(\mathbb{R}) \quad E \rightarrow \mathbb{R} \]

\[ A \mapsto \sigma_A \quad \mapsto \langle \sigma_A \rangle \]

where \( E \) maps a probability distribution to its expectation value. The linearity follows from (1.4). Recall also that Borel functions of observables are observables, so moments of \( \sigma_A \) are computed by applying (11.19) to powers of \( A \). A probability measure is not determined by its moments in general, though it is if the moments satisfy a bound on their growth. In any case we can take this as heuristic motivation for the idea that states can be defined as linear functionals of observables.

Based on such considerations, and many more as explained in the references, one often postulates that in quantum mechanics the complex vector space \( \mathcal{O} \) (of complex observables) is a \( C^* \)-algebra. We pause to give basic definitions.

**Definition 11.20.** Let \( \mathcal{O} \) be a complex algebra

1. A Banach algebra structure on \( \mathcal{O} \) is a Banach norm compatible with the algebra structure:

\[ \|A_1 \circ A_2\| \leq \|A_1\| \cdot \|A_2\|, \quad \text{for all } A_1, A_2 \in \mathcal{O}, \]

\[ \|1\| = 1 \]

2. If \( \mathcal{O} \) is a \( \ast \)-algebra, then the Banach norm induces the structure of a Banach \( \ast \)-algebra if

\[ \|A^\ast\| = \|A\| \quad \text{for all } A \in \mathcal{O}. \]

3. A Banach \( \ast \)-algebra is a \( C^\ast \)-algebra if, in addition,

\[ \|A^\ast A\| = \|A\|^2 \quad \text{for all } A \in \mathcal{O}. \]
Example 11.24.

(1) Let \( X \) be a compact Hausdorff topological space, and let \( C(X; \mathbb{C}) \) denote the vector space of continuous complex-valued functions on \( X \). It is an \textit{commutative} algebra by pointwise addition and multiplication. The sup norm \( \| f \| = \max_{x \in X} |f(x)| \) is a Banach space norm. Complex conjugation provides a \( \ast \)-structure. Every commutative \( C^* \)-algebra is isomorphic to \( C(X; \mathbb{C}) \) for some compact Hausdorff space \( X \).

(2) Let \( \mathcal{H} \) be a complex Hilbert space, finite dimensional or separable infinite dimensional. Then the \( \ast \)-algebra \( \text{End} \mathcal{H} \) of bounded linear operators on \( \mathcal{H} \), equipped with the operator norm \( \| T \| = \sup_{\xi \in \mathcal{H}, \| \xi \| = 1} \| T \xi \| \), is a \( C^* \)-algebra. It is not commutative as long as \( \dim \mathcal{H} > 1 \).

Basic operator theory of bounded operators on Hilbert space generalize to \( C^* \)-algebras. The following is just a taste.

Definition 11.25. Let \( \mathcal{O} \) be a \( C^* \)-algebra. Fix \( A \in \mathcal{O} \).

(1) The \textit{spectrum} of \( A \) is

\[
\text{spec } A = \{ \lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible in } \mathcal{O} \}.
\]

(2) \( A \) is \textit{self-adjoint} if \( A^* = A \).

(3) \( A \) is \textit{nonnegative} if \( A \) is self-adjoint and \( \text{spec } A \subset \mathbb{R} \).

The spectrum is a compact subset of \( \mathbb{C} \); the spectrum of a self-adjoint element is a subset of \( \mathbb{R} \subset \mathbb{C} \). An element \( A \) is nonnegative iff it has the form \( A = B^*B \) for some \( B \in \mathcal{O} \).

The GNS (Gelfand-Naimark-Segal) construction implies that any \( C^* \)-algebra is isomorphic to a closed \( \ast \)-subalgebra of \( \text{End} \mathcal{H} \) for some Hilbert space \( \mathcal{H} \), though this presentation is not canonical. There are many references on \( C^* \)-algebras; for example, Chapter 1 of the lecture notes [Pu] summarizes the basics.

With this background we can define states in terms of observables, assuming observables form a \( C^* \)-algebra.

Definition 11.27. Let \( \mathcal{O} \) be a \( C^* \)-algebra. A \textit{state} on \( \mathcal{O} \) is a continuous linear map \( \omega : \mathcal{O} \rightarrow \mathbb{C} \) such that (i) \( \omega(A) \geq 0 \) for all nonnegative \( A \in \mathcal{O} \), and (ii) \( \omega(1) = 1 \).

States form a convex space; as before, pure states are the extreme points.

Remark 11.28. Algebra-like structures have been a recent focus in Wick-rotated field theory: factorization algebras [CG1, CG2] and, in a topological context, \( E_n \)-algebras.

Lecture 12: Affine spacetimes

So far in our discussion of mechanics we have treated time and space separately, to the extent that we’ve treated space at all. In this lecture we combine time and space into a single geometric structure: spacetime. The spacetime structures we introduce are affine: structures on affine space. (As with other affine geometric structures, we can transport them to smooth manifolds.)
The classical version is a *Galilean structure*, whose signal feature is simultaneity. This is the natural home for Newton’s equations, which we touch upon only briefly: it is non-relativistic, hence non-realistic, hence non-relevant; its interest for us is mostly historical and pedagogical. (But see [Ar] for applications to Newtonian mechanics.) One of the most important developments of early 20th century physics is Einstein’s introduction of *Minkowski spacetime*, based on a Lorentz metric. This requires the specification of a universal physical constant: the speed of light $c$. (A Galilean spacetime has separate metrics for time and space, whereas a Lorentz metric mixes them; the speed of light is needed to convert from lengths to time intervals.) In the second part of this lecture we develop the linear algebra underpinning: the geometry of a real vector space equipped with a Lorentz metric. Some statements, such as the Cauchy-Schwarz inequality for timelike vectors, are different than in the positive definite case.

Our spacetimes are equipped with a time-orientation. This then orients worldlines that represent motions: there is a *forward* direction. In the dual spacetime the time-orientation induces a notion of *positive energy*, which we saw in Lecture 11 is what enables Wick rotation. Also, typically only time-orientation preserving symmetries of spacetime act as linear symmetries of a quantum mechanical system; time-orientation preserving symmetries of spacetime typically act antilinearly (see Lemma 11.3).

### 12.1 Euclidean space

Fix the dimension of space to be $d \in \mathbb{Z}_{\geq 0}$.

Let $W$ be a $d$-dimensional inner product space, and suppose $E$ is an affine space over $W$. Then $E$ is called a *Euclidean space*. Its symmetry group $O(E)$ is the classical group of Euclidean isometries, and it fits into a group extension

\[(12.1) \quad 0 \longrightarrow W \longrightarrow O(E) \longrightarrow O(W) \longrightarrow 1\]

in which $O(W)$ is the group of linear orthogonal transformations of the vector space $W$. The group $O(E)$ has two components, distinguished by whether a transformation preserves or reverses the orientation of space. Some mechanical systems have the full group $O(E)$ of isometries as symmetries; others only have the subgroup $SO(E)$ of orientation-preserving isometries.

**Example 12.2.**

1. A “free” particle moving in $E$ (with zero potential energy function) as in (1.16) has the full group $O(E)$ of isometries.
2. Recall the particle on a ring (Example 3.56). The second term in the Lagrangian density (3.57) is not invariant under spatial reflection $x \mapsto -x$.

### 12.2 Galilean spacetime

We use the notation $n = d + 1$ for spacetime dimension.

**Definition 12.3.** Let $V$ be a real vector space of dimension $n$. A *Galilean structure* $\Gamma$ on $V$ is:

1. a codimension one subspace $W \subset V$ equipped with an inner product; and
2. an orientation
on $V/W$. A Galilean spacetime $M$ is an affine space over a vector space equipped with a Galilean structure.

Figure 26. Galilean spacetime $M$ and the simultaneity foliation $S$

A Galilean spacetime $M$ is depicted in Figure 26. The orbits of translation by $W \subset V$ on $M$ give a codimension 1 foliation $S$ by cooriented affine subspaces: the simultaneity foliation. Spacetime points on the same leaf represent simultaneous “events” in spacetime. The coorientation, which is induced from the orientation of $V/W$, gives a notion of forward time. The inner product on $W$ induces a Euclidean structure on each leaf, which then is an affine model of space as in the previous subsection. The quotient affine space $M/W$ is an affine model of time, as in §11.1. Note that we do not fix an inner product—a “clock”—on $V/W$; see Remark 11.2. Observe that in a Galilean spacetime $M$ there is no distinguished complement to $W \subset V$, which would give a distinguished time direction in $M$. Let

$$(12.4) \quad \pi : M \longrightarrow M/W$$

be the quotient map.

**Definition 12.5.** Let $M$ be a Galilean spacetime. A worldline in $M$ is a connected 1-dimensional submanifold $C \subset M$ that is transverse to the simultaneity foliation $S$. The worldline has constant velocity if $C \subset M$ is an open interval in an affine line.

As depicted in Figure 26, the quotient (12.4) maps $C$ diffeomorphically onto a submanifold $I \subset M/W$, an open interval. The inverse $\gamma : I \rightarrow C \hookrightarrow M$ is a canonical parametrization of $C$.

The tangent line $U$ at some point of a worldline is a line in $V$ that is transverse to $W$. Lines transverse to $W$ form an affine space over $\text{Hom}(V/W, W)$. Therefore, two affine worldlines have a relative velocity in $\text{Hom}(V/W, W)$. (Observe that vectors in $V/W$ have units of time and vectors in $W$ have units of length, so an element of $\text{Hom}(V/W, W)$ has units of velocity. If we fix an inner product—a clock—on $V/W$, then there is a distinguished nonzero vector in $V/W$ which is oriented an of unit norm, in which case we can identify the velocity as a vector in $W$.) However, there is no absolute velocity of a single worldline. Let $\gamma : I \rightarrow M$ be the canonical parametrization of a worldline. Then $V/W \xrightarrow{d\gamma} V \xrightarrow{d\pi} V/W$ has constant value the identity map, hence $(V/W)^{\otimes 2} \xrightarrow{d^2\gamma}$.
$V \xrightarrow{d\tau} V/W$ is zero. So $d^2\gamma$ factors to a map $(V/W)^{\otimes 2} \to W$, which is the acceleration. In summary, acceleration is defined for a Galilean worldline, whereas velocity is not. (Zero acceleration, or constant velocity, is defined, as in Definition 12.5.) As a consequence, Newtonian mechanics can be formulated in a Galilean spacetime. For example, Newton’s second law $F = ma$ uses the acceleration.

Let $\mathcal{G}^\uparrow(M)$ denote the subgroup of affine symmetries of $M$ that preserve the Galilean structure $\Gamma$. There is a larger group $\mathcal{G}(M)$ of symmetries that preserve the simultaneity foliation and inner products but not necessarily the time-orientation. These groups fit into extensions

$\begin{tikzcd}
1 & V & \mathcal{G}^\uparrow(M) & \mathcal{G}^\uparrow(V) & 1 \\
1 & V & \mathcal{G}(M) & \mathcal{G}(V) & 1
\end{tikzcd}$

If $n \geq 2$ then $\mathcal{G}^\uparrow(M)$ has two components, distinguished by whether a transformation preserves or reverses orientation of space. The group of components of $\mathcal{G}(M)$ is the Klein four-group, isomorphic to the product of two cyclic groups of order two, with the additional distinction that tracks time-orientation preservation or reversal. These symmetries include temporal and spatial translations as well as Euclidean transformations of space. There is also a distinguished subgroup of affine transformations $f: M \to M$ whose differential $df \in \text{Aut}(V)$ induces the identity map on the subspace $W \subset V$ and the identity map on the quotient $V/W$. These are shearing transformations: $df \in \text{Hom}(V/W, W)$ is a velocity that measures the distance the boost boosts in unit time. Figure 27 illustrates the effect of a Galilean boost on a constant velocity worldline.

![Figure 27. A Galilean boost](image)

**Remark 12.7.** The Galilean group is the spin double cover of the identity component of $\mathcal{G}^\uparrow(M)$. It has a nontrivial central extension related to mass in nonrelativistic physics [GS], [DF2, §1.4]. See also [FM1] for more about Galilean geometry and symmetry in nonrelativistic quantum mechanics.

### 12.3 Lorentz geometry

As preparation for our discussion of Minkowski spacetime in the next lecture we discuss the linear geometry of a Lorentz metric.
Definition 12.8. Let $V$ be a real $n$-dimensional vector space. A *Lorentz structure* on $V$ is a nondegenerate symmetric bilinear form

(12.9) \[ \langle -, - \rangle : V \times V \rightarrow \mathbb{R} \]

of signature $(1, n-1)$.

The standard model is $V = \mathbb{R}^{1,n-1}$ with standard basis $e_0, e_1, \ldots, e_{n-1}$ such that

(12.10) \[ \langle e_i, e_j \rangle = \begin{cases} 0, & i \neq j; \\ +1, & i = j = 0; \\ -1, & i = j \neq 0. \end{cases} \]

Definition 12.11. Let $V$ be a Lorentzian vector space with metric $\langle -, - \rangle$.

1. For $\xi \in V$ set $|\xi| = \sqrt{|\langle \xi, \xi \rangle|}$.
2. Set

(12.12) \[ V^> = \{ \tau \in V : \langle \tau, \tau \rangle > 0 \}, \]
 \[ N(V) = \{ \eta \in V : \langle \eta, \eta \rangle = 0 \}, \]
 \[ V^< = \{ \sigma \in V : \langle \sigma, \sigma \rangle < 0 \}. \]

Then $V = V^> \sqcup N(V) \sqcup V^<$.

Vectors in $N(V)$ are called *null* or *lightlike*, those in $V^>$ *timelike*, and those in $V^<$ *spacelike*. Each of these subsets is a cone (i.e., is closed under scalar multiplication by $\mathbb{R}^>0$). In Figure 28 the timelike vectors lie inside the null cone and the spacelike vectors lie outside of it.

![Figure 28. Lorentzian vector spaces](image)

Remark 12.13. The terminology suggests that $\langle \tau, \tau \rangle$ has units of time squared ($T^2$) for a timelike vector $\tau$, whereas $\langle \sigma, \sigma \rangle$ has units of length squared ($L^2$) for a spacelike vector $\sigma$. We choose $\langle -, - \rangle$ to have units of $L^2$, but then need a universal constant to convert from length to time, as we introduce in a Minkowski spacetime.
If \( U \subset V \) is a timelike line, then there is an orthogonal splitting

\[
V \cong U \oplus U^\perp
\]

in which \( U^\perp \) is spacelike: the metric on \( U^\perp \) is negative definite; see Figure 28.

**Proposition 12.15.** Let \( V \) be a Lorentzian vector space of dimension \( n \).

1. \( V^{>0} \) has two components.
2. \( V^{<0} \) is empty if \( n = 1 \), has two components if \( n = 2 \), and is connected if \( n \geq 3 \).
3. \( N(V) \setminus \{0\} \) is empty if \( n = 1 \), has 4 components if \( n = 2 \), and has two components if \( n \geq 3 \).

**Proof.** Fix a timelike vector \( \tau_0 \), and as in (12.14) write vectors in \( V \) as a multiple of \( \tau_0 \) plus an orthogonal spacelike vector. Then define deformation retractions

\[
\begin{align*}
(12.16) \quad f_t(a\tau_0 + \sigma) &= \begin{cases}
((1-t)a + t \text{sign}(a))\tau_0 + (1-t)\sigma, & \text{case (1)}; \\
(1-t)a\tau_0 + \frac{(1-t)|\sigma| + t}{|\sigma|}\sigma, & \text{case (2), } n \geq 2; \\
\frac{a\tau_0 + \sigma}{(1-t) + t|a|}, & \text{case (3), } n \geq 2
\end{cases}
\end{align*}
\]

onto \( S^0 \), \( S^{n-2} \), and \( S^0 \times S^{n-2} \) respectively. Here \( a \in \mathbb{R} \).

**Definition 12.17.** Let \( V \) be a Lorentzian vector space. A choice of component \( P \subset V^{>0} \) is a time-orientation. Vectors in \( P \) are forward timelike. The closure \( \overline{P} \) consists of forward vectors, which are timelike or lightlike.

**Proposition 12.18.** Let \( V \) be a Lorentzian vector space of dimension \( n \geq 2 \). Then the orthogonal group \( O(V) \) has 4 components, distinguished by the action of an orthogonal transformation on the two-element sets \( \pi_0 V^{>0} \) and \( \pi_0 B(V) \), where \( B(V) \) is the space of bases of \( V \).

In other words, we track whether an orthogonal transformation preserves time-orientation and overall orientation. (In place of the latter we often track space-orientation).

**Proof.** For \( V = \mathbb{R}^{1,1} \) with metric \( \langle e_0, e_0 \rangle = 1 \), \( \langle e_1, e_1 \rangle = -1 \), \( \langle e_0, e_1 \rangle = 0 \), an orthogonal transformation has the form

\[
(12.19) \quad \begin{pmatrix}
\epsilon_1 \cosh \varphi & \epsilon_2 \sinh \varphi \\
\epsilon_1 \sinh \varphi & \epsilon_2 \cosh \varphi
\end{pmatrix},
\]

where \( \varphi \in \mathbb{R} \) and \( \epsilon_i = \pm 1 \). For \( n > 2 \) we proceed by induction. Let \( \dim V > 2 \). The subgroup \( \text{SO}^+(V) \) of transformations that act trivially on both two element sets acts transitively on the connected space \( \{\sigma \in V : |\sigma|^2 = -1\} \); the stabilizer subgroup at \( \sigma \) is isomorphic to \( \text{SO}^+(\mathbb{R}\langle \sigma \rangle^\perp) \), which is connected by induction. (Note that \( \mathbb{R}\langle \sigma \rangle^\perp \) is Lorentzian.)

The next proposition is analogous to the usual Cauchy-Schwarz and triangle inequalities in a Euclidean vector space, but with an important difference: the inequalities are reversed.
Proposition 12.20. Let $\tau, \tau'$ be forward vectors in a time-oriented Lorentzian vector space. Then

(i) $\langle \tau, \tau' \rangle \geq |\tau| |\tau'|$; and
(ii) $\tau + \tau'$ is a forward vector, and $|\tau + \tau'| \geq |\tau| + |\tau'|$.

Equality holds if and only if $\tau, \tau'$ are proportional.

Proof. Choose $\tau_0$ forward timelike with $|\tau_0| = 1$. Write $\tau = a\tau_0 + \zeta$ and $\tau' = a'\tau_0 + \zeta'$, where $\langle \tau_0, \zeta \rangle = \langle \tau_0, \zeta' \rangle = 0$. Then $a, a' > 0$, $a \geq |\zeta|$, and $a' \geq |\zeta'|$. Thus by the ordinary Cauchy-Schwarz inequality $aa' \geq |\zeta| |\zeta'| \geq -\langle \zeta, \zeta' \rangle$, from which $\langle \tau, \tau' \rangle = aa' + \langle \zeta, \zeta' \rangle \geq 0$ with equality if and only if $\tau, \tau'$ are lightlike and proportional. Now the affine function $t \mapsto \langle \tau + t\tau', \tau_0 \rangle$ is not identically zero, so for some $t_0$ we have $\langle \tau + t_0\tau', \tau_0 \rangle = 0$. Then either $\tau + t_0\tau' = 0$ or $\tau + t_0\tau'$ is spacelike. In the former case we are done, so assume the latter. Then the quadratic function $t \mapsto \langle \tau + t\tau', \tau + t\tau' \rangle$ has a strictly negative minimum, so its discriminant $\langle \tau, \tau' \rangle^2 - \langle \tau, \tau \rangle \langle \tau', \tau' \rangle$ is strictly positive. This proves (i). The deduction of (ii) from (i) is as in the Euclidean case: expand $\langle \tau + \tau', \tau + \tau' \rangle$. □

Definition 12.21. Suppose $\tau, \tau'$ are forward timelike vectors in a time-oriented Lorentzian vector space. The hyperbolic angle $\varphi$ between $\tau$ and $\tau'$ is defined by

\begin{equation} \cosh \varphi = \frac{\langle \tau, \tau' \rangle}{|\tau| |\tau'|}. \end{equation}

Figure 29. A Lorentzian vector space $V$ and its dual $V^*$, with forward timelike ($P \subset V$) and positive energy ($P^* \setminus \{0\} \subset V^*$) vectors colored. Regions are labeled by units: time (T), length (L), energy (E), and momentum (p)

A nondegenerate symmetric bilinear form on a vector space $V$ induces a canonical isomorphism

\begin{equation} \phi: V \xrightarrow{\cong} V^* \end{equation}

and, by transport, a nondegenerate symmetric bilinear form on $V^*$. Therefore, the dual to a Lorentzian vector space $V$ is a Lorentzian vector space $V^*$. The dual to time (T) has units of $T^{-1}$, which is frequency, and the dual to length (L) has units of $L^{-1}$, which is wave number. In a quantum theory we have Planck’s constant $\hbar$ available; see (1.25). Multiplying by $\hbar$ we see that the dual
to time has units $ML^2/T^2$ of energy, and the dual to length has units $ML/T$ of momentum. Thus a point of $V^*$ is sometimes called an energy-momentum (vector). A time-orientation $P \subset V^{>0}$ induces the same structure $P^* \subset (V^*)^{>0}$. Nonzero vectors in $P^*$ are said to have positive energy. See Figure 29 for an illustration.

**12.4 Spin groups**

The spin group is a double cover of the identity component of an orthogonal group. For a finite dimensional real vector space equipped with a definite bilinear form, the orthogonal group is compact and consists of two components. The identity component for the standard model is the special orthogonal group $SO_n$, and for $n \geq 3$ the fundamental group is cyclic of order 2. In that case the spin group $Spin_n$ is a simply connected compact Lie group. The fundamental group of $SO_2$ is infinite cyclic, and $Spin_2$ is a connected double covering group. The group $SO_1$ consists only of the identity element, so $Spin_1 = \mu_2 = \{\pm 1\}$ is the cyclic group of order 2. In Lorentz signature $(1, n-1)$ the identity component $SO^1_{1,n-1}$ of the orthogonal group is noncompact if $n \geq 2$; it deformation retracts onto its maximal compact subgroup is $SO_{n-1}$. In low dimensions there are exceptional isomorphisms among Lie groups that allow us to identify the compact and Lorentz spin groups in terms of other Lie groups:

\[
\begin{array}{c|c|c}
 n & Spin_n & Spin_{1,n-1} \\
1 & \mu_2 & \mu_2 \\
2 & T & \mathbb{R}^{>0} \times \mu_2 \\
3 & SU_2 & SL_2 \mathbb{R} \\
4 & SU_2 \times SU_2 & SL_2 \mathbb{C} \\
5 & Sp_2 & Sp_{1,1} \\
6 & SU_4 & SL_2 \mathbb{H} \\
\end{array}
\]

(12.24)

**12.5 The speed of light**

A Galilean structure (Definition 12.3) equipped with a “clock” has two inner products: one for time $(V/W)$ and one for space $(W)$. They have separate units of time and length. The cornerstone of special relativity is the merging of these two inner products into a single entity: a Lorentz metric. It gives measurements in units of length, say, and therefore we need a “mechanism” to convert to units of time. That mechanism is a universal constant of nature: the speed of light $c$. The constancy of the speed of light was Einstein’s starting point for his special theory of relativity. The constant $c$ has units of speed, i.e., of length over time: $[c] = \frac{L}{T}$. In a quantum theory there is another universal constant, $\hbar$, and $[\hbar] = \frac{ML^2}{T}$. In a relativistic quantum theory we have both universal constants, and use them to convert

\[
L \sim T \sim \frac{\hbar/M^2}{c^2} \sim M^{-1}
\]

(12.25)
**Bonus Lecture: Symmetry and families**

Two powerful techniques that permeate mathematics are the use of symmetry and of parametrized families. Indeed, as we will see the former is a special case of the latter via the *equivariance* → *families* principle. Both are powerful problem solving techniques, and more fundamentally they capture important aspects of structure. Not surprisingly, this is true for quantum systems as well.

We begin with a few examples of symmetry and families in quantum mechanics. Then, as background and perspective, we meander lightly over their occurrence in several mathematical situations. Some of our examples resonate with ideas in quantum theory, such as phase transitions. We explain in an elementary example how to convert a situation with symmetry into a parametrized family.

We conclude the lecture with mathematical background on fiber bundles, groupoids, and stacks.

**B.1 Examples in quantum mechanics**

**Example B.1** (symmetry in a QM system). Recall the simple harmonic oscillator, which was introduced in Lecture 5. The classical theory describes a particle on the Euclidean line $\mathbb{E}^1$ with potential function $V(x) = x^2/2$. The classical Hamiltonian for a motion $\gamma: \mathbb{R} \to \mathbb{E}^1$ is

$$H = \frac{1}{2}(\dot{\gamma}^2 + \gamma^2),$$

and the classical phase space is a plane. The motion via concentric circles is depicted in Figure 12. Evidently, the rotation group $O_2$ acts as symmetries of this system. Notice that the identity component $SO_2$ acts by time-preserving transformations, whereas reflections—which comprise the off-component of $O_2$—act by time-reversing transformations.

For the quantum harmonic oscillator, we take the state space $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$ and Hamiltonian

$$H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2\right),$$

though based on the analysis in Lecture 5 we can give an abstract description. As we will show, the group $O_2$ acts on this quantum theory, but we must understand quantum theory as *projective* as explained in Lecture 8. Later in this lecture we address the question: Is there a linearization of this projective action, i.e., a lift to a linear action on $\mathcal{H}$?

The next examples are of families in quantum mechanics. First, we indicate how we encounter a family in a fixed quantum mechanical (QM) system. Then we give examples of families of quantum systems. The distinction between these two paradigms is sometimes illusory; see the remark at the end of Example B.6 and also Example B.17.

**Example B.4** (families in a fixed system). Suppose $(\mathcal{H}, H)$ defines a QM system. Recall the definition of correlation functions from the end of Lecture 5. For example, fix a unit norm vector $\psi \in \mathcal{H}$, which determines a pure state, and fix times $t_0 < t_1 < t_2 < t_f$ and observables $A_1, A_2$. Then the 2-point correlation function

$$f(t_1, t_2) = \langle \psi, U_{t_f-t_2} A_2 U_{t_2-t_1} A_1 U_{t_1-t_0} \psi \rangle$$
as a function of $t_1, t_2$ is an example of a family of computations in a single theory.

**Example B.6** (families of QM systems). A system may have parameters—in quantum field theory they are often called *coupling constants*—which we allow to vary, and thus embed the system in a family. For example, the simple harmonic oscillator with mass $m$ and spring constant $k$ has Hamiltonian

\[(B.7) \quad H_{m,k} = \frac{1}{2}(m\dot{\gamma}^2 + k\gamma^2).\]

This defines a family of systems with parameter space $(\mathbb{R}^0)^2$.

In quantum field theory it is oft fruitful to turn a coupling “constant” into a scalar field. In this way one can consider a single system with these scalar fields, or a family of system parametrized by the values of these scalar fields. (A classical analog appears in **Example B.17** below.)

**Figure 30.** Phases of $H_2O$

**Example B.8** (another family of quantum systems). The chemical compound $H_2O$ can be described as a quantum system with parameters of temperature and pressure, as illustrated in Figure 30. We return to this picture later in the lecture.

**B.2 Examples of families in mathematics**

We give simple, but illustrative, examples from different parts of mathematics.

**Example B.9** (algebraic geometry). Even for a simple arithmetic equation, such as

\[(B.10) \quad x^4 + y^4 - z^4 = 0, \quad x, y, z \in \mathbb{Z},\]
it is fruitful to fiber over $\text{Spec}(\mathbb{Z})$, i.e., to consider the equation over finite fields $\mathbb{F}_p$, $p$ prime. In complex geometry, or in algebraic geometry more generally, one encounters nice families of varieties: flat families. Consider a deformation of a singularity, as in the equation

(B.11) \[ y^2 - x^2 = \epsilon. \]

Here $x, y \in \mathbb{C}$, and for each fixed $\epsilon \in \mathbb{C}$ the equation (B.11) describes a curve in the complex affine $x, y$-plane. The curve is a nonsingular conic if $\epsilon \neq 0$, and it is a union of two lines if $\epsilon = 0$. Alternatively, (B.11) defines a surface in complex affine 3-space with coordinates $x, y, \epsilon$. The surface projects to the complex affine $\epsilon$-line, and the fibers of the projection are the plane curves just introduced; see Figure 31.

![Figure 31. A family of conics](image)

**Example B.12** (algebra). Define

(B.13) \[ A = \mathbb{C}[x, \epsilon]/(x^2 - \epsilon). \]

If $\epsilon$ is a fixed complex number, then (B.13) is a family of algebras $A_\epsilon$ parametrized by $\epsilon \in \mathbb{C}$. For $\epsilon \neq 0$ the algebra $A_\epsilon$ is isomorphic to $\mathbb{C} \times \mathbb{C}$; for $\epsilon = 0$ it is the nonsemisimple algebra $\mathbb{C}[x]/(x^2)$. If we regard $\epsilon$ as an indeterminate, then (B.13) defines a single algebra which is also a module over the polynomial algebra $\mathbb{C}[\epsilon]$. Pass to a geometric picture by defining spaces on which these algebras are the algebras of complex-valued functions: the planar conic $\{x^2 - \epsilon = 0\}$ in the affine $(\epsilon, x)$-plane projecting to the $\epsilon$-line, as depicted in Figure 32. The fiber over $\epsilon \neq 0$ is a set of two distinct points, whereas the fiber over $\epsilon = 0$ is a double point.

**Example B.14** (topology). There are many notions of a family of topological spaces. The nicest is a fiber bundle, in which the homeomorphism type of fibers is locally constant on the base. We emphasize that a fiber bundle is a property or condition on a map $\pi: X \to S$; it is not extra structure or data. (See Definition B.25 below.) There are weaker properties of a continuous map which also tell that its fibers vary “nicely” in some weaker, often homotopical, sense: fibration, Serre fibration, and quasifibration.
Example B.15 (differential topology). The nicest notion is that of a smooth fiber bundle. The fibers of a submersion are submanifolds, but the diffeomorphism type can jump if the fibers are not all compact. (A proper submersion is a fiber bundle.) Here is an example.

Work in the affine space $\mathbb{A}^3$ with coordinates $(x, y, z)$. Define

$$X = \{ (x, y, z) \in \mathbb{A}^3 : y^2 + z^2 = 1 \} \setminus \{ (0, 0, +1), (0, 0, -1) \}.$$ 

This is a cylinder with two points $n, s$ deleted. Let $P$ denote the space of affine planes in $\mathbb{A}^3$ which contain the $z$-axis; then $P$ is diffeomorphic to $\mathbb{R}P^1$, the space of lines through the origin in the $x, y$-plane. Define $\pi : X \rightarrow P$ to be the map which takes $p \in X$ to the plane containing the distinct non-collinear points $n, s, p$, as depicted in Figure 33. Then $\pi$ is a surjective submersion. However, $\pi$ is not a fiber bundle. The typical fiber of $\pi$ is an ellipse minus the points $n, s$, whereas the fiber over the $x, z$-plane $\Pi_{x,z}$ is the union of two affine lines minus $n, s$. The former has two components while the latter has four components. The special fiber over $\Pi_{x,z}$ is not diffeomorphic to the other fibers, and therefore $\pi$ cannot be locally trivial at $\Pi_{x,z}$. Variation: compactify the generic fiber by blowing up the points $n, s$ into circles to obtain a new submersion whose generic fiber is a single circle and whose special fiber is two lines.

Example B.17 (differential equations). First, ordinary differential equations (ODEs). A vector field $\xi$ on a smooth manifold $M$ determines the first-order ODE

$$\dot{\gamma}(t) = \xi(\gamma(t)).$$
for a motion $\gamma: \mathbb{R} \to M$. (Assume $M$ is compact or $\xi$ is suitably bounded so that integral curves exist for infinite time.) An initial condition is $\gamma(0) = m, m \in M$, so we obtain a family of initial value problems parametrized by $M$.

Consider now a family of vector fields $\xi(s), s \in S$, for $S$ a smooth manifold. This defines a family of ODEs parametrized by $S$. Consider the vector field $\eta(s, m) = (0, \xi(s)(m))$ on the Cartesian product manifold $S \times M$. The ODE at a particular $s$ is solved by integral curves of $\eta$ on $\{s\} \times M$. More generally, if $\sigma: \mathbb{R} \to S$ is a “scalar field” with values in $S$, there is an ODE

\begin{equation}
\dot{\gamma}(t) = \xi(\sigma(t))(\gamma(t))
\end{equation}

which we solve as follows: pull back $\eta$ via $\sigma \times \text{id}_M: \mathbb{R} \times M \to S \times M$ to a vector field $\eta_\sigma$ on $\mathbb{R} \times M$, and construct integral curves of $\partial/\partial t + \eta_\sigma$. This is a classical dynamical systems variant of a maneuver in quantum field theory.

The continuity method (or method of continuity) for partial differential equations [GT] is another instance of families. Consider a PDE $\mathcal{E}$ that one wants to prove has solutions. The technique is to embed it into a family $\mathcal{E}_t, t \in [0, 1]$, in which $\mathcal{E}_1 = \mathcal{E}$ and existence for $\mathcal{E}_0$ is straightforward. Then

\begin{equation}
\{t \in [0, 1] : \mathcal{E}_t \text{ has a solution}\}
\end{equation}

is proved to be open by an implicit function theorem argument. The art comes in proving that (B.20) is also closed, which usually requires delicate \textit{a priori} estimates.

### B.3 Moduli spaces

Families often arise from considering (equivalence classes of) mathematical objects of a specified type. Here are a few examples.

**Example B.21.** Fix a vector space $V$. Consider the family of lines $L \subset V$. The parameter space for lines is the projective space $\mathbb{P}V$. Furthermore, there is a universal line bundle $\mathcal{L} \to \mathbb{P}V$ whose fiber at $L \to \mathbb{P}V$ is canonically identified with the line $L \subset V$. From that description, it is clear that $L \to \mathbb{P}V$ comes as a subbundle $L \subset V$ of the constant vector bundle $V \to \mathbb{P}V$ with fiber $V$.

One can generalize to the moduli space of subspaces of $V$ of any fixed dimension. The parameter space is the Grassmannian, and again there is a universal vector bundle over the Grassmannian, and it is embedded in the constant vector bundle with fiber $V$.

This example illustrates that in many moduli problems one fixes \textit{discrete parameters}, here the dimension of a subspace.

Important moduli spaces in geometry include the moduli space of algebraic curves of a fixed genus, the moduli space of algebraic surfaces with fixed topological invariants, etc.

The following example from differential geometry is a geometric cousin of moduli spaces of QM systems.

**Example B.22** (1-dimensional Riemannian manifolds). Consider the heuristic “moduli space of 1-dimensional spaces with metric data”. One interpretation is the moduli space of smooth 1-dimensional Riemannian manifolds. To begin, fix the diffeomorphism type to be a circle. Then the
Riemannian structure has a single invariant: the total length. (This follows using parametrization by arclength.) So the moduli space is $\mathbb{R}^+\rangle$ and the universal Riemannian manifold $X$ is a cone minus the cone point. The resulting fiber bundle is depicted in Figure 34. There is a similar picture for the diffeomorphism type $(S^1)^{\mathbb{Z}_2}$ of two disjoint circles: the moduli space is $(\mathbb{R}^+angle)^2$.

Are there more general “moduli spaces of 1-dimensional spaces with metric data” in which a single circle can be connected to two circles by a smooth path? Figure 35 illustrates two ways in which one circle can morph into two circles. The first mechanism passes through a compact

---

35This should be a Riemannian fiber bundle, but what is a Riemannian structure on a fiber bundle? It must generalize a Riemannian structure on a single Riemannian manifold—a Riemannian manifold over a point. Over a general base $S$, a Riemannian structure on a fiber bundle $\pi: X \to S$ should certainly include an inner product on the relative tangent bundle $T(X/S) = \ker \pi_\ast \to X$. But that is not sufficient to produce a parametrized Levi-Civita covariant derivative. For that one also needs a horizontal distribution on the fiber bundle $\pi$: a subbundle of $TX \to X$ transverse to the relative tangent bundle. A horizontal distribution—compatible with other structures present—is usually required on a fiber bundle equipped with differential geometric data.

36 Moduli of 1-dimensional spaces embedded into a Euclidean plane might be a route to tighten these heuristics.
singular space, the figure 8. The second scenario is essentially that in Example B.15; it contains only smooth spaces, but passes through a noncompact manifold. The first scenario is analogous to a first-order phase transition in quantum mechanics. The second is analogous to a higher-order phase transition. Note in the second that the spectrum of the Laplace operator is pure point off of the noncompact fiber of the family. At the noncompact fiber the spectral gap disappears, and it is continuous spectrum which has violated the gap. One might imagine that in the first scenario the gap is violated by an additional eigenvalue which comes down to the minimum.

The general lesson is that a moduli space $\mathcal{M}$ may contain a singular locus $\Delta \subset \mathcal{M}$, and often the moduli space of interest is $\mathcal{M} \setminus \Delta$. In Example B.22 we do not pin down a moduli space $\mathcal{M}$ which includes noncompact 1-manifolds, but one can imagine such a space to be path connected via processes analogous to the second one in Figure 35. (That one may get a connected moduli space when allowing certain “surgery-like” processes is familiar in other contexts. For example, Reid [Re] conjectured such a connectedness statement for the moduli of Calabi-Yau 3-folds.) The moduli space $\mathcal{M} \setminus \Delta$ of closed smooth Riemannian 1-manifold has a nontrivial set of components in bijection with $\mathbb{Z}_{\geq 0}$. In fact, the isomorphism $\pi_0(\mathcal{M} \setminus \Delta) \xrightarrow{\sim} \mathbb{Z}_{\geq 0}$ takes a manifold $M$ to the cardinality of $\pi_0 M$. This is typical: information about the homotopy type of the moduli space $\mathcal{M}$ (or $\mathcal{M} \setminus \Delta$) can often be expressed in terms of topological invariants of the objects it parametrizes.

Another general lesson. There are two general types of “questions” we might ask and express in terms of a moduli space. Geometric questions lead to functions which are not locally constant. In Example B.22 the total length of a closed Riemannian 1-manifold is such a function. Topological questions are those that only depends on the underlying homotopy type of the moduli space. The first such query is the set of deformation classes of parametrized objects, which is the set of path components of the moduli space. That remembers coarse structure, in Example B.22 the number of circles (rather than their total length).

![Figure 36. The moduli space of ordered pairs of real numbers](image)

**Example B.23.** A simple illustration of these general features is the moduli space $\mathcal{M} \cong \mathbb{R}^2$ of ordered pairs $(x_1, x_2)$ of real numbers. One might consider the diagonal $\Delta \subset \mathcal{M}$ of pairs $(x, x)$, $x \in \mathbb{R}$, to be singular, as in Figure 36. This is the “gapless” subspace on which the distance between the points on the real line vanishes. Thus $\mathcal{M} \setminus \Delta$ parametrizes gapped pairs. A geometric
function is \( x_1 - x_2 : \mathcal{M} \setminus \Delta \to \mathbb{R} \neq 0 \). The composition with \( \mathbb{R} \neq 0 \to \pi_0 \mathbb{R} \neq 0 \) factors to an isomorphism \( \pi_0 (\mathcal{M} \setminus \Delta) \to \pi_0 \mathbb{R} \neq 0 \) which only remembers the ordering of the points.

### B.4 Symmetry as families

The cyclic group \( C_2 \) of order 2 acts on \( \mathcal{M} \) by exchanging the order of the points. The action is reflection about the diagonal \( \Delta \subset \mathcal{M} \), as depicted in Figure 37. The fixed point set is \( \Delta \), and the action is free on \( \mathcal{M} \setminus \Delta \). The quotient space is contractible. However, the quotient space loses information about the stabilizer subgroups at points where the action fails to be free. For example, consider the path in \( \mathcal{M} \) in Figure 37; it is a loop in the quotient. One can imagine the path as moving in a double cover of the quotient, and as it crosses the fixed point set \( \Delta \) it should jump sheets. In other words, the lift of the loop in the quotient to the double cover is not a closed loop. This indicates that the quotient should have nonzero fundamental group, so in particular cannot be contractible.

![Figure 37. The \( C_2 \)-action on \( \mathcal{M} \); a loop in the quotient](image)

There is a quotient construction which captures this information: the action groupoid or, better, the quotient stack. See §B.6.2 for an introduction to groupoids and stacks. The intuitive idea of the action groupoid is that if \( X \) is a space equipped with the action of a group \( G \), then instead of identifying points \( x \) and \( g \cdot x \) for \( x \in X, \ g \in G \), we adjoin a morphism from \( x \) to \( g \cdot x \). The morphism is labeled by the group element, and composition of morphisms follows the group law in \( G \). We use the notation \( 'X//G' \) for the action groupoid. Notice that in the example at hand, there are nonidentity automorphisms of points of \( \Delta \). A stack is the invariant object underlying a groupoid, much as a smooth manifold is the invariant object underlying an open cover and its gluings. (In fact, the latter is an example of how a groupoid determines a stack.) One can define groupoids and stacks for discrete sets, topological spaces, and smooth manifolds (and beyond). We often implicitly use the smooth version. Associated to a groupoid or stack is a topological space, its geometric realization, and so too is associated a homotopy type.

The quotient groupoid \( *//G \) of a group \( G \) acting on a point \( * \) is a rigid version of the classifying space of a group. Its geometric realization has the usual homotopy type of the classifying space \( BG \).

If \( G \subset X \) is a space with a left group action, then there is a fiber bundle

\[
X//G \longrightarrow *//G
\]

(B.24)
of stacks, and the fiber over the basepoint $* \hookrightarrow *//G$ is the space $X$. This is the strong form of a group action: an object\footnote{The picture is quite general: the object may be a (higher) category in place of a topological space.} fibered over the classifying space $*//G$. This is the equivariance $\rightarrow$ families principle that sees objects with symmetry as special cases of parametrized families.

**B.5 A moduli space in quantum mechanics**

After this long digression we return to physics, specifically to the family of QM systems pictured in Figure 30. This is the phase diagram of $H_2O$. Now we understand it as a moduli space $\mathcal{M}$ of quantum systems. One can ask geometric questions, which often lead to functions on $\mathcal{M}$, such as correlation functions as functions of temperature and pressure. But one can also ask topological questions, such as: What is the set of phases? Here the phase of a quantum system is defined to be a path component, and so the set of phases is $\pi_0$ of the moduli space. This moduli space is manifestly contractible, so there seem to be no interesting phases. However, there is a singularity locus $\Delta$ in this problem, namely the locus of points in $\mathcal{M}$ at which the quantum system is gapless. (Recall the basic dichotomy of Definition 5.7(2).) In Figure 30 these points are on the dark black curve, which extends to the origin and off to infinite pressure past the point D in the figure. The other branch terminates in a “critical point” C. These gapless points are where the system undergoes a phase transition. Here the transition is first-order except at the point C. The first-order and higher-order phase transitions are analogous to the two geometric transitions in Figure 35. The set $\pi_0(\mathcal{M}\setminus\Delta)$ of phases has cardinality 2. Note, however, if we restrict to standard room pressure (which is 101 kPa in the diagram), then there are the 3 standard phases of our everyday experience: solid, liquid, gas.

There are similar families in other parts of quantum theory: quantum field theory and string theory, for example. One usually fixes discrete parameters, such as dimension and symmetry type. The set of phases is a macroscopic quantity, as topological quantities (independent of scale) tend to be.

**B.6 Mathematical background**

**B.6.1 Fiber bundles.** The following, which is due to Steenrod [Ste] can be given in the topological or smooth context. For definiteness we work in the smooth category.

**Definition B.25.** A smooth map $\pi: X \rightarrow S$ of smooth manifolds is a fiber bundle if for all $s \in S$ there exists an open neighborhood $U \subset S$ of $s$ and a diffeomorphism which fits into the commutative diagram

\[
\begin{array}{ccc}
U \times \pi^{-1}(s) & \stackrel{\sim}{\longrightarrow} & \pi^{-1}(U) \\
pr_1 \downarrow & & \downarrow \pi \\
U & \rightarrow & \\
\end{array}
\]

(B.26)

We might also require that $\pi$ be surjective. Notice that the only data in Definition B.25 is the smooth map $\pi$. The definition is a condition on a smooth map, as are the definitions of an immersion, embedding, submersion, etc.
B.6.2 Groupoids and stacks. I have gone on too long in these notes, so I will simply give references. You will find a leisurely exposition of categories, classifying spaces, groupoids, etc. in the course notes [F4]. I also recommend [FHT1, §§A.1–A.2], which emphasizes the topological case. For smooth groupoids, called Lie groupoids, and the associated stacks see [BX]. I also recommend the expository article [We1].

Lecture 13: The geometry of Minkowski spacetime

In the first part of the lecture we consider the geometry of special relativity, whose home is Minkowski spacetime. A Minkowski spacetime is a Lorentz version of Euclidean space, and it is also equipped with a time-orientation and a speed of light $c \in \mathbb{R}^{>0}$. We study worldlines of particles, for which we define acceleration as the second fundamental form. We also define their energy-momentum, which lives in the dual space to translations. The limit as $c \to \infty$ of a Minkowski spacetime is a Galilean spacetime, as we briefly show. (See [DF2, §1.4] for a more generous exposition.) We also introduce the Poincaré group, which is a double cover of the identity component of isometries of Minkowski spacetime.

13.1 Minkowski spacetime

Relativistic spacetime is an affine space over a Lorentzian vector space. We include as well the speed of light and a time-orientation.

Definition 13.1. Let $V$ be a Lorentzian vector space. A Minkowski spacetime $(M, P, c)$ over $V$ is an affine space $M$ over $V$, a time-orientation $P \subset V^{>0}$, and positive real number $c > 0$.

Figure 38. A Minkowski spacetime $M$ over a Lorentzian vector space $V$

See Figure 38 for a depiction. The standard example $\mathbb{M}^n$ is standard affine space $\mathbb{A}^n$ with standard affine coordinates $t, x^1, \ldots, x^{n-1}$ and the translationally invariant metric

\[
\begin{equation}
\begin{aligned}
c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - \cdots - (dx^{n-1})^2.
\end{aligned}
\end{equation}
\]
Write $x^0 = ct$. Forward timelike vectors $\tau \in \mathbb{R}^{1,n-1}$ are required to have positive $\partial/\partial t$ component. We typically abbreviate $(M, P, c)$ as $M$.

**Definition 13.3.** Let $M$ be a Minkowski spacetime over a Lorentzian vector space $V$. Suppose $p, q \in M$. Write $q = p + \xi$ for $\xi \in V$.

(i) Then we say that $p, q$ are timelike/lightlike/spacelike separated according as $\xi$ is timelike/lightlike/spacelike.

(ii) If $p, q$ are spacelike separated, define the distance between $p$ and $q$ as

\begin{equation}
\text{dist}(p, q) = \sqrt{-\langle \xi, \xi \rangle}.
\end{equation}

(iii) If $p, q$ are timelike separated, define the proper time between $p$ and $q$ as

\begin{equation}
\text{ptime}(p, q) = \frac{1}{c} \sqrt{\langle \xi, \xi \rangle}.
\end{equation}

The proper time between two points (spacetime events) on the path is the elapsed time an observer traveling on this path would measure on his clock. Consider a triangle $p, q, r$ such that any pair of vertices are timelike separated. Then the reverse triangle inequality in Proposition 12.20 implied that

\begin{equation}
\text{ptime}(p, q) + \text{ptime}(q, r) < \text{ptime}(p, r)
\end{equation}

unless $q$ lies on the affine line connecting $p$ and $r$. This is known as the twin paradox: an evil twin traveling in a spaceship from $p$ to $q$ to $r$ ages less than the hapless chap traveling the straight and narrow from $p$ to $r$.

The following is the Minkowski spacetime analog of Definition 12.5.

**Definition 13.7.** Let $M$ be a Minkowski spacetime over a Lorentzian vector space $V$. A worldline is a connected 1-dimensional submanifold $C \subset M$ such that $T_p C \subset V$ is timelike for all $p \in C$. The worldline has constant velocity if $C \subset M$ is an open interval in an affine line.

A constant velocity motion is sometimes called an inertial path or inertial motion.

![Figure 39. A worldline $C$ in a Minkowski spacetime $M$](image)
Lemma 13.8. Up to time translation, a worldline has a unique parameterization \( s \mapsto p(s) \) such that \( |\dot{p}(s)| = c \) for all \( s \) and \( \dot{p}(s) \) is forward timelike.

The proof is essentially the same as that of unit speed parametrizations of motions on Euclidean space. The parameter \( s \) is called the proper time along \( C \); it is determined up to shifts by a constant.

For \( C \subset M \) a worldline, let \( U_p = T_p C \subset V \) denote the tangent line at \( p \in C \). Then \( U_p \) is a timelike line in \( V \); it is independent of \( p \) for inertial paths. Let \( \tau_p \in U_p \) denote the forward vector of proper time \( |\tau_p| = c \). Note that \( \tau_p = \dot{p}(s) \) for any distinguished parametrization, where \( p(s) = p \). This is not a velocity. (Recall from the discussion following Definition 12.5 that worldlines in Galilean spacetimes do not have a velocity either.) The second derivative of a distinguished parametrization satisfies \( \ddot{p}(s) \in U_p \perp \). The subspace \( U_p \perp \subset V \) is spacelike, and \( \ddot{p}(s) \) is the acceleration of the worldline at \( p = p(s) \). Note that it vanishes identically iff the motion is inertial.

The following is a preliminary to an intrinsic definition of acceleration. Let \( M \) be an affine space over a vector space \( V \), and suppose \( C \subset M \) is a submanifold of dimension \( k \). Let \( \text{Gr}_k(V) \) be the Grassmannian of \( k \)-dimensional subspaces of \( V \). Define the Gauss map

\[
\Gamma: C \longrightarrow \text{Gr}_k(V)
\]

\[
p \mapsto T_p C
\]

The Gauss map is constant iff \( C \) lies in an affine subspace of dimension \( k \). If \( U \in \text{Gr}_k(V) \), then there is a canonical isomorphism \( T_U \text{Gr}_k(V) \cong \text{Hom}(U,V/U) \). Use it to write the differential of the Gauss map at \( p \in C \) as a linear map

\[
d\Gamma_p: T_p C \longrightarrow \text{Hom}(T_p C,V/T_p C).
\]

Equivalently, this is a bilinear map

\[
\Pi_p: T_p C \times T_p C \longrightarrow V/T_p C.
\]

Lemma 13.12. The bilinear form \( \Pi_p \) is symmetric.

We leave the proof to the homework. The symmetric bilinear form \( \Pi_p \) is the second fundamental form of \( C \subset M \) at \( p \). It is typically defined when \( M \) is a Euclidean space, in which case we identify the quotient \( V/T_p C \) with the normal space \( T_p C \perp \subset V \). In this form it generalizes to submanifolds of Riemannian manifolds.

Return now to a worldline \( C \subset M \) in a Minkowski spacetime. The second fundamental form at \( p \in C \) is a linear map

\[
\alpha_p: U_p^{\otimes 2} \rightarrow U_p \perp,
\]

where the Lorentz metric identifies the quotient \( V/U_p \) with the spacelike hyperplane \( U_p \perp \subset V \). Then \( \alpha_p \) is the (intrinsic) acceleration. Observe that it has units \( L/T^2 \) since \( U_p \) has units of time and \( U_p \perp \) has units of length. We leave the reader to reconcile this definition with the one above using a distinguished parametrization.

If we attach a mass \( m \) to a worldline, then we can define its energy-momentum. Recall (12.23) that the Lorentz metric determines an isomorphism \( \varphi: V \xrightarrow{\cong} V^* \).
**Definition 13.14.** Let $M$ be a Minkowski spacetime over a Lorentzian vector space $V$, and let $C \subset M$ be the worldline of a particle with (rest) mass $m$. Then the energy-momentum of $(C, m)$ is

\begin{equation}
\mu: C \rightarrow V^*
\quad p \mapsto m\phi(\tau_p)
\end{equation}

where $\tau_p \in T_pC \subset V$ is the forward timelike vector with $|\tau_p| = c$.

Note that $\phi$ is an isometry, hence the norm square of $\mu(p)$ is the constant $m^2c^2$. At $p \in C$ we have the instantaneous time-space splitting $V = U_p \oplus U^p_\perp$, and so a decomposition $\mu(p) = E\tau + \vec{p}$ into energy plus momentum. The norm square is then

\begin{equation}
E^2/c^2 - |\vec{p}|^2 = m^2c^2.
\end{equation}

This is the famous equation of Einstein.

**Remark 13.17.** The map $\phi$ is structural; it has no units. The vector $\tau_p$ has units of velocity $(L/T)$, since we normalized it via $|\tau_p| = c$. Hence $\mu(p)$ in (13.15) has units of momentum $(ML/T)$.

The energy-momentum has constant norm, interpreted as the mass, and it has positive energy so lies in the cone $P^* \subset V^*$ of positive norm square vectors that evaluate positively on vectors in $P \subset V$.

**Figure 40.** The mass shell (mass hyperboloid) $O_m$

**Definition 13.18.** The **mass shell** or **mass hyperboloid** is the subset of positive energy vectors of fixed norm:\footnote{Recall that the norm on $V^*$ has units of momentum.}

\begin{equation}
O_m = \{ \theta \in P^*: |\theta| = mc \}.
\end{equation}

The mass shell is depicted in Figure 40. The image of a timelike line in $V$ under $\phi: V \rightarrow V^*$ intersects $O_m$ in a single point, and this intersection defines a diffeomorphisms of the manifold of timelike lines in $V$ with $O_m$.  

\footnote{Recall that the norm on $V^*$ has units of momentum.}
13.2 The nonrelativistic limit

Consider the family of standard Minkowski spacetimes \((\mathcal{M}^n, c)\) parametrized by \(c \in \mathbb{R}^>0\). The nonrelativistic limit is \(c \to \infty\). The metric (13.2) blows up, but the inverse metric

\[
(13.20) \quad \frac{1}{c^2} \left( \frac{\partial}{\partial t} \right)^2 - \left( \frac{\partial}{\partial x^1} \right)^2 - \left( \frac{\partial}{\partial x^2} \right)^2 - \cdots - \left( \frac{\partial}{\partial x^{n-1}} \right)^2
\]

has a limit which is a degenerate symmetric bilinear form on \(V^*\), where \(V = \mathbb{R}^{1,n-1}\). Its kernel is a line \(K \subset V^*\) whose annihilator is a codimension one subspace \(W \subset V\). The time orientation \(P\) induces an orientation of \(V/W\). Furthermore, \(W\) inherits a definite inner product from the \(c \to \infty\) limit of (13.20). In other words, we obtain a Galilean structure (Definition 12.3). Therefore, Minkowski spacetime is a deformation of Galilean spacetime. The latter should be regarded as singular, whereas Minkowski spacetime is generic (or stable). We refer to [DF2, §1.4] for more details.

13.3 The Poincaré group

Let \(M\) be a Minkowski spacetime over a Lorentzian vector space \(V\). The group \(O(V)\) of linear isometries of \(V\) has a subgroup \(O^+(V)\) of isometries that preserve the forward timelike cone \(P \subset V\). If the dimension \(n\) is at least 2, then \(O(V)\) has 4 components and \(O^+(V)\) has two components. The identity component of \(O(V)\) (and of \(O^+(V)\)) is \(SO^+(V)\). These groups are depicted in Figure 41. There are corresponding affine isometry groups \(SO^+(M) \subset O^+(M) \subset O(M)\). The double cover of \(SO^+(V)\) is the spin group \(Spin(V)\); see §12.4 for a very brief introduction.

Figure 41. The four components of \(O(V)\) (assuming \(n \geq 2\))

**Definition 13.21.** Let \(M\) be a Minkowski spacetime over a Lorentzian vector space \(V\). The *Poincaré group* \(\mathcal{P}(M)\) is a double cover \(\mathcal{P}(M) \to SO^+(M)\) that projects to the spin double cover \(Spin(V) \to SO^+(V)\).
These isometry groups of Minkowski spacetime fit into the following diagram of group extensions:

\[
\begin{array}{ccccccccc}
1 & \rightarrow & V & \rightarrow & \mathcal{P}(M) & \rightarrow & \text{Spin}(V) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow^{2:1} & & \downarrow^{2:1} & & \\
1 & \rightarrow & V & \rightarrow & \text{SO}^\uparrow(M) & \rightarrow & \text{SO}^\uparrow(V) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & V & \rightarrow & \text{O}^\uparrow(M) & \rightarrow & \text{O}^\uparrow(V) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & V & \rightarrow & \text{O}(M) & \rightarrow & \text{O}(V) & \rightarrow & 1
\end{array}
\]

(13.22)

Each of the groups \( \text{SO}^\uparrow(M) \subset \text{O}^\uparrow(M) \subset \text{O}(M) \) acts effectively on \( M \): every non-identity group element moves some point of \( M \). The Poincaré group \( \mathcal{P}(M) \) has a special non-identity element \( \epsilon \) that covers the identity element of \( \text{SO}^\uparrow(M) \); it projects to a similar element of \( \text{Spin}(V) \). Then \( \epsilon \) acts as the identity map on \( M \): it does not move any points of spacetime. Such a symmetry is called internal. We consider more general groups of symmetries, including more elaborate internal symmetries, in upcoming lectures.

Remark 13.23. In Galilean geometry there is a distinguished subgroup of boost symmetries; see Figure 27 and the nearby text. There is no such distinguished subgroup in Minkowski geometry unless we fix a time-space splitting (12.14).

Lecture 14: Classical relativistic particle; relativistic systems; relativistic QM

In the first part of this lecture we consider a classical particle in Minkowski spacetime. The classical motions have worldlines that are timelike affine lines. The collection of such lines is a fiber bundle of affine spaces, and it carries a symplectic form for which the fibers are Lagrangian. We indicate a Lagrangian density from which this symplectic manifold can be derived, and we also show how its nonrelativistic limit is the standard Lagrangian of Newtonian mechanics.

The classical relativistic particle is an example of a relativistic system. This is a modification of Definition 1.11 that replaces time evolution of states and observables by an action of the Poincaré group on states and observables. (Later we consider more general symmetry groups.) Then we specialize to relativistic quantum mechanics, for which the data is a (projective) Hilbert space and a unitary representation of the Poincaré group. We recall the spectral theorem for the restriction of the representation to the subgroup of translations. The spectrum of the representation is the support of the spectral measure, which is a subset of energy-momentum vectors. In subsequent lectures we explore relativistic QM systems in more depth and give examples. (For instance, the next lecture begins with the relativistic quantum particle.)

We conclude the lecture with mathematical background on smooth densities.
14.1 The classical relativistic particle

Let $M$ be a Minkowski spacetime over a vector space $V$ with Lorentzian inner product $\langle -,- \rangle$. This data includes a convex cone $P \subset V$ of forward timelike vectors and a constant $c \in \mathbb{R}^>0$, the speed of light. Recall (Definition 13.7) that a particle trajectory in this geometry is a worldline: a smooth connected 1-dimensional submanifold $C \subset M$ whose tangent lines are timelike.

**Definition 14.1.** A *classical trajectory* is an affine worldline $C \subset M$. Denote the space of all classical trajectories as $\mathcal{M}$.

Thus $\mathcal{M}$ is the manifold of all timelike affine lines in $M$. It is the *phase space* of the classical relativistic particle. We will see that it has a nice geometric structure. A classical trajectory is depicted in Figure 42.

![Figure 42. A classical trajectory](image)

Fix a mass $m$, which is a positive real number, and suppose $C \subset M$ is a worldline. We interpret $C$ as the record of a spacetime motion of a particle of mass $m$. Introduce the (Lagrangian) density

\begin{equation}
- mc d\mu_C
\end{equation}

on $C$, where $d\mu_C$ is the smooth density of the induced Riemannian metric (§14.4.1). The *action* on a compact connected subset $J \subset C$ is $-mc \text{Length}(J)$. Therefore, the worldlines that extremize the action are locally length-minimizing, which in affine space implies that they are affine lines, i.e., classical trajectories as in Definition 14.1. It is convenient to rewrite (14.2) in terms of an arbitrary parametrization $\gamma: I \rightarrow M$ of $C$:

\begin{equation}
- mc d\mu_C = - mc \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle^{1/2} ds, \quad s \in I.
\end{equation}

Introduce standard coordinates $t, x^1, \ldots, x^{n-1}$ on $M$ so that the metric is

\begin{equation}
c^2 dt^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2.
\end{equation}

Parametrize $C$ by $t$, so via functions $t, x^1(t), \ldots, x^{n-1}(t)$. Write

\begin{equation}
v = \left[ (\dot{x}^1)^2 + \cdots + (\dot{x}^{n-1})^2 \right]^{1/2}.
\end{equation}
Then write (14.3) as

\[-mc \, d\mu_C = -mc^2 \sqrt{1 - v^2/c^2} \, dt\]

\[= \left[ -mc^2 + \frac{1}{2}mv^2 + O\left(\frac{v^4}{c^2}\right) \right] \, dt.\]

The first two terms in brackets are the standard action for a Newtonian/Galilean particle of mass \(m\) and potential energy \(mc^2\) (recall (13.16)).

Let \(C \subset M\) be a classical trajectory, i.e., a timelike affine line. Its tangent line \(U \subset V\) is a timelike line. Write \(\tau_U \in U\) for the forward vector with \(|\tau_U| = c\). The energy-momentum of \(C\) is the constant \(\theta_U = m \phi(\tau_U)\) given by (13.15); note that \(|\theta_U| = mc\). Recall the mass shell defined introduced in Definition 13.18. The map \(U \mapsto \theta_U\) from timelike affine lines in \(V\) to \(\mathcal{O}_m\) is a diffeomorphism, and so there is a surjective map

\[\pi: \mathcal{M} \longrightarrow \mathcal{O}_m\]

that takes a classical trajectory to its energy-momentum. In fact, \(\pi\) is a fiber bundle of affine spaces. Namely, the fiber over \(\theta_U\) is the space of affine lines parallel to \(U\), which is an affine space over \(V/U\).

**Proposition 14.8.** \(\mathcal{M}\) carries a symplectic form with respect to which the fibers of \(\pi\) are Lagrangian.

The key observation is that \(\pi\) is an affine bundle over the cotangent bundle of \(\mathcal{O}_m\). Then we transport the natural symplectic form on the total space of the cotangent bundle to the total space of \(\pi\) via a natural family of sections of \(\pi\), after proving that the result is independent of the section.

**Proof.** First, recall that if \(X\) is a smooth manifold, then the total space \(T^*X\) of its cotangent bundle \(\varpi: T^*X \rightarrow X\) carries a canonical 1-form \(\gamma\); see Example 2.29(2). Namely, at a cotangent vector \(\lambda \in T^*_xX\) we have

\[\gamma_\lambda(\dot{\lambda}) = \lambda(\varpi_\ast \dot{\lambda}), \quad \dot{\lambda} \in T^*_x(T^*X).\]

The differential \(d\gamma\) is a nondegenerate closed 2-form on \(T^*X\), i.e., it is a symplectic form. (We say that \(T^*X\) is an exact symplectic manifold since the symplectic form is exact.) Note that the fibers of the cotangent bundle \(\varpi: T^*X \rightarrow X\) are Lagrangian.

Next, suppose that \(\pi: E \rightarrow X\) is an affine bundle over \(\varpi: T^*X \rightarrow X\), i.e., for each \(x \in X\) the fiber \(E_x\) is an affine space over \(T^*_xX\). A section \(s\) of \(\pi\) induces an identification \(\varphi_s: E \xrightarrow{\cong} T^*_xX\), and the difference of two sections \(s', s\) is a section of \(\varpi\), i.e., a 1-form \(\alpha\). An easy check shows that

\[\varphi^*_s \gamma - \varphi^*_s \gamma = \alpha.\]

(Of course, one can carry out these constructions locally on \(X\).) Now if given a set \(S\) of sections of \(\pi\) such that the 1-form difference between any \(s', s \in S\) is closed, then it follows from (14.10) that the transport \(\varphi^*_s(d\gamma)\) of the symplectic form to \(E\) is independent of \(s \in S\).
Apply this to the affine bundle (14.7), whose total space is the space of classical trajectories of a particle of mass $m$ in a Minkowski spacetime $M$. First, $\mathcal{O}_m$ is defined by the equation

$$\langle \theta, \theta \rangle = m^2 c^2, \quad \theta \in V^*,$$

so differentiating we find a canonical identification $T_{\theta_U} \mathcal{O}_m \cong (U^*)^\perp \cong (U^\perp)^*$. Hence we can and do identify $T^{\star} \mathcal{O}_m \cong U^\perp \cong V/U$. The fiber of (14.7) at $\theta_U \in \mathcal{O}_m$ is affine over $V/U$; see the text following (14.7). Hence we are in the position envisioned in the previous paragraph. Furthermore, $S = M$ parametrizes a natural space of sections of $\pi$: given a timelike line $U \subset V$ and a point $p \in M$, there is a unique timelike affine line parallel to $U$ that passes through $p$. If $p', p \in M$, then write $p' = p + \xi$ for $\xi \in V$. Now $\xi$ is a constant 1-form on $V^*$, so it restricts to a closed 1-form on $\mathcal{O}_m \subset V^*$, and this is the difference of the sections labeled by $p', p$. This completes the construction of a symplectic form on $M$. The fact that the fibers of $\pi$ are Lagrangian follows from the corresponding fact for the cotangent bundle to $\mathcal{O}_m$. □

Observe that the isometry group $O(M)$ of Minkowski spacetime acts on the manifold $\mathcal{M}$ of timelike affine lines. We leave as an exercise that the subgroup $O^{\uparrow}(M)$ acts preserving the symplectic form and its complement reverses the sign of the symplectic form.

The symplectic manifold $M$ almost defines a classical mechanical system, according to the discussion in Lecture 1. What is missing is a 1-parameter group of symplectic diffeomorphisms of $\mathcal{M}$. What is naturally there instead is the action of $O^{\uparrow}(M)$ by symplectic diffeomorphisms. If we fix a timelike affine line $U_0 \subset V$, and so obtain a splitting $V = U_0 \oplus U_0^\perp$ as in (12.14), then we can restrict the action to $U \subset O^{\uparrow}(M)$ to obtain time evolution. But in relativistic systems we shun time-space splittings, and so we are led a relativistic modification of a mechanical system.

14.2 Relativistic systems

The following is a modification of Definition 1.11. Recall Definition 13.1 of a Minkowski spacetime and Definition 13.21 of the Poincaré group.

**Definition 14.12.** Let $M$ be a Minkowski spacetime. A relativistic system on $M$ consists of DOSO (data of states and observables) as in Axiom System 1.1 together with a representation of the Poincaré group $\mathcal{P}(M)$ by automorphisms of states and observables.

**Remark 14.13.**

1. As mentioned several times, we will define a more general notion of symmetry group of a relativistic system, and there is a corresponding variation of Definition 14.12 in which that symmetry group acts on states and observables.

2. If $\dim M = 1$, then Definition 14.12 reduces to Definition 1.11.

3. As opposed to a mechanical system, a relativistic system can admit a local theory of observables. We will develop this for relativistic quantum mechanical systems in a future lecture.

**Definition 3.1** of a Hamiltonian system also generalizes to relativistic systems.

**Definition 14.14.** Let $M$ be a Minkowski spacetime.
(1) The data of a classical relativistic system on $M$ is a symplectic manifold $(N,\omega)$ and an action of the Poincaré group $\mathcal{P}(M)$ by symplectic automorphisms of $N$.

(2) A system is free if $(N,\omega)$ is an affine symplectic space and $\mathcal{P}(M)$ acts by affine symplectic automorphisms.

The classical relativistic particle is a classical relativistic system, but it is not free according to Definition 14.14. We will see free classical systems of fields.

Remark 14.15. Just as there may be a Lagrangian formulation of a Hamiltonian mechanical system (§8.2), so too there may be a Lagrangian formulation of a classical relativistic system.

Definition 14.16. Let $M$ be a Minkowski spacetime. The linear data of a relativistic quantum mechanical (QM) system is a pair $(\mathcal{H},U)$ consisting of a complex separable Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation $U: \mathcal{P}(M) \to U(\mathcal{H})$ of the Poincaré group.

We explore Definition 14.16 in the next section.

Remark 14.17.

(1) The discussion in Lecture 8 is centered around the notion that quantum theory is projective—recall the slogan (8.1)—hence we should better say that a relativistic QM system is a projective unitary representation of $\mathcal{P}(M)$. However, any central extension by $T$ of $\mathcal{P}(M)$ splits, and except in low dimensions there are no nontrivial characters—the splitting of the central extension is unique—so there is no loss in using linear representations. For other symmetry groups, there may be nontrivial central extensions. In any case conceptually one should replace the linear space $\mathcal{H}$ by a projective space. We have chosen to ease the exposition by using linear spaces, but the reader should keep this important point in mind.

(2) We should also recall the discussion in Lecture 10, which allows bosonic and fermionic states and so replaces $\mathcal{H}$ by a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. The unitary representation $U$ is by even operators, so it preserves each homogeneous subspace.

14.3 Relativistic quantum mechanics

Recall the discussion of spectral decompositions in Lecture 4.

Suppose given a relativistic QM system on a Minkowski spacetime $M$, as in Definition 14.16. One might construct the spectrum of a representation of $\mathcal{P}(M)$ by decomposing it into irreducible representations. Such a decomposition may have discrete and continuous parts. Instead, it is advantageous to restrict to the abelian translation subgroup $V \subset \mathcal{P}(M)$ and decompose according to unitary characters of $V$, which are identified with $V^*$ via the isomorphism:

\[
V^* \longrightarrow \text{Hom}(V,T)
\]

\[
\theta \longmapsto (\xi \mapsto e^{-i\theta(\xi)/\hbar})
\]

(Observe $[\xi] = L$ and $[\theta] = ML/T$, so the exponent is dimensionless.)
Theorem 14.19. Let \( U : V \to U(\mathcal{H}) \) be a unitary representation of the vector group \( V \). Then there exists a spectral measure \( \pi_U \) on \( V^* \) such that

\[
U(\xi) = \int_{V^*} e^{-i\theta(\xi)/\hbar} d\pi_U(\theta)
\]

for all \( \xi \in \mathcal{H} \).

Let \( P \subset V \) denote the cone of forward timelike vectors, part of the data of the Minkowski spacetime \( M \). Its dual \( P^* \subset V^* \) is the cone of energy-momentum vectors of positive energy. The support of a measure is the complement of the union of open subsets with measure zero; see Definition 1.55.

Definition 14.21. Let \( U : \mathcal{P}(M) \to U(\mathcal{H}) \) be a unitary representation of the Poincaré group \( \mathcal{P}(M) \).

1. The spectrum \( \text{spec}(U) \subset V^* \) of \( U \) is the support of the spectral measure \( \pi_U \).
2. We say that \( U \) has positive energy if \( \text{spec}(U) \subset \overline{P^*} \).
3. A relativistic particle is an irreducible unitary positive energy representation of \( \mathcal{P}(M) \).

The spectrum of \( U \) is a closed subset of \( V^* \), and it is invariant under the action of \( \text{SO}^+(V) \). We study irreducible representations of \( \mathcal{P}(M) \) in a future lecture.

The decomposition of \( \overline{P^*} \) into \( \text{SO}^+(V) \)-orbits is roughly given by the mass; see Figure 43. If \( n = \dim M \geq 3 \), then that decomposition is precisely

\[
\overline{P^*} = \{0\} \sqcup N(V^*)^+ \sqcup \bigcup_{m > 0} \mathcal{O}_m,
\]

where \( N(V^*)^+ = (N(V^*) \cap \overline{P^*}) \setminus \{0\} \) is the space of positive energy null vectors. (For \( \dim M = 2 \) the positive energy null vectors \( N(V^*)^+ \) decompose into two orbits, each of which is a ray, and for \( \dim M = 1 \) there are no nonzero null vectors: \( N(V^*)^+ = \emptyset \).)

There is an important dichotomy of relativistic quantum mechanical systems: gapped vs. gapless.

Figure 43. The decomposition of \( \overline{P^*} \) into \( \text{SO}^+(V) \)-orbits
Definition 14.23. Let $U : \mathcal{P}(M) \to U(\mathfrak{h})$ be a unitary representation of the Poincaré group. Then $U$ is gapped if the minimal mass subset is isolated in $\text{spec}(U)$ and the restriction of $U$ to the corresponding closed subspace of $\mathfrak{h}$ is a finite sum of irreducible representations. If $U$ is not gapped, then we say that $U$ is gapless.

If $U : V \to U(\mathfrak{h})$ is a unitary representation of the translation group, then elements of $\text{Sym}^k(V)$ also act by unitary operators for any $k \in \mathbb{Z}^\geq 1$: the symmetric product $\xi_1 \ast \cdots \ast \xi_k$ acts by the composition $U(x_1) \circ \cdots \circ U(x_k)$. The inverse Lorentz metric on $V^*$ is an element of $\text{Sym}^2(V)$; the corresponding unitary operator $M$ is the mass square operator (after multiplication by $\hbar^2/c^2$). It commutes with any representation $U$ of the Poincaré group. If $U$ is a positive energy representation of Poincaré and $0 \in \text{spec}(U)$, then $0 \in \text{spec}(M)$, and $U$ is gapped iff $M$ has a spectral gap in the sense of Definition 5.7(2).

Finally, we remark that some positive energy unitary representations of $\mathcal{P}(M)$ extend to a positive energy unitary representation of a double cover of $O^+(M)$. (Recall the various groups in (13.22).) There are two such double covers, known as pin groups, and they embed in Clifford algebras. See [FH1, Appendix A] for more detail.

14.4 Mathematical background

14.4.1 Smooth densities. Let $V$ be a finite dimensional real vector space of dimension $N$. Canonicaly associated to $V$ are several real lines. For example, the determinant line $\text{Det} V = \bigwedge^N V$ is the highest exterior power of $V$. A nonzero vector in $\text{Det} V$ represents an oriented parallelepiped in $V$. A nonzero vector in $\text{Det} V^*$ is a notion of oriented volume on $V$. There is also a “line” of densities, though it is not a vector space; it is diffeomorphic to the real line and carries an analogous trichotomy of positive/zero/negative numbers.

Definition 14.24.

(1) Let $V$ be a finite dimensional real vector space of dimension $N$, and let $\mathcal{B}(V)$ denote its right $\text{GL}_N \mathbb{R}$-torsor of bases. The line of densities on $V$ is

$$|\text{Det} V^*| = \{ \mu : \mathcal{B}(V) \longrightarrow \mathbb{R} : \mu(b \cdot g) = |\det g| \mu(b) \text{ for all } b \in \mathcal{B}, g \in \text{GL}_N \mathbb{R} \}$$

$$= \mathcal{B}(V) \times_{\text{GL}_N \mathbb{R}} \mathbb{R},$$

where in the second line $\text{GL}_N \mathbb{R}$ acts on $\mathbb{R}$ by $g \mapsto |\det g|$. There is a partition

$$|\text{Det} V^*| = |\text{Det} V^*|^+ \cup \{0\} \cup |\text{Det} V^*|^−$$

according to the sign of $\mu$. Elements of $|\text{Det} V^*|^+$ are positive densities.

(2) Let $M$ be a smooth manifold. Then a smooth positive density on $M$ is a section of the fiber bundle $|\text{Det} T^* M|^+ \to M$. 

A positive density on $M$ is also called a smooth measure. A partition of unity argument proves that smooth measures exist; the fibers of $|\text{Det} T^* M|^+ \to M$ are convex.

If $G$ is a Lie group that acts smoothly and transitively on $M$, then a $G$-invariant measure exists iff for any $x \in M$ the action of the stabilizer subgroup $G_x \subset G$ on $|\text{Det} T^* M|^+$ is trivial. (In general, the action is by a character $G_x \to \mathbb{R}^>0$. For example, if $G_x$ is compact, then the action is necessarily trivial. This argument proves the existence of Haar measure on compact Lie groups.)

Now suppose that $V$ is an inner product space. Since the representation $g \mapsto |\text{det} g|$ restricts to the trivial representation on $O_N \subset \text{GL}_n \mathbb{R}$, the line $|\text{Det} V|^+$ is canonically diffeomorphic to $\mathbb{R}^>0$. There is a canonical element: $1 \in \mathbb{R}^>0$. This density encodes the notion of volume on $V$ induced from the lengths and angles computed via the inner product. Applied pointwise on a Riemannian manifold $M$, we learn that $M$ carries a canonical smooth measure, the Riemannian measure $\mu_M$.

**Lecture 15: Relativistic quantum mechanics**

We already gave basic definitions in §14.3. Namely, the data of a relativistic QM system is a unitary representation of the Poincaré group. Its restriction to the vector space of translations can be “diagonalized” according to Theorem 14.19. The spectral measure that expresses this diagonalization has a support, which leads to the notion of a positive energy representation (Definition 14.21). We always consider relativistic QM systems that satisfy the positive energy condition.

We begin in this lecture with an example: the quantum relativistic particle of mass $m$. We treated the classical relativistic particle in §14.1, where we construct a symplectic manifold $\mathcal{M}$ equipped with an action of time-orientation preserving isometries of Minkowski spacetime. Here we construct a quantum system which is meant to be its quantization. This is an irreducible representation of the isometry group, so is a relativistic particle (Definition 14.21). In subsequent examples we construct examples of relativistic QM systems from fields (rather than particles).

We then turn to a few other general topics in relativistic quantum mechanics. The first are constraints when one has the underlying quantum mechanical system of a relativistic quantum field theory. It is only in subsequent lectures that we define the latter, but here it is convenient to state the spin-statistics theorem and the misnamed CPT theorem. We conclude with a general discussion of affine symmetry types and of a relativistic symmetry type.

15.1 The quantum relativistic particle

We do not give a systematic discussion of quantization in these lectures, so only loosely discuss the passage from the symplectic manifold $\mathcal{M}$ of classical trajectories (Definition 14.1) of a particle

---

39: ‘T’ is ‘time-reversal’ and ‘P’ is ‘parity’. We replace ‘P’ with ‘R’, which is ‘reflection’, since in any dimension reflection in a hyperplane reverses orientation, whereas parity—reflection in the origin—preserves or reverses orientation depending on the parity of the dimension. The ‘C’ is ‘conjugation’. Usually this is ‘charge conjugation’, but there is no charge to conjugate for representations of the Poincaré group. Rather, we can think it is ‘complex conjugation’ which evokes the antilinearity of time-reversing transformations, but in truth that is not its intended meaning. Best to simply call it the ‘RT theorem’. Also, ‘C’, ‘R’, and ‘T’ do not refer to particular elements of the Poincaré group but rather to entire components.
of mass \(m\) in a Minkowski spacetime \(M\) to a unitary representation of the Poincaré group \(P(M)\). It turns out that this representation is irreducible.

Recall that \(\mathcal{M}\) is the total space of an affine bundle over the mass shell \(\mathcal{O}_m\). An important ingredient in the quantization of a symplectic manifold is a polarization, which in this case can be taken to be a foliation by (real) Lagrangian submanifolds, namely the fibers of \(\pi: \mathcal{M} \to \mathcal{O}_m\). Then, roughly, one builds a Hilbert space of functions which are constant along the leaves of the foliation, which in this case amounts to a Hilbert space of functions on \(\mathcal{O}_m\). As we show in the next lecture, there is a ray of \(SO^{\uparrow}(\mathcal{V})\)-invariant measures on \(\mathcal{O}_m\), and we let the quantization \(\mathcal{H} = L^2(\mathcal{O}_m; \mathbb{C})\) be the Hilbert space of complex-valued \(L^2\) functions on \(\mathcal{O}_m\). (There are only bosonic states in this system: \(\mathcal{H}\) is even.) Note that each \(\xi \in \mathcal{V}\) determines a smooth function

\[(15.1) \quad \theta \mapsto e^{-i\theta(\xi)/\hbar}\]

on \(\mathcal{O}_m\), and \(\xi \in \mathcal{V}\) acts as multiplication by this function. To see the action of the entire Poincaré group, it is convenient to make a different model. Observe that \(P(M)\) acts on the Cartesian product \(\mathcal{O}_m \times M\), and so too acts linearly on functions on \(\mathcal{O}_m \times M\). These actions factor through the quotient \(P(M) \to SO^{\uparrow}(M)\). Restrict to functions \(f\) that are \(L^2\) in the \(\mathcal{O}_m\)-direction and that satisfy

\[(15.2) \quad f(\theta_U, p + \xi) = e^{i\theta_U(\xi)/\hbar} f(\theta_U, p), \quad (\theta_U, p) \in \mathcal{O}_m \times M.\]

We can interpret these functions as \(L^2\) sections of a hermitian line bundle over \(\mathcal{O}_m\). The isometry group \(SO^{\uparrow}(M)\) manifestly acts on this Hilbert space. This is the desired representation.

The spectrum of this representation is the mass shell \(\mathcal{O}_m \subset \mathcal{V}^*\). We prove in the next lecture that \(SO^{\uparrow}(\mathcal{V})\) acts transitively on \(\mathcal{O}_m\). It follows that this representation is irreducible, so defines a relativistic particle.

**15.2 Two theorems in quantum field theory**

Relativistic quantum field theories form a special class of relativistic quantum mechanical theories. For these, there are two basic theorems which restrict the representations of the Poincaré group that occur.

Recall from (10.1) that states form a disjoint union of two convex spaces: bosonic states and fermionic states. A state is either bosonic or fermionic, a feature which is termed its statistics.

An irreducible representation of the spin group \(Spin(\mathcal{V})\) has a spin, which is a half-integer. We define it in the next lecture. For now we just need the value modulo the integers. Recall that the central element \(\epsilon \in Spin(\mathcal{V})\), which generates the kernel of the double cover \(Spin(\mathcal{V}) \to SO^{\uparrow}(\mathcal{V})\), has order 2.

**Definition 15.3.** Let \(\epsilon \in Spin(\mathcal{V})\) be the central element. An irreducible representation of \(Spin(\mathcal{V})\) has integer spin if \(\epsilon\) acts as +1, and it has half-integer spin if \(\epsilon\) acts as −1.

A representation of integer spin factors to a representation of \(SO^{\uparrow}(\mathcal{V})\).
Theorem 15.4 (spin-statistics). Suppose a positive energy unitary representation \( U : \mathcal{P}(M) \to U(\mathcal{H}^0 \oplus \mathcal{H}^1) \) is the underlying QM systems a relativistic quantum field theory. Then \( U(\epsilon) \) is the grading operator

\[
\text{id}_{\mathcal{H}^0} \oplus -\text{id}_{\mathcal{H}^1}
\]
on \mathcal{H}^0 \oplus \mathcal{H}^1.

In other words, \( U(\epsilon) = +1 \) on \( \mathcal{H}^0 \) and \( U(\epsilon) = -1 \) on \( \mathcal{H}^1 \). In the physics literature, the notation ‘\((-1)^F\)’ is often used for the grading operator (15.5).

The second theorem tells about symmetries that reverse time-orientation. For simplicity we consider quantum field theories that only have bosonic states, so by Theorem 15.4 are representations of the quotient \( \text{SO}^\uparrow(M) \) of the Poincaré group. Note that \( \text{SO}^\uparrow(M) \) is the identity component of the 2-component Lie group \( \text{SO}(M) \) whose off-component consists of isometries of \( M \) that reverse time-orientation (T) and reverse space-orientation (R). The following theorem states that time-orientation reversing symmetries are always implemented in a in a quantum field theory, albeit by antilinear operators—see Lemma 11.3.

Theorem 15.6 (CRT theorem). Suppose a positive energy unitary representation \( U : \text{SO}^\uparrow(M) \to U(\mathcal{H}) \) is the underlying QM systems a relativistic quantum field theory. Then \( U \) extends to a homomorphism \( \text{SO}(M) \to \text{Q}(\mathcal{H}) \) in which elements of \( \text{SO}(M) \setminus \text{SO}^\uparrow(M) \) act by an antiunitary transformation. If the quantum field theory has a positive energy unitary representation of the Lie group \( \text{O}^\uparrow(M) \), then there is an extension to \( \text{O}(M) \to \text{Q}(\mathcal{H}) \) such that time-orientation reversing transformations act antiunitarily.

Recall the Lie group \( Q \) in (8.46) which is the union of unitary and antiunitary transformations. There is a version of Theorem 15.6 for representations of the Poincaré group as well; the extensions are representations of Lie groups whose quotients by translations embed in Clifford algebras; see [FH1, Appendix A]. That reference discusses the extension of the theorem to arbitrary relativistic symmetry types.

15.3 Relativistic symmetry types

We continue the discussion of §7.4.

Let \( M \) be a space on which we are doing some geometry. Imagine, for example, that \( M \) is a topological space and our geometry consists of a finite cover \( \pi : \tilde{M} \to M \). Then symmetries of our geometry induce symmetries of \( M \). In other words, there is a homomorphism from the symmetry group \( \mathcal{G} \) of the geometric structure to the group \( \text{Aut}(M) \) of symmetries of \( M \). In our motivating example of a finite cover, this is a homomorphism \( \text{Aut}(\pi) \to \text{Homeo}(M) \) from the group of homeomorphisms of \( \tilde{M} \) that preserve the fibers of \( \pi \) to the group of homeomorphisms of the base \( M \). In general, one has an exact sequence of group homomorphisms

\[
1 \to K \to \mathcal{G} \to \text{Aut}(M)
\]
in which the kernel \( K \) consists of symmetries that fix \( M \) pointwise. These are *internal symmetries* that act on internal geometric structures without moving the points of \( M \).
Specialize now to the case of an affine space $M$ over a vector space $V$. Affine symmetries lie in a group extension

$$(15.8) \quad 1 \rightarrow V \rightarrow \text{Aut}(M) \rightarrow \text{Aut}(V) \rightarrow 1$$

in which the kernel is the normal subgroup of translations and the quotient is the group of linear symmetries of the vector space of translations. Affine geometry is the geometry of global parallelism, and therefore any geometric structure on $M$ should be invariant under translation. So a Lie group $\mathcal{G}$ of symmetries of such a structure contains $V$ as a normal subgroup, and so fits into a diagram

$$(15.9) \quad \begin{array}{c}
1 \\ K \\
\downarrow \\
\downarrow \\
\downarrow \\
1
\end{array}
\quad \begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\quad \begin{array}{c}
1 \\ \mathcal{G} \\
\downarrow \tilde{\lambda} \\
\downarrow \lambda \\
1 \\
\longrightarrow
\end{array}
\quad \begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\quad \begin{array}{c}
1 \\ \text{Aut}(M) \\
\downarrow \\
\downarrow \\
\downarrow \\
\longrightarrow
\end{array}
\quad \begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\quad \begin{array}{c}
1 \\ \text{Aut}(V) \\
\downarrow \\
\longrightarrow
\end{array}$$

in which the long rows are group extensions and the columns are exact: the group $K$ of internal symmetries is the kernel of the homomorphisms $\lambda, \tilde{\lambda}$. The groups $\mathcal{G}$ of affine symmetries and $G$ of linear symmetries determine each other, as do the group homomorphisms $\tilde{\lambda}$ and $\lambda$.

**Example 15.10.** If $V$ carries an inner product, then the inclusion $\lambda: O(V) \rightarrow \text{Aut}(V)$ encodes the induced Euclidean structure on $M$. A spin structure is encoded in a homomorphism $\lambda: \text{Spin}(V) \rightarrow \text{Aut}(V)$ which is a nontrivial $^{40}$ double cover of its image $\text{SO}(V)$, though in this case it is less clear how to describe the geometric structure preserved by the group $\text{Spin}(V)$. As another example, we might have the projection $\lambda: \text{Aut}(V) \times K \rightarrow \text{Aut}(V)$ for some Lie group $K$. This is the symmetry of a geometry $M \times F$ in which $F$ is some manifold with $K$ acting as a group of symmetries.

As in §7.4 we use a model space to define a general type of geometry.

**Definition 15.11.** Let $n$ be a positive integer. An **affine symmetry type** of dimension $n$ consists of a Lie group $G_n$ and a group homomorphism $\lambda_n: G_n \rightarrow \text{GL}_n \mathbb{R}$.

If this data exists coherently for all positive integers $n$, then we say that the affine symmetry type is **stable**. A Euclidean structure is an example:

$$\cdots \longrightarrow O_n \longrightarrow O_{n+1} \longrightarrow \cdots$$

$$\begin{array}{c}
\lambda_n \\
\downarrow \\
\lambda_{n+1}
\end{array}$$

$$\cdots \longrightarrow \text{GL}_n \mathbb{R} \longrightarrow \text{GL}_{n+1} \mathbb{R} \longrightarrow \cdots$$

$^{40}$at least if $\dim V \geq 2$. 
Use Definition 7.45 to define (families of) geometries of type $\lambda_n$.

A symmetry type for relativistic quantum mechanics is a special case, but we insist that the symmetry group maps onto the identity component of the isometry group of Minkowski spacetime. This ensures that the symmetry group describes “relativistic invariance”.

**Definition 15.13.** Let $n$ be a positive integer. A relativistic symmetry type of spacetime dimension $n$ is a Lie group $G_n$ equipped with a Lie group homomorphism $\lambda_n : G_n \to O_{1,n-1}^\uparrow$ whose image contains the identity component $SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow$.

The kernel $K \subset G_n$ of $\lambda_n$ is the model group of internal symmetries. Let $I_{1,n-1}^\uparrow = O^\uparrow(M^n)$ denote the group of isometries of model Minkowski spacetime $M^n$ that preserve time-orientation. Then analogous to (15.9) is the commutative diagram

\[
\begin{array}{c}
1 \\
\downarrow \\
K \\
\downarrow \\
G_n \\
\downarrow \\
I_{1,n-1}^\uparrow \\
\downarrow \\
O_{1,n-1}^\uparrow \\
\downarrow \\
1
\end{array}
\]

in which the long rows are group extensions and the columns are exact.

**Remark 15.15.** Relativistic symmetry types partition into two classes according to the image of $\lambda_n$, which is either $SO_{1,n-1}^\uparrow$ or $O_{1,n-1}^\uparrow$. There are various appellations one can ascribe to these classes. Perhaps the most common is telling whether there are $(O_{1,n-1}^\uparrow)$ or are not $(SO_{1,n-1}^\uparrow)$ time-reversing symmetries in the theory. This is counterintuitive at first, and one can tell instead whether or not there are symmetries that reverse the orientation of space. For relativistic field theories, Theorem 15.6 provides the time-reversal symmetries if orientation-reversing symmetries of space are present.

**Example 15.16.** The Poincaré group $G_n = P_n$ provides an example of a relativistic symmetry type. Namely, we take

\[
G_n = \text{Spin}_{1,n-1} \longrightarrow SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow.
\]

The internal symmetry group is the center $\mu_2 \subset \text{Spin}_{1,n-1}$ of the spin group, generated by the element $\epsilon$ in Definition 15.3. Another important example is

\[
G_n = \text{Spin}_{1,n-1}^c \longrightarrow SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow
\]

in which the internal symmetry group is the circle group $T$. Theories with this symmetry type have an integer-valued charge operator, the representative of a nonzero element in the Lie algebra of $T$. 


Analogous to Definition 14.16 is the following.

**Definition 15.19.** Let $\lambda_n: G_n \to O_{1,n-1}^\uparrow$ be a relativistic symmetry type, and suppose $M$ is a Minkowski spacetime with $\lambda_n$-structure. The linear data of a *relativistic quantum mechanical* (QM) system of type $\lambda_n$ on $M$ is a pair $(\mathcal{H}, U)$ consisting of a complex separable Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation $U: \mathcal{G}(M) \to U(\mathcal{H})$.

As in Definition 7.45 the Minkowski spacetime is determined by a right $\mathcal{G}_n$-torsor, and that is also used to define the Lie group $\mathcal{G}(M)$ of symmetries.

**Remark 15.20.** It is oft said that the Poincaré group is a subgroup of the group $\mathcal{G}$ of symmetries of a relativistic QM system on a Minkowski spacetime $M$. This discussion emphasizes that, rather, the quotient group $SO_{1,n-1}^\uparrow(M)$ receives a map from $\mathcal{G}$ or from an index 2 subgroup.

There are analogs of the spin-statistics Theorem 15.4 and CRT Theorem 15.6 for general symmetry groups. The first appears in [FH1, §2] as part of a structure theory for relativistic symmetry types. The second is [FH1, Theorem A.23].

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**Lecture 16: The Lorentz group action on energy-momentum vectors**

Recall (Definition 14.21) that a relativistic particle is an irreducible unitary positive energy representation of the Poincaré group. The Poincaré group (13.22) and its cousins (15.14) sit in extensions with abelian kernel. We use this to analyze representations in two stages. First, the restriction of a representation to the kernel decomposes into characters, or rather the spectral theorem (Theorem 14.19) gives a spectral measure on the space of characters. Second, we look at the action of the quotient group on the characters and its implementation on the representation. As preparation for this analysis in the next lecture, in this lecture we study a finite group analog that eliminates the infinite dimensional linear analysis. One point is that the action of the quotient is via a central extension, not of the group but rather of a groupoid associated to the extension. Therefore, we begin with a preliminary discussion on groupoids,\textsuperscript{41} useful in its own right.

In the second part of the lecture we deduce the orbits and stabilizers of the group $SO_{1,n-1}^\uparrow$ acting on positive energy vectors $P^*_1,\ldots,n_{n-1} \subset (\mathbb{R}^{1,n-1})^*$. (These are roughly the quotient group and characters in the group extension in which Poincaré sits.) The mass shells $O_m$ for $m \in \mathbb{R}^{>0}$ are orbits, and the stabilizer subgroup is isomorphic to $SO_{n-1}$. The situation is slightly more complicated for null vectors, which split into a few orbits, depending on the dimension. The stabilizer groups are noncompact, and for a nonzero null vector the stabilizer is a form of a Euclidean group. This deduction fits into conformal geometry, and we digress to expose the beautiful quadric that conformally compactifies a real vector space with a conformal structure (of any signature).

### 16.1 Groupoids, central extensions, and vector bundles

We begin with the definition of a groupoid.

\textsuperscript{41}Groupoids appeared briefly in §B.4 in the bonus lecture.
Definition 16.1. A groupoid \( \mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1) \) is the data of a pair of sets (objects and morphisms), a map \( \mathcal{G}_0 \to \mathcal{G}_1 \) (identity morphisms), two maps \( \mathcal{G}_1 \to \mathcal{G}_0 \) (source and target), a map \( \mathcal{G}_1 \to \mathcal{G}_1 \) (inverse), and a map \( \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1 \) (composition). There are several conditions: associativity, identity morphisms, inverse morphisms, etc.

The fiber product \( \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \) is the set of composable pairs of morphisms. A groupoid is a category in which every morphism is invertible. If \( \mathcal{G}_0 = * \) is a point, then \( \mathcal{G} \) is a group. One can define groupoids in categories other than the category of sets. For example, see [FHT1, Appendix A] for topological groupoids. The data of a groupoid is depicted thus:

\[
\begin{array}{c}
\mathcal{G}_0 \xleftarrow{s} \mathcal{G}_1 \xrightarrow{m} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1
\end{array}
\]

Here \( s, t \) are source and target maps, \( i \) is inversion, and \( m \) is the composition law (multiplication).

Example 16.3. A group \( \mathcal{G}^1 \) is a groupoid with \( \mathcal{G}^0 = * \) a point.

Example 16.4. Suppose \( X \) is a set equipped with the left action of a group \( G \). The action groupoid of \( G \rtimes X \) is a groupoid with \( \mathcal{G}^0 = X \) and \( \mathcal{G}^1 = G \times X \): the source and target maps are \( s(g, x) = x, t(g, x) = g \cdot x \); see Example 11.16.

As a preliminary, let \( A \) be an abelian group. Then multiplication \( A \times A \to A \) is a group homomorphism. Hence if \( L_1, L_2 \) are right \( A \)-torsors, then so too is

\[
L_1 \otimes L_2 := (L_1 \times L_2) \times_{(A \times A)} A.
\]

For \( A = \mathbb{C}^x \) this duplicates the usual tensor product of the associated complex lines.

In the next definition we can use either \( \mathbb{T} \)-torsors or hermitian lines.

Definition 16.6. Let \( \mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1) \) be a groupoid. A central extension \( \tau = (L, \theta) \) of \( \mathcal{G} \) (by \( \mathbb{T} \)) is a principal \( \mathbb{T} \)-bundle \( L \to \mathcal{G}_1 \) and an isomorphism \( \theta \) of principal \( \mathbb{T} \)-bundles over \( \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \) that maps

\[
\begin{array}{c}
\theta_{\gamma_2, \gamma_1} : L_{\gamma_2} \otimes L_{\gamma_1} \xrightarrow{\cong} L_{\gamma_2 \gamma_1}
\end{array}
\]
For each triple \( \gamma_1 \rightarrow \gamma_2 \rightarrow \gamma_3 \) of composable arrows, impose the condition that the diagram

\[
\begin{array}{ccc}
L_{\gamma_3 \circ \gamma_2 \circ \gamma_1} & \xrightarrow{\theta_{\gamma_2 \circ \gamma_1}} & L_{\gamma_3} \otimes L_{\gamma_2 \circ \gamma_1} \\
\downarrow & & \downarrow \\
L_{\gamma_3 \circ \gamma_2} \otimes L_{\gamma_1} & \xrightarrow{\theta_{\gamma_3 \circ \gamma_2 \circ \gamma_1}} & L_{\gamma_3} \otimes L_{\gamma_2} \otimes L_{\gamma_1}
\end{array}
\]

commute.

See Figure 45 for an illustration. Definition 16.6 reduces to a central extension of the group \( \mathcal{G}^1 \) in case \( \mathcal{G}^0 = * \).

**Figure 46. A (twisted) vector bundle over a groupoid**

**Definition 16.9.** Let \( \mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1) \) be a groupoid.

1. A vector bundle \( E \rightarrow \mathcal{G} \) is a vector bundle \( E \rightarrow \mathcal{G}_1 \) together with isomorphisms

\[
\psi_{\gamma} : E_x \xrightarrow{\cong} E_y, \quad (x \xrightarrow{\gamma} y) \in \mathcal{G}_1,
\]

that satisfy \( \psi_{\gamma_2 \circ \gamma_1} = \psi_{\gamma_2} \circ \psi_{\gamma_1} \) for all pairs of composable arrows \( x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z \). A vector bundle over \( \mathcal{G} \) is also called a representation of \( \mathcal{G} \).
(2) Let $\tau = (L, \theta)$ be a central extension of $\mathcal{G}$. Then a $\tau$-twisted vector bundle over $\mathcal{G}$ is a vector bundle $E \to \mathcal{G}_1$ together with isomorphisms

$$\psi_\gamma: L_\gamma \otimes E_x \xrightarrow{\cong} E_y, \quad (x \xrightarrow{\gamma} y) \in \mathcal{G}_1,$$

that are required to satisfy the condition that the diagram

$$\begin{array}{ccc}
L_{\gamma_2} \otimes L_{\gamma_1} \otimes E_x & \xrightarrow{\psi_{\gamma_1}} & L_{\gamma_2} \otimes E_y \\
\downarrow{\theta_{\gamma_2 \gamma_1}} & & \downarrow{\psi_{\gamma_2}} \\
L_{\gamma_2 \circ \gamma_1} \otimes E_x & \xrightarrow{\psi_{\gamma_2 \circ \gamma_1}} & E_z
\end{array}$$

commute for each pair of composable arrows $x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z$. A $\tau$-twisted vector bundle is also called a $\tau$-twisted representation of $\mathcal{G}$.

If $\mathcal{G} = X//G$ is the action groupoid of $G \subset X$, then a vector bundle over $\mathcal{G}$ is a $G$-equivariant vector bundle over $X$.

Next, we prove an equivalence between equivariant vector bundles and representations of a stabilizer subgroup. Let $G_0$ be a compact Lie group, and denote by $\text{Irr}_f(G_0)$ the set of isomorphism classes of finite dimensional irreducible representations of $G_0$. We require compactness so that $\text{Irr}_f(G_0)$ is a finite or countable discrete set.

![Figure 47. Equivariant vector bundles ↔ representations of stabilizer](image-url)

**Proposition 16.13.** Let $X$ be a smooth manifold, and suppose a Lie group $G$ acts smoothly and transitively on $X$ with compact stabilizers. Let $G_x \subset G$ be the stabilizer subgroup at $x \in X$. Then evaluation at $x$ is a 1:1 correspondence

$$\begin{array}{cccc}
\text{Vect}_{\text{irr}}(X//G) & \longrightarrow & \text{Irr}_f(G_x) \\
E & \mapsto & E_x
\end{array}$$

(16.14)
between isomorphism classes of irreducible finite rank vector bundles over $X/G$ and isomorphism classes of irreducible finite dimensional representations of $G_x$.

Definition 11.27 illustrates the correspondence (16.14). As corollaries, $\text{Vect}_{\text{irr}}(X/G)$ is discrete and $\text{Irr}_f(G_x)$ is canonically independent of $x$, which in any case is easy to verify directly.

Proof. The map

$$
(16.15) \quad G \longrightarrow X
$$

$$
g \longmapsto g \cdot x
$$

is a principal $G_x$-bundle. The inverse map to (16.14) applies the mixing construction: an irreducible representation $G_x \circlearrowleft E_x$ maps to the $G$-equivariant vector bundle $G \times_{G_x} E_x \rightarrow X$. □

16.2 Finite group extensions

Consider a group extension

$$
(16.16) \quad 1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1
$$

of finite groups in which $G'$ is abelian. Our task is to classify irreducible unitary representations of $G$. A complex irreducible representation of $G'$ is 1-dimensional, so is given by a unitary character $\chi: G' \rightarrow \mathbb{T}$. Let $X$ be the abelian group of such characters, the Pontrjagin dual group to $G$. Use the extension (16.16) to construct a homomorphism

$$
(16.18) \quad \alpha: G'' \rightarrow \text{Aut}(G').
$$

Namely, if $g'' \in G'$ and $g \in G$ is a lift of $g''$, then conjugation by $g$ is an automorphism of the normal subgroup $G'$, and since $G'$ is abelian this automorphism of $G'$ is independent of the lift. Now use (16.18) to construct a right action of $G''$ on $X$: if $\chi \in X$ and $g'' \in G''$, then define $\chi \cdot g''$ as the composition

$$
(16.19) \quad \chi \cdot g'': G' \xrightarrow{\alpha(g'')} G' \xrightarrow{\chi} \mathbb{T}
$$

Example 16.20. The symmetric group on 3 letters is the smallest nonabelian group. It fits into the group extension

$$
(16.21) \quad 1 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \text{Sym}_3 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1
$$

---

42Every representation (real or complex) of a finite group is unitarizable: average an inner product over the group. We treat only complex representations; one can also investigate real and quaternionic representations.

43The relation between $G'$ and $X$ is symmetric: the nondegenerate pairing

$$
(16.17) \quad X \times G' \longrightarrow \mathbb{T}
$$

$$
\chi \cdot g' \longmapsto \chi(g')
$$

identifies each group as the Pontrjagin dual of the other.
The Pontrjagin dual $X$ to $\mathbb{Z}/3\mathbb{Z}$ is the cyclic group $\mu_3 = \{1, \omega, \omega^2\}$ of cube roots of unity. (For definiteness, take $\omega = e^{2\pi i/3}$.) Observe that $\mathbb{Z}/3\mathbb{Z}$ and $\mu_3$ are isomorphic, but not canonically so.\footnote{It is not always true that an abelian group and its Pontrjagin dual are isomorphic: the Pontrjagin dual to $\mathbb{Z}$ is $T$. But for a finite group there are such (noncanonical) isomorphisms.}

The action of the non-identity element of $\mathbb{Z}/2\mathbb{Z}$ on $X$ fixes $1$ and exchanges $\omega \leftrightarrow \omega^2$. The quotient groupoid $X \sslash (\mathbb{Z}/2\mathbb{Z})$ is depicted in Figure 48.

![Figure 48. The groupoids induced from (16.21) and Example 16.22.](image)

**Example 16.22.** The 8-element quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ sits in a group extension

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow Q \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

The groupoid of the action of the quotient on the Pontrjagin dual to the kernel is also indicated in Figure 48.

Now suppose $\rho: G \rightarrow U(V)$ is a unitary representation of the finite group $G$ in the extension (16.16). “Sheafify” the extension over $X$ as follows. For each $\chi \in X$ define the isotypical subspace

$$V_\chi = \operatorname{Hom}_G(\mathbb{C}_\chi, V) \otimes \mathbb{C}_\chi \subset V,$$

where $\mathbb{C}_\chi$ is the line $\mathbb{C}$ with the linear $G$-action defined by the character $\chi$. The vector spaces $V_\chi$ are the fibers of a vector bundle $E_V \rightarrow X$, and the space $\bigoplus_\chi V_\chi$ of sections of $E_V \rightarrow X$ is canonically isomorphic to $V$. The support of $E_V \rightarrow X$ (the subset of $X$ for which the fiber is nonzero) is a union of $G''$-orbits, and if $\rho$ is an irreducible representation then the support is a single $G''$-orbit. However, the action of $G''$ on $X$ does not necessarily lift to the total space $E_V$.

**Example 16.25.** The symmetric group $\text{Sym}_3$ has 3 irreducible representations $V^{(1)}, V^{(e)}, V^{(s)}$. For the trivial representation $V^{(1)}$ the vector bundle $E_{V^{(1)}} \rightarrow X$ has support $\{1\} \subset X$ and the fiber is 1-dimensional. Similarly for the sign representation $V^{(s)}$, which is pulled back from the nontrivial character of the quotient $\mathbb{Z}/2\mathbb{Z}$ in (16.21). The action of the stabilizer $\mu_3$ at $1 \in \chi$ is trivial for the trivial representation and is nontrivial for $V^{(s)}$. The representation $V^{(s)}$ is 2-dimensional, the support of $E_{V^{(s)}}$ is $\{\omega, \omega^2\} \subset X$, and the fibers of $E_{V^{(s)}} \rightarrow X$ are 1-dimensional at each of $\omega, \omega^2$. However, there is no natural action of the quotient $\mathbb{Z}/2\mathbb{Z}$ in (16.21) on $E_{V^{(s)}} \rightarrow X$.\footnote{It is not always true that an abelian group and its Pontrjagin dual are isomorphic: the Pontrjagin dual to $\mathbb{Z}$ is $T$. But for a finite group there are such (noncanonical) isomorphisms.}
Example 16.26. The quaternion group $Q$ has 5 irreducible representations. Four of them are characters, pulled back from characters of the quotient group—the Klein 4-group—in (16.23). The remaining irreducible complex representation is 2-dimensional: it is the complex representation that underlies the action of $Q$ on the quaternions $\mathbb{H}$ by left multiplication. For that representation the bundle $E \to X$ has support $\{e\} \subset X$ and the fiber is 2-dimensional. But notice that the stabilizer group $G'' = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ does not act linearly on the fiber—it only acts through a central extension

\begin{equation}
1 \to \mathbb{T} \to \widetilde{G}'' \to G'' \to 1,
\end{equation}

namely the central extension induced from the central extension (16.23) by extension from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{T}$.

The example of the quaternion group typifies the general case: the group extension (16.16) determines a central extension of the groupoid $X//G''$, and it is this central extension that lifts to act linearly on $E_V \to X$. We defer to [FM1, §9] for the construction and its generalizations to extensions with nonabelian kernel and to extensions for which the kernel or quotient is a compact Lie group or lattice. What is more immediate to construct is that, in particular, for each $\chi \in X$ the stabilizer $G''_\chi \subset G''$ acts on the fiber $(E_V)_\chi = V_\chi$ through a centrally extended group $\widetilde{G}''_\chi$. That extension is defined via the diagram

\begin{equation}
1 \to \mathbb{T} \xrightarrow{\chi} \widetilde{G}''_{\chi} \to G''_\chi \to 1
\end{equation}

The second row is the restriction of the third row over the stabilizer group, and the first row is a “pushout” using the character $\chi$. That is, the image of $G'$ in $G_\chi \times \mathbb{T}$ is normal, and $\widetilde{G}''_\chi$ is defined to be the quotient group. A splitting of the group extension (16.16) induces a splitting of the central extension of $G''_{\chi}$.

Remark 16.29. In our application to the Poincaré group, the extension

\begin{equation}
1 \to V \to \mathcal{P}(M) \to \text{Spin}(V) \to 1
\end{equation}

splits—but not canonically—and so the action of the stabilizers also splits noncanonically.

This discussion results in the following.

Proposition 16.31. There is a bijection between (i) isomorphism classes of irreducible unitary representations $\rho$ of $G$, and (ii) pairs $(O, \lambda)$ of a $G''$-orbit on the set $X$ of characters of $G'$ and an isomorphism class of projective representations of the stabilizer group $G''_\chi$ for some $\chi \in O$. The projectivity of the representation of the stabilizer is computed in (16.28).
Remark 16.32. One can pass from representations to the representation ring of virtual representations. Then the bijection becomes an isomorphism of the representation ring of G with a twisted equivariant K-theory group of X, as explained in [FM1, §9].

16.3 Conformal groups and conformal compactification

This is a digression, another preliminary to the central material in this lecture. Let \( V \) be a real vector space with a conformal structure, i.e., with a nondegenerate symmetric bilinear form up to multiplication by a positive real number. Suppose the form has signature \((p+1, q+1)\) for \( p, q \in \mathbb{Z}_{\geq 0} \). In particular, the form is indefinite. Let \( Q(V) \subset \mathbb{P}V \) be the projectivization of the null cone \( N(V) \subset V \). Recall that any \( N \in \mathbb{P}V \) there is a natural identification \( T_N \mathbb{P}V \cong \text{Hom}(N, V/N) \). Differentiate the equation that cuts out \( N(V) \subset V \) to deduce that if \( N \subset V \) is a null line, then

\[(16.33) \quad T_N Q(V) \cong \text{Hom}(N, N^\perp/N).\]

The conformal structure on \( V \) induces a conformal structure on \( N^\perp/N \): the form restricts on \( N^\perp \) to have kernel \( N \), and the induced form on the quotient induces a form on \( N^\perp/N \) up to a positive constant; its signature is \((p, q)\). Evaluation at a nonzero vector in \( N \) produces an isomorphism \( \text{Hom}(N, N^\perp/N) \cong N^\perp/N \), and the transport of the conformal structure on \( N^\perp/N \) to \( \text{Hom}(N, N^\perp/N) \) is independent of the choice of nonzero vector in \( N \). Altogether, then, \( Q(V) \) has a natural conformal structure of signature \((p, q)\). The projective orthogonal group \( \text{PO}(V) = O(V)/\mu_2 \) acts by conformal symmetries on the quadric \( Q(V) \). Observe that \( Q(V) \subset \mathbb{P}V \) is a closed submanifold of a compact manifold, hence is a compact manifold. In fact, \( Q(V) \approx S^p \times S^q / \mu_2 \) where the involution is the simultaneous antipodal map, as we leave the reader to verify. Then \( \text{PO}(V) \cong O_{p+1,q+1}/\mu_2 \).

![Figure 49. V = W ⊕ H](image)

Let \( H \) be a hyperbolic plane—a 2-dimensional real vector space with a bilinear form of signature \((1, 1)\). Let \( W \) be a finite dimensional real vector space equipped with a nondegenerate symmetric bilinear form, and set \( V = W \oplus H \); see Figure 49. The quadric \( Q(H) \) consists of two points. Fix a basis \( \{e_1, e_2\} \) of \( H \) of null vectors with \( \langle e_1, e_2 \rangle = 1 \). The following proposition exhibits \( Q(V) \) as a conformal compactification of \( W \).
Proposition 16.34. There is a partition

\begin{equation}
Q(V) = W \amalg N(W) \amalg Q(W),
\end{equation}

where

\begin{equation}
W \hookrightarrow Q(V) \quad N(W) \hookrightarrow Q(V) \quad Q(W) \hookrightarrow Q(V)
\end{equation}

\begin{align*}
\xi &\mapsto [\xi; 1, -\|\xi\|^2] & \xi &\mapsto [\xi; 0, 1] & \xi &\mapsto [\xi; 0, 0]
\end{align*}

In the conformal compactification \(Q(V)\), the codimension one submanifold \(N(W)\) and the codimension two submanifold \(Q(W)\) live at infinity. If \(W\) has a positive definite conformal structure, then \(V\) is Lorentzian. In this case \(N(W) = \{0\}\) is a single point, \(Q(W) = \emptyset\) is empty, and the conformal compactification of \(W\) is the sphere.

**Remark 16.37.** Let \(N = \mathbb{R} \cdot e_1\) be the span of \(e_1 \in H\). Then there is a natural isomorphism \(N^\perp / N \cong W\), and so we can identify \(Q(V)\) as the conformal compactification of \(N^\perp / N\).

### 16.4 Orbits and stabilizers of the \(\text{SO}^+(V)\)-action on \(P^* \subset V^*\)

Let \(V\) be a Lorentz vector space equipped with a time-orientation \(P \subset V\); the dual space \(V^*\) has a cone \(P^* \subset V^*\) of positive energy-momentum vectors. Since \(\text{SO}^+(V)\) acts on \(V^*\) by isometries, the norm square is preserved. Recall the definitions of the mass shell (13.19) for \(m \in \mathbb{R}^{>0}\) and of the positive null cone (16.38):

\begin{align}
O_m &= \{ \theta \in P^* : |\theta| = mc \}, \\
N(V^*^+) &= \{ \theta \in P^* \setminus \{0\} : \langle \theta, \theta \rangle = 0 \}.
\end{align}

The following was stated loosely in (14.22). As usual, that \(n = \dim M\).

![Figure 50. The orbits of \(\text{Spin}(V)\) on \(P^*\)](image)

**Proposition 16.39.**
(1) $SO^\uparrow(V)$ acts transitively on $O_m$.
(2) $SO^\uparrow(V)$ acts transitively on $N(V^*)^+$ if $n \geq 3$. For $n = 2$ there are two orbits

$(16.40)$

$$N(V^*)^+ = L^* \sqcup R^*,$$

each of which is a ray. For $n = 1$, $N(V^*)^+ = \emptyset$.

This leads to the enumeration of $SO^\uparrow$-orbits in $P^*$, depicted in Figure 50:

$(16.41)$

$$P^* = \begin{cases} 
\{0\} \sqcup N(V^*)^+ \sqcup \bigcup_{m>0} O_m, & n \geq 3; \\
\{0\} \sqcup L^* \sqcup R^* \sqcup \bigcup_{m>0} O_m, & n = 2; \\
\{0\} \sqcup \bigcup_{m>0} O_m, & n = 1. 
\end{cases}$$

Figure 51. The decompositions used in the proof of Proposition 16.39

Proof. For (1) choose a splitting $V = U \oplus U^\perp$ with $U$ a timelike line, and then there is an induced dual splitting $V^* = U^* \oplus (U^\perp)^*$. Let $\tau_U \in U$ be the forward timelike vector with $|\tau_U| = c$, and let $\theta_U \in U^*$ be the dual positive energy vector: $|\theta_U| = c$. For $\theta = \frac{E}{c}\theta_U + \theta^\perp \in O_m$ compute

$(16.42)$

$$m^2 c^2 = |\theta|^2 = \frac{E^2}{c^2} - |\theta^\perp|^2.$$ 

For fixed $E$, the subgroup $SO(U^\perp) \subset SO^\uparrow(V)$ acts transitively on the sphere in $U^\perp$ of radius $|\theta^\perp|$. This allows us to reduce to the 2-dimensional case. Namely, fix a line $K \subset U^\perp$ and assume, by
the aforementioned transitivity, that \( \theta^\perp \in K \). Fix \( \ell_K^* \in K \) such that \(|\ell_K^*| = 1\). An element of \( \text{SO}^\uparrow(U \oplus K) \) acts on \( U^* \oplus K^* \) by

\[
\begin{pmatrix}
\cosh \varphi & \sinh \varphi \\
\sinh \varphi & \cosh \varphi
\end{pmatrix}, \quad \varphi \in \mathbb{R},
\]

in the basis \( \{\theta_U, \ell_K^*\} \). These matrices act transitively on \( \mathcal{O}_m \cap (U^* \oplus K^*) \):

\[
(16.43) \quad \begin{pmatrix} mc \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} mc \cosh \varphi \\ mc \sinh \varphi \end{pmatrix}
\]

For (2) use the orthogonal decompositions (see Figure 51)

\[
(16.44) \quad \begin{pmatrix} mc \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} mc \cosh \varphi \\ mc \sinh \varphi \end{pmatrix}
\]

and observe that any vector in \( N(V^*)^+ \) transforms to a vector with vanishing \( X^* \)-component under a suitable element of \( \text{SO}(K \oplus X) \). In the chosen basis, an element of \( N(V^*)^+ \cap (U^* \oplus K^*) \) has the form \( \begin{pmatrix} E/c \\ p \end{pmatrix} \) where \( E/c = \pm p \). The matrices (16.43) act transitively on the two rays of \( N(V^*)^+ \), and if \( n \geq 3 \) we can use an element of \( \text{SO}(K \oplus X) \) to transform from one ray to the other. \( \square \)

Next we turn to the stabilizer groups, continuing with the notation in the proof of Proposition 16.39.

**Proposition 16.46.** The stabilizer groups of the \( \text{SO}^\uparrow(V) \)-action on \( \overline{P}^* \) are as follows.

1. At the point \( m\theta_U \in \mathcal{O}_m \), the stabilizer group is \( \text{SO}(U^\perp) \).
2. If \( \theta \in N(V^*)^+ \) spans a null line \( N^* \), for \( N \subset V \) a null line, then the stabilizer group is \( \text{Euc}^0(N^\perp/N) \), where \( \text{Euc}^0 \) is the component of the identity of the Euclidean group.

**Proof.** The proof of (1) is essentially contained in the proof of Proposition 16.39(1): a time-orientation preserving isometry that fixes a forward vector in \( U \) also fixes \( U \), hence is an isometry of \( U^\perp \).

For (2), the projectivization \( Q(V^*) = \mathbb{P}N(V^*) \subset \mathbb{P}V^* \) is diffeomorphic to the \((n - 2)\)-sphere, the conformal compactification of \( N^\perp/N \); see Remark 16.37. \( \text{SO}^\uparrow(V) \) acts as the group of orientation-preserving conformal transformations, as in §16.3. The stabilizer of a point on the sphere—a lightlike line—is the identity component of the Euclidean group together with dilations. The stabilizer group of a point on the lightlike ray does not include the dilations. \( \square \)

Finally, the stabilizer group at \( \{0\} \subset V^* \) of the action of \( \text{SO}^\uparrow(V) \) is the entire group \( \text{SO}^\uparrow(V) \).

**Lecture 17: Irreducible representations of the Poincaré group**

In relativistic quantum mechanics a **relativistic particle** is an irreducible positive energy unitary representation of the Poincaré group (Definition 14.21(3)). In this lecture we give the classification
of these representations, which is usually attributed to Wigner [Wig]. It fits into a general theory
developed for finite groups by Frobenius and generalized to locally compact groups in a long series
of papers by Mackey; see [Ma2, Ma3, Ma4] for a sample. In brief, this is part of the theory of
induced representations. We construct the representations and sketch an argument for why these
are all irreducible positive energy unitary representations.

Let $M$ be a Minkowski spacetime over a Lorentzian vector space $V$. Recall that the Poincaré
group fits into the group extension

$$1 \rightarrow V \rightarrow \mathcal{P}(M) \rightarrow \text{Spin}(V) \rightarrow 1$$

in which the kernel $V$ is abelian, and the extension is split by a choice of $p \in M$. (Hence $\mathcal{P}(M)$ is
a semidirect product group, though not canonically.) We begin with a definition of the little
groups. Then we discuss the classification in question. We make the hypothesis that the stabilizer
subgroup acts via a finite dimensional representation, which leads to a brief discussion of irreducible
representations of the spin group. We review the constraints from quantum field theory already
introduced in Lecture 15—the spin-statistics theorem and the CRT theorem—and we introduce
another constraint due to Weinberg-Witten [WW]. We indicate how the theory generalizes to
other symmetry groups in relativistic quantum mechanics.

17.1 Little groups and invariant measures

We resume the usual relativistic setup and notation.

Recall from §16.4 that there are three basic types of orbits of the action of $\text{SO}^\uparrow(V) \circ \mathcal{P}^\uparrow$: (1) the
mass shell $\mathcal{O}_m$ of positive mass $m \in \mathbb{R}^+0$, (2) an orbit in $N(V^*)^+$ of positive energy null energy
momentum vectors$^{45}$, and (3) the singleton orbit $\{0\}$. The stabilizer groups of $\text{SO}^\uparrow(V) \circ \mathcal{P}^\uparrow$ are

$^{45}$If $n \geq 3$ then $\text{SO}^\uparrow(V)$ acts transitively on $N(V^*)^+$, for $n = 2$ there are two orbits, and $N(V^*)^+ = \emptyset$ for $n = 1$.}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure52.png}
\caption{A timelike line $U$ and a null line $N$}
\end{figure}
computed in Proposition 16.46. For representations of the Poincaré group \( P(M) \) we are interested in the action \( \text{Spin}(V) \circlearrowleft P^* \), and for this spin action the stabilizers are double covers of the previous. Hence in case (1), at \( m\theta_U \in \mathcal{O}_m \) the stabilizer group is

\[
\text{Spin}(U^\perp) \cong \text{Spin}_{n-1},
\]

a compact spin group. In case (2) suppose \( N \subset N(V) \) is a null line, \( N^* \subset N(V^*) \) is the dual null line, and \( \theta \in N^* \cap N(V^*)^+ \). Then the stabilizer group at \( \theta \) fits into the group extension

\[
1 \rightarrow N^\perp/N \rightarrow \widetilde{\text{Euc}}^0(N^\perp/N) \rightarrow \text{Spin}(N^\perp/N) \rightarrow 1;
\]

it is the double cover of the identity component of the Euclidean group of the \((n-2)\)-dimensional inner product space \( N^\perp/N \). In case (3) the stabilizer group is the noncompact Lorentz spin group \( \text{Spin}(V) \).

We are interested in finite dimensional unitary representations of the stabilizer groups. In case (2) we have the following.

**Lemma 17.4.** A finite dimensional unitary representation of \( \widetilde{\text{Euc}}^0(N^\perp/N) \) factors through \( \text{Spin}(N^\perp/N) \).

**Proof.** Any unitary representation restricts to a representation of the translation subgroup \( N^\perp/N \), and its spectrum is a closed subset of the Pontrjagin dual group \( (N^\perp/N)^* \) of characters. That is, the spectrum is a union of orbits of \( \text{SO}(N^\perp/N) \circlearrowleft N^\perp/N \), i.e., is a union of concentric spheres. If the representation is finite dimensional, the spectrum must be \( \{0\} \).

In our study of irreducible representations of the Poincaré group \( P(M) \) it is the action of the spin group \( \text{Spin}(V) \) on \( V^* \) which enters; see (17.1). The stabilizers are the double covers of the groups in Proposition 16.46. Only finite dimensional representations of the stabilizers are relevant, as we will explain, and this motivates the following.

**Definition 17.5.** The little group at \( \theta \in \overline{P^*} \setminus \{0\} \) is \( \text{Spin}(U^\perp) \) for \( \theta = m\theta_U \) and \( \text{Spin}(N^\perp/N) \) for \( \theta \in N^* \cap \overline{P^*} \) a nonzero null vector.

These groups are isomorphic to \( \text{Spin}_{n-1} \) and \( \text{Spin}_{n-2} \), respectively, but not canonically.

**Remark 17.6.** The identification of a stabilizer group with the standard group \( \text{Spin}_{n-1} \) or \( \text{Spin}_{n-2} \) is well-defined up to the outer automorphism induced by an element in the off-component of the corresponding orthogonal group, i.e., up to conjugation by a reflection. That automorphism can act nontrivially on \( \text{Irr}_f(\text{Spin}_m) \). For example, if \( m \) is even it exchanges the self-dual and anti-self-dual irreducible components of the middle exterior power of the vector representations; it also exchanges the two half-spin representations. Also, if \( m = 2 \) it maps the representation of spin \( j \) to the representation of spin \(-j\) (see below). In theories which require a spatial orientation—for example, quantum field theories which are not time-reversal invariant—there is no ambiguity and we can label vector bundles by isomorphism classes of representations of \( \text{Spin}_m \).

The stabilizer group of \( \text{Spin}(V) \circlearrowleft \overline{P^*} \) at \( \{0\} \subset \overline{P^*} \subset V^* \) is the entire group \( \text{Spin}(V) \).
Proposition 17.7. If \( n \geq 3 \), then the only unitary finite dimensional irreducible unitary representation of \( \text{Spin}(V) \) is the one-dimensional representation on which every element of \( \text{Spin}(V) \) acts as the identity.

The proof is left as a homework problem.

Now we turn to invariant smooth measures on the orbits.

Proposition 17.8. Let \( \mathcal{O} \subset \mathbb{P}^* \) be a nonzero orbit of \( \text{Spin}(V) \). Then there is a ray of \( \text{Spin}(V) \)-invariant smooth measures on \( \mathcal{O} \).

Proof. By the argument in the penultimate paragraph of §14.4.1, if \( \mathcal{O} = \mathcal{O}_m \) is a mass shell for some \( m > 0 \), then the invariant smooth measure exists since the stabilizer group (17.2) is compact. For a positive energy null vector, the stabilizer group is the double cover of the identity component of the Euclidean group in (17.3), and it acts on the ray of positive densities on the tangent space to the orbit, which is \( N^\perp \) in the notation of (17.3). Recall the discussion in §16.3: the quadric \( Q(V^*) \) of null lines is an \( (n-2) \)-sphere, and if \( N^* \in Q(V^*) \) is identified as the point at infinity, then the subspace \( N^\perp / N \) in (17.3) acts on the tangent space at infinity as infinitesimal translations at zero, which act as zero. So the action factors to the quotient spin group, which is compact and so acts trivially on densities on the tangent space. \( \square \)

17.2 Irreducible representations of Poincaré

We first construct irreducible unitary representations. Then we state a theorem that these are all of them. The discussion in §16.2 motivates restricting a unitary representation of \( \mathcal{P}(M) \) to the vector subgroup \( V \), and then Theorem 14.19 determines a spectral measure on \( V^* \). If the representation is irreducible and has positive energy, then the support of the measure is a single \( \text{Spin}(V) \)-orbit in \( \mathbb{P}^* \). This is the starting point of the construction. (Recall the analog Proposition 16.31 for finite groups.)

Fix a \( \text{Spin}(V) \)-orbit \( \mathcal{O} \subset \mathbb{P}^* \). By Proposition 16.13 a finite rank \( \text{Spin}(V) \)-equivariant hermitian vector bundle \( E \to \mathcal{O} \) is equivalent to a finite dimensional unitary representation of the stabilizer subgroup at any point of \( \mathcal{O} \). If \( \mathcal{O} = \{0\} \) and \( n \geq 3 \), then by Proposition 17.7 the only finite dimensional irreducible unitary representation is the one-dimensional trivial representation. The corresponding representation of \( \mathcal{P}(M) \) is also the trivial one-dimensional representation. If \( \mathcal{O} \neq \{0\} \), then any finite dimensional unitary representation of the stabilizer factors through the little group, which is a spin group. Fix an invariant smooth measure on \( \mathcal{O} \), which exists and is unique up to a positive scalar by Proposition 17.8. Then \( \text{Spin}(V) \) acts unitarily on the Hilbert space

\[
\mathcal{H} = L^2(\mathcal{O}; E)
\]

of \( L^2 \) sections of \( E \to \mathcal{O} \).

Recall the construction. On the space \( C^\infty(\mathcal{O}; E) \) of smooth sections, define the hermitian form

\[
\langle \psi_1, \psi_2 \rangle = \int_{\mathcal{O}} \langle \psi_1(\theta), \psi_2(\theta) \rangle \, d\mu_{\mathcal{O}}(\theta), \quad \psi_1, \psi_2 \in C^\infty(\mathcal{O}; E),
\]
where the integrand is the pointwise hermitian inner product and $\mu_\mathcal{O}$ is the invariant measure on the orbit. Then (17.9) is the Hilbert space completion of the hermitian vector space $C^\infty(\mathcal{O}; E)$. If $\psi: \mathcal{O} \to E$ is a smooth section of $E \to \mathcal{O}$, and if $g \in \text{Spin}(V)$, then

$$
(17.11) \quad (g \psi)(\theta) = g \cdot \psi(g^{-1} \cdot \theta), \quad \theta \in \mathcal{O}.
$$

The action extends to the Hilbert space completion.

Next, a vector $\xi \in V$ determines a smooth function $\theta \mapsto e^{-i\theta(\xi)/\hbar}$ whose value at $\theta \in V^*$ is the unitary character $\theta$ on the vector $\xi$. The restriction of (17.12) to $\mathcal{O} \subset V^*$ is also smooth, and smooth complex-valued functions on $\mathcal{O}$ act as multiplication operators on $L^2$ sections of vector bundles.

The actions (17.11) of $\text{Spin}(V)$ and (17.12) of $V$ do not commute. Rather, an easy computation shows that they combine to an action of the semidirect product group

$$
(17.13) \quad \text{Spin}(V) \ltimes V
$$

Remark 17.14. The Poincaré group $\mathcal{P}(M)$ is not canonically this semidirect product; it is naturally the extension (17.1). The discussion in §16.2 illustrates how to construct representations of group extensions with abelian kernels. Namely, the extension determines a central extension of the action groupoid $V^* \rightharpoonup \text{Spin}(V)$, and it is vector bundles over this central extension that are needed. Irreducible positive energy bundles have support on a positive energy orbit, and they are determined by an irreducible extension of the central extension of stabilizer subgroup; see (16.28). In this case that is a central extension of a spin group,\footnote{or for nonzero null vectors of a double cover of a Euclidean group} which necessarily splits. Or, more simply, we observe that the Poincaré group is isomorphic to the semidirect product (17.13) upon fixing a point $p_0 \in M$. So the classification of representations of $\mathcal{P}(M)$ is equivalent to the classification of representations of the semidirect product (17.13).

Let $\mathcal{U}^+_\mathcal{P}(M)$ denote the set of isomorphism classes of irreducible positive energy unitary representations of the Poincaré group $\mathcal{P}(M)$. The main theorem, which we do not prove here, asserts that the foregoing construction accounts for all desired representations.

Remark 17.15. We assume that the representations are induced from finite dimensional representations of the stabilizer group, i.e., are presented as sections of a finite rank vector bundle over an orbit of $\text{Spin}(V)$ on $V^*$. This finite dimensionality is equivalent to the assertion that there are a finite number of states at a fixed energy-momentum.

Theorem 17.16. There is a surjective map

$$
(17.17) \quad \mathcal{U}^+_\mathcal{P}(M) \longrightarrow \mathbb{R}^\geq.
$$

The fiber at $m > 0$ is isomorphic to $\text{Vect}_{\text{irr}}(\mathcal{O}_m \rightharpoonup \text{Spin}(V))$. If $n \geq 3$, then the fiber over $m = 0$ is the union $\text{Vect}_{\text{irr}}(N(V^*)^+ \rightharpoonup \text{Spin}(V)) \cup \{\ast\}$. For $n = 2$ the fiber over $m = 0$ has 5 points.
Here ‘∗’ denotes the trivial representation. For \( n = 2 \) there are three massless orbits, and the nonzero ones each admit two equivariant line bundles: the stabilizer subgroup \( \mu_2 \) acts trivially or by the sign representation.

I would like to see a proof of Theorem 17.16 along the lines of the outline in the first paragraph of this section. Wigner’s paper is [Wig], and there are several papers and books of Mackey that discuss induced representations for semidirect products and group extensions; a sample is [Ma2, Ma3, Ma4, Ma5, Ma6]. There is also a discussion in [Sg, §3.9] in a volume that is of broader interest.

Remark 17.18. As explained in Lecture 10 a quantum mechanical system has bosonic and fermionic states, so all representations should be on \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces. This amounts to a pair of representations of the Poincaré group, and so a pair of equivariant vector bundles over an orbit.

### 17.3 Representations of the spin group

So the classification comes down to a classification of representations of the compact spin group \( \text{Spin}_m \) for \( m = n - 1 \) (massive representations) and \( m = n - 2 \) (massless representations). We only scratch the surface.

First, since \( \text{Spin}_2 \cong \mathbb{T} \) is abelian its irreducible complex representations are 1-dimensional. The character group of \( \mathbb{T} \) is infinite cyclic, and for \( \text{Spin}_2 \) we identify that infinite cyclic group with \( \frac{1}{2} \mathbb{Z} \) and so label an irreducible representation by its spin \( j \in \frac{1}{2} \mathbb{Z} \): the representation labeled by \( j \) is \( \lambda \mapsto \lambda^{2j} \). The spin 0 representation is the trivial one and the spin 1/2 representation is the representation on which \( \text{Spin}_2 \cong \mathbb{T} \) acts by scalar multiplication.

For \( m \geq 2 \) we define the spin of an irreducible representation as follows.

**Definition 17.19.** Let \( \text{Spin}_m \to U(W) \) be an irreducible unitary representation. The spin of the representation is the largest spin \( |j| \) that occurs in the decomposition into irreducibles under restriction to \( \text{Spin}_2 \subset \text{Spin}_m \).

The decomposition is independent of the choice of \( \text{Spin}_2 \subset \text{Spin}_m \). The spin is a measure of the complexity of the representation. The spin modulo integers agrees with Definition 15.3: an irreducible representation of integer spin factors through \( \text{Spin}_m \to \text{SO}_m \) whereas in a representation with half-integer spin the central element \( \epsilon \in \text{Spin}_m \) acts as \(-1\).

**Remark 17.20.** For massless little group representations, the ‘spin’ is usually called ‘helicity’.

There is a unique spin 0 irreducible representation of \( \text{Spin}_m \): the trivial representation.

A spin 1/2 representation is called a spin representation; it extends to a module over the even Clifford algebra. For \( m \) odd there is a unique irreducible spin representation \( S \) of dimension \( 2^{(m-1)/2} \); for \( m \) even there are two “half-spin” representations \( S^\pm \), each of dimension \( 2^{m/2-1} \). In low dimensions we can use the special isomorphisms (12.24) to describe/construct the spin representations. Thus \( \text{Spin}_2 \cong \mu_2 \) and the spin representation is the sign representation on \( S = \mathbb{C} \). As above, \( \text{Spin}_2 \cong \mathbb{T} \) and the two half-spin representations are the spin \( \pm 1/2 \) representations on \( \mathbb{C} \); they are

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\[47\] That is not a very helpful description without an explicit isomorphism \( \text{Spin}_2 \cong \mathbb{T} \). Better: \( \text{Spin}_2 \) embeds in the real Clifford algebra \( \text{Cliff}_{-2} \) generated by \( e_1, e_2 \) with \( e_1^2 = -1 \) as elements \( \cos(\theta/2) + \sin(\theta/2)e_1e_2, 0 \leq \theta \leq 4\pi \), and the spin \( j \) representation sends the element labeled \( \theta \) to multiplication by \( e^{\sqrt{-1}\theta/2} \). See [ABS] for the basics of Clifford algebras and spin groups.
complex conjugates. Then $\text{Spin}_3 \cong \text{SU}_2$, and the spin representation $S = \mathbb{C}^2$ is the defining representation of $\text{SU}_2$. Finally, $\text{Spin}_4 \cong \text{SU}_2 \times \text{SU}_2$, and the two half-spin representations $S^\pm \cong \mathbb{C}^2$ factor through projection onto a factor. Each of these representations is self-conjugate (quaternionic).

The basic spin 1 representation is the vector representation $\text{Spin}_m \rightarrow \text{SO}_m \circ \mathbb{C}^m$. The exterior power representations

\begin{equation}
\bigwedge^\ell \mathbb{C}^m, \quad \ell = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \tag{17.21}
\end{equation}

also have spin 1. If $m$ is even, then the middle exterior power representation splits into a sum

\begin{equation}
\bigwedge^{m/2} \mathbb{C}^m = \bigwedge_{+}^{m/2} \mathbb{C}^m \oplus \bigwedge_{-}^{m/2} \mathbb{C}^m \tag{17.22}
\end{equation}


The symmetric power $\text{Sym}^s \mathbb{C}^m, s \in \mathbb{Z} \geq 0$ has spin $s$.

Remark 17.23. For the Poincaré group, then, a particle—an irreducible positive energy unitary representation—has a mass $m \in \mathbb{R}^>0$ and a spin $j \in \frac{1}{2} \mathbb{Z}^\geq 0$. These are the quantum numbers of the particle. However, these quantum numbers do not determine the particle, as the examples above illustrate.

17.4 Constraints from relativistic quantum field theory

The two basic theorems of relativistic quantum field theory—Theorem 15.4 and Theorem 15.6—constrain the particle representations that occur in relativistic quantum field theory.

First, as in Remark 17.18 a quantum field theory has a $\mathbb{Z}/2\mathbb{Z}$-graded unitary representation $\rho: \mathcal{P}(M) \rightarrow \text{U}(\mathbb{H}^0 \oplus \mathbb{H}^1)$, which amounts to a pair of unitary representations. The spin-statistics theorem implies the following.

Corollary 17.24. In a quantum field theory the little group $(\text{Spin}_m)$ representation has integer spin for bosonic states and half-integer spin for fermionic states.

The CRT theorem states that in a quantum field theory the representation of the Poincaré group extends to a representation of a 2-component Lie group that double covers the middle group in

\begin{equation}
1 \rightarrow V \rightarrow \text{SO}(M) \rightarrow \text{SO}(V) \rightarrow 1 \tag{17.25}
\end{equation}

and that the off-component acts antiunitarily. Thus the $\text{SO}^+(V)$-action on $V^*$ extends to an action of $\text{SO}(V)$, but since the off-component is to act antilinearly, it conjugates the character and so the usual action of the off-component on $V^*$ is multiplied by $-1$. If $n$ is even, then $-1 \in \text{SO}(V)$, and so the stabilizer group of the $\text{SO}(V)$-action at a point $\theta \in \overline{F}$ is the direct product of the $\text{SO}^+(V)$ stabilizer group and the cyclic group $\mu_2$. Furthermore, the generator of $\mu_2$ acts antilinearly on the fiber of an equivariant vector bundle. In other words, that generator is a real structure that commutes with the little group representation. For the Poincaré group we obtain a central extension, and the new generator may act as a quaternionic structure. This proves the following.
Corollary 17.26. If $n$ is even, then in a quantum field theory the little group representation is self-conjugate.

This is an important constraint. For example, the self-dual and anti-self-dual representations (17.22) of $\text{SO}_2$ are not self-conjugate (they are each other’s conjugate), so they cannot occur, whereas they are self-conjugate in the case of $\text{SO}_4$.\(^{48}\)

There is another basic constraint due to Weinberg-Witten [WW]. It states, roughly, that massless particles in a quantum field theory have helicity $\leq 1$ (and in a theory of quantum gravity they have helicity $\leq 2$).

17.5 More examples

We illustrate the classification by exhibiting continuous families of representations. Recall that the representations of a compact Lie group do not deform: the space of isomorphism classes of irreducible representations is discrete. Not so for a noncompact Lie group, such as the Poincaré group. We also illustrate Corollary 17.26 in this discussion. The basic idea is to consider a continuous family of massive representations with $m > 0$ and to consider the limit $m \to 0$. This produces a massless (mass zero) representation. We are led to construct (different) massless representations that do not deform to massive representations. We only describe a few illustrative cases.

Example 17.27. Set $n = 4$ so that the massive little group is isomorphic to $\text{Spin}_3 \cong \text{SU}_2$ and the massless little group is isomorphic to $\text{Spin}_2 \cong \mathbb{T}$. Every representation of $\text{SU}_2$ is self-conjugate—either real or quaternionic—and so there is no constraint on massive representations from Corollary 17.26. However, irreducible massless representations are parametrized by $j \in \frac{1}{2}\mathbb{Z}$, and if $j \neq 0$ the representation is not self-conjugate: the conjugate of the helicity $j$ representation is the helicity $-j$ representation. On the other hand, for any $j \in \frac{1}{2}\mathbb{Z}^\geq 0$ the 2-dimensional representation $j \oplus -j$ is self-conjugate: it admits either a real or a quaternionic structure.

There is a unique spin 0 particle for any mass $m \geq 0$ and these form a continuous family of irreducible representations.

The vector representation of the massive little group is 3-dimensional, so gives rise to a rank 3 equivariant vector bundle over each mass shell $O_m$, $m > 0$, and these form a continuous family. The limit $m \to 0$ is again rank 3, now supported over the positive energy null vectors $N(V^*)^\perp$. The corresponding representation of the massless little group is reducible: it is the sum of the vector representation of $\text{Spin}_2$ and the trivial 1-dimensional representation.

We claim that the 2-dimensional vector representation of $\text{Spin}_2$, which is real,\(^{49}\) gives rise to a massless representation that does not deform to a massive representation. For if it did, the central element $\epsilon \in \text{Spin}_3$ acts trivially, and so the representation drops to $\text{SO}_3$. However, there is no 2-dimensional real representation of $\text{SO}_3$.

This means that the massless vector particle is rigid: it does not deform. (Note in general that the space of isomorphism classes of irreducible representations of Poincaré with fixed mass is discrete.) We will see that this representation occurs in 4-dimensional Maxwell theory, the quantum theory of electromagnetism. We conclude: that theory does not have a massive deformation.

\(^{48}\)The latter occur as massless particles in some 6-dimensional quantum field theories. There are similar massless particles in 10-dimensional quantum field theory approximations to superstring theories.

\(^{49}\)It is $1 \oplus -1$ in the notation above.
Example 17.28. Now set $n = 3$. The massless vector particle is induced from the trivial 1-dimensional representation of the little group $\text{Spin}_1 \cong \mathbb{Z}_2$. This representation does deform to the massive vector particle induced from either the spin 1 or the spin $-1$ representation of the little group $\text{Spin}_2 \cong \mathbb{T}$. (These representations are not self-conjugate, but the restriction in Corollary 17.26 from CRT only applies in even spacetime dimensions.) We will see (in a homework problem) that this massive deformation comes about from a Chern-Simons term that is added to the Maxwell theory.

17.6 General relativistic symmetry types

Recall from Definition 15.13 that the general symmetry type in $n$-dimensional relativistic quantum mechanics is specified by a Lie group $G_n$ and a homomorphism $\lambda: G_n \to O_{1,n-1}^\uparrow$ whose image contains the identity component $SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow$. The homomorphism $\lambda$ fits into the diagram

\[
\begin{array}{ccccccccc}
1 & & 1 & & & & & & \\
& & & & & & & & \\
& & & & K & & K & & \\
& & & & & & & & \\
1 & & & & \mathbb{R}^{1,n-1} & & \mathcal{G}_n & & G_n & & 1 \\
& & & & & & & & \\
& & & & 1 & & \mathcal{I}_{1,n-1}^\uparrow & & \mathcal{O}_{1,n-1}^\uparrow & & 1 \\
\end{array}
\]

(17.29)

in which the rows are group extensions and $K$ is the kernel of $\lambda$. The group $K$ is the group of internal symmetries. If $K$ is a compact Lie group, as often occurs, then the (new) little group representations at fixed positive energy-momentum are still discrete, but there are additional discrete invariants—quantum numbers—that label the representations. The induced irreducible positive energy unitary representations of $\mathcal{G}$ are the particles for this symmetry type, and they carry the additional quantum numbers.

Example 17.30. Consider $G_n = \text{Spin}_{1,n-1} \times \mathbb{Z}_2 \mathbb{T} = \text{Spin}_{c,1,n-1}^c$, and so $\mathcal{G}_n = \mathcal{P}_n \times \mathbb{Z}_2 \mathbb{T}$. Note that $\text{Spin}_{c,1,n-1}^c$ is noncompact; it is the Lorentz spin-c group. On the other hand, the little group is the compact Lie group $\text{Spin}_m^c$, where as usual $m = n - 1$ in the massive case and $m = n - 2$ in the massless case. Irreducible representations of $\text{Spin}_m^c = \text{Spin}_m \times \mathbb{Z}_2 \mathbb{T}$ are the tensor product of an irreducible representation of $\text{Spin}_m$ and an irreducible representation of $\mathbb{T}$. The latter is indexed by an integer, which in many contexts is an electric charge. Regardless, that integer is a new quantum number for this symmetry type. That the central element $\epsilon \in \text{Spin}_m$ and $-1 \in \mathbb{T}$ act equally in the representation enforces a constraint, once we apply Corollary 17.24: bosonic states have even electric charge and fermionic states have odd electric charge.
**Lecture 18: Locality and fields**

A relativistic quantum mechanical system (Definition 14.16) is a unitary positive energy representation of the Poincaré group, or of some other symmetry group of a system on a Minkowski spacetime. Quantum field theories are an important class of examples; they have more data than the representation. In particular, they have fields. In this lecture we introduce a very general concept of a field, and then we give an example of a free classical field theory, the theory of a real massive scalar field. In the next lecture we quantize this theory and we also give examples of other free field theories.

The signal property of a field is locality. Local fields also occur in differential geometry (vector fields, tensor fields, spinor fields), and so we begin with a general discussion of local objects in geometry: sheaves. Sheaves on a fixed space are more familiar than sheaves on a class of spaces—here manifolds of a fixed dimension—and so we introduce that more general concept. Fields in both geometry and physics often have internal symmetries—gauge symmetries—and so form groupoids or higher groupoids. The associated sheaf theory has a homotopical flavor, which we only hint at here and leave for references (and references in references).

Following the general discussion we specialize to fields on Minkowski spacetime, which have an additional property: Poincaré invariance. (As in many situations, what sounds like a condition is in fact data.) The classical wave equations that cut out classical fields are also local, and so classical solutions form a subsheaf of all fields. The space of global classical solutions on Minkowski spacetime has a symplectic structure. We illustrate all of this for the massive scalar field.

### 18.1 Fields as sheaves

Here are some “local quantities” on a smooth manifold:

1. real-valued functions
2. double covers
3. orientations
4. Riemannian metrics
5. connections on principal $G$-bundles for a fixed Lie group $G$
6. spin structures

How do we encode locality in a mathematical structure? One common feature is a \textit{global} $\rightarrow$ \textit{local} operation: given any of these objects on a manifold, we can restrict to an open submanifold (open subset). What is more or less the same, we can pull back under a local diffeomorphism. Some of these—(1), (2), (5)—admit a pullback operation for \textit{any} smooth map of smooth manifolds, but that is not true of the others. A second common feature is a \textit{local} $\rightarrow$ \textit{global} operation. Given an open cover of a manifold, an object of one of these types on each open set, and isomorphisms on intersections of open sets, then we can glue the local objects uniquely into a global object.

**Example 18.1.** Consider the map $M \mapsto H^1(M;\mathbb{Z}/2\mathbb{Z})$ which assigns to a smooth manifold $M$ the abelian group of isomorphism classes of double covers over $M$. This has a \textit{global} $\rightarrow$ \textit{local} structure: isomorphism classes of double covers restrict to isomorphism classes of double covers. But it does not satisfy a \textit{local} $\rightarrow$ \textit{global} condition. For example, cover the circle $S^1$ by two overlapping open intervals $U_1, U_2$, as in Figure 53. Then $H^1(-;\mathbb{Z}/2\mathbb{Z}) = 0$ on $U_1, U_2$, and $U_1 \cap U_2,$
but globally we have $H^1(S^1;\mathbb{Z}/2\mathbb{Z}) \neq 0$. To construct the nontrivial double cover of $S^1$ from local data we must remember how that local data is glued, not just that there exists a gluing. In other words, isomorphisms of double covers must be part of the data, not merely a condition. Yet again: double covers are local objects only if we treat the collection of double covers as a groupoid, not as a set. This lesson—that isomorphisms and automorphisms (gauge transformations) must be tracked and not merely noted—is crucial for building local quantum field theories out of connections (gauge fields) and their cousins of higher categorical depth.

Fix one of the “local quantities” in the list above, and denote by $\mathcal{F}(M)$ the collection of all such objects on a smooth manifold $M$. What mathematical structure does $\mathcal{F}(M)$ have? In cases (1), (3), and (4) it is a set. But in cases (2), (5), and (6) the objects in $\mathcal{F}(M)$ have internal symmetries: a double cover has deck transformations, and a connection may have automorphisms. In these cases $\mathcal{F}(M)$ is a groupoid. There are other examples of fields that have additional layers of structure—automorphisms of automorphisms of . . . , etc.—and in those cases the collection of fields is a higher groupoid. Technically, it is easier to use simplicial sets than higher groupoids; see the discussion in [FH2, §4] and the references therein.

The following very general definition is used to encode the global $\rightarrow$ local property.

**Definition 18.2.** Let $C$ be a category, let $\textbf{Set}$ be the category of sets. A presheaf of sets on $C$ is a functor $\mathcal{F}: C^{\text{op}} \rightarrow \textbf{Set}$.

The opposite category $C^{\text{op}}$ has the same objects and morphisms as $C$, but the source and target of a morphism are reversed, as is the order of composition. There is also a category $\textbf{Gpd}$ of groupoids, a category $\textbf{Set}_\Delta$ of simplicial sets, and corresponding presheaves of groupoids and simplicial presheaves

$$\mathcal{F}: C^{\text{op}} \rightarrow \textbf{Gpd}$$

$$\mathcal{F}: C^{\text{op}} \rightarrow \textbf{Set}_\Delta$$

---

50 A double cover is the special case of (5) in which $G$ is cyclic of order 2. If $M$ is connected, then the automorphism group of a $G$-connection on $M$ is noncanonically isomorphic to the centralizer in $G$ of the holonomy group with respect to some basepoint. A spin structure is a “twisted version” of a double cover; it too has automorphisms.

51 Let $\Delta$ be the category whose objects are nonempty totally ordered finite sets and whose morphisms are order-preserving maps. A simplicial set is a functor $\Delta^{\text{op}} \rightarrow \textbf{Set}$, which explains the notation $\textbf{Set}_\Delta$. 

---

**Figure 53.** A covering of the circle by two open sets.
Example 18.4. Let $X$ be a smooth manifold. Define the category $\text{Open}(X)$ whose objects are open subsets of $X$ and whose morphisms are inclusions of open sets. A presheaf on $X$ is a presheaf on the category $\text{Open}(X)$. For example, there are presheaves of sets $\mathcal{F}_1, \mathcal{F}_2$ which assign to an open set $\mathcal{U} \subset X$

$$\begin{align*}
\mathcal{F}_1(\mathcal{U}) &= \{\text{orientations of } \mathcal{U}\} \\
\mathcal{F}_2(\mathcal{U}) &= \{\text{Riemannian metrics on } \mathcal{U}\}
\end{align*}$$

Example 18.6. Let $C = \text{Man}$ be the category whose objects are smooth manifolds and whose morphisms are smooth maps. The following are presheaves of sets or groupoids on $\text{Man}$:

$$\begin{align*}
\mathcal{F}_1(M) &= \Omega^*_M \\
\mathcal{F}_2(M) &= \text{Map}(M, X) \\
\mathcal{F}_3(M) &= \{\text{groupoid of } G\text{-connections on } M\} \\
\mathcal{F}_4(M) &= H^1(M; \mathbb{Z}/2\mathbb{Z}) \\
\mathcal{F}_5(M) &= \{\text{groupoid of double covers of } M\}
\end{align*}$$

Here $X$ is a fixed smooth manifold and $G$ is a fixed Lie group. The following assignments are not presheaves on $\text{Man}$ since there is no pullback under arbitrary smooth maps:

$$\begin{align*}
M &\mapsto \{\text{Riemannian metrics on } M\} \\
M &\mapsto \{\text{orientations of } M\} \\
M &\mapsto \{\text{groupoid of spin structures on } M\}
\end{align*}$$

Example 18.9. For $n \in \mathbb{Z}^\geq 0$ let $C = \text{Man}_n$ be the category whose objects are smooth manifolds of dimension $n$ and whose morphisms are local diffeomorphisms. There is an “inclusion” functor $\text{Man}_n \to \text{Man}$, so any presheaf on $\text{Man}$, such as those in (18.7), pulls back to a presheaf on $\text{Man}_n$. In addition, the assignments (18.8) are presheaves of sets or groupoids on $\text{Man}_n$.

Remark 18.10.

(1) In Example 18.6 and Example 18.9 we can think of a presheaf as analogous to a distribution, but on manifolds rather than on functions, in which case $M$ may be regarded as a “test manifold”.

(2) Fields in relativistic field theory fall under Example 18.4 ($X$ is Minkowski spacetime), and fields in a Wick-rotated $n$-dimensional field theory form a presheaf on $\text{Man}_n$, as in Example 18.9.

Now we turn to the local $\to$ global condition: the sheaf condition. In the general setting of presheaf on an arbitrary category $C$ additional structure on $C$ is required: a Grothendieck topology. For the examples $C = \text{Open}(X), C = \text{Man}, C = \text{Man}_n$ the definition is more familiar.
Definition 18.11. Let \( \mathcal{F} : \text{Man}^{\text{op}} \to \text{Set} \) be a presheaf. Then \( \mathcal{F} \) is a sheaf if for every manifold \( M \) and every open cover \( \{ U_\alpha \}_{\alpha \in A} \) of \( M \)

\[
\mathcal{F}(M) \longrightarrow \prod_{\alpha_0} \mathcal{F}(U_{\alpha_0}) \longrightarrow \prod_{\alpha_0, \alpha_1} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1})
\]

is an equalizer diagram.

This is the usual gluing property of a sheaf: given \( x_{\alpha_0} \in \mathcal{F}(U_{\alpha_0}) \) such that the restrictions of \( x_{\alpha_0} \) and \( x_{\alpha_1} \) to \( U_{\alpha_0} \cap U_{\alpha_1} \) agree, there is a unique \( x \in \mathcal{F}(M) \) which restricts on \( U_{\alpha_0} \) to \( x_{\alpha_0} \). We often say \( \mathcal{F}(U) \) is the set of sections of the sheaf \( \mathcal{F} \) on the open set \( U \). There is a similar definition for sheaves of sets on \( \text{Open}(X) \) or on \( \text{Man}_q \).

For a sheaf of groupoids or a sheaf of simplicial sets the analog of Definition 18.11 uses a bit of homotopy theory: see [FH2, §5].

Example 18.13. The presheaves in (18.5), (18.7), and (18.8) are all sheaves except for \( \mathcal{F}_4 \) in (18.7), as already emphasized in Example 18.1.

18.2 Fields on Minkowski spacetime

Recall the background data of a relativistic field theory: a Lorentzian vector space \( V \), an affine space \( M \) over \( V \), a time-orientation \( P \subset V \), a speed of light \( c \in \mathbb{R}^{>0} \), and a relativistic symmetry type (Definition 15.13): a Lie group \( G \) and a homomorphism \( \lambda : G \to O^+(V) \) whose image contains \( SO^+(V) \). There results a Lie group \( \mathcal{G} \) and a homomorphism \( \tilde{\lambda} : \mathcal{G} \to O^+(M) \) that fit into the diagram (15.14).

A relativistic field theory also needs fields. Recall that sheaves push forward under maps.

Definition 18.14. A (collection of) relativistic field(s) on \( M \) is a simplicial sheaf \( \mathcal{F} : \text{Open}(M) \to \text{Set}_\Delta \) which is \( \mathcal{G} \)-invariant in the sense that it is equipped with isomorphisms

\[
\mathcal{F} \xrightarrow{\mathcal{G}} \tilde{\lambda}(g)_* \mathcal{F}, \quad g \in \mathcal{G},
\]

that satisfy appropriate transitivity for \( g_1, g_2 \in \mathcal{G} \).

For sheaves of sets the transitivity is a simple equality; for sheaves of groupoids or simplicial sets it involves more data to express higher coherence.

Example 18.16. Here are typical examples of fields. The symmetry types are either \( O^+(V) \xrightarrow{\text{id}} O^+(V) \) or \( \text{Spin}(V) \to O^+(V) \). Variations and many more examples exist.

1. Fix a smooth manifold \( X \) and let \( \mathcal{F}(U) = \text{Map}(U, X) \). In general, this is called a scalar field or \( \sigma \)-model field. For \( X = \mathbb{R} \) it is a real scalar field, for \( X = \mathbb{C} \) it is a complex scalar field, for \( X = \mathfrak{g} \) a Lie algebra it is a Lie algebra-valued scalar field, etc.

\[\text{A map } X \to Y \text{ induces a pullback } \text{Open}(Y) \to \text{Open}(X) \text{ on open sets, and a presheaf } \text{Open}(X)^{\text{op}} \to \text{Set}_\Delta \text{ pushes forward to } \text{Open}(Y)^{\text{op}} \to \text{Set}_\Delta \text{ by composition with } \text{Open}(Y)^{\text{op}} \to \text{Open}(X)^{\text{op}}.\]
(2) Now suppose $X$ is a smooth manifold equipped with an action of $G$. Define the sheaf $\mathcal{F}(U) = \text{Map}(U, X)$ as in (1), but now $g \in G$ acts on $\mathcal{F}(U)$ by

$$\mathcal{F}(U) \xrightarrow{\pi(g)} \mathcal{F}(\tilde{\lambda}(g)U)$$

where $\pi(g) \in G$ is the image of $g$ under the homomorphism $\pi : G \to G$.

(3) In classical Maxwell theory we have the classical electromagnetic field $\mathcal{F}(U) = \Omega^2_{\text{closed}}(U)$.

This description does not include gauge invariance, so in many ways it is advantageous to take instead the groupoid-valued field $\mathcal{F}(U) = \{ \mathbb{R}^{>0} \text{-connections on } U \}$.

(4) For a Lie group $G$ we have the groupoid-valued sheaf $\mathcal{F}(U) = \{ G \text{-connections on } U \}$ of gauge fields with gauge group $G$.

(5) This is an important special case of (2). Let $\mathcal{S}$ be a real spinor representation of $\text{Spin}(V)$, i.e., a real module over a real Clifford algebra built from the Lorentzian vector space $V$.

Set $\mathcal{F}(U) = \text{Map}(U, \mathcal{S})$. Then $\mathcal{F}$ is called a spinor field. It should be treated as an odd (fermionic) field and so has a quite different nature from bosonic fields.\footnote{We do not elaborate on fermionic fields here; see [F5, §1] for a heuristic account.}

The symmetry type here is $\text{Spin}_{1,n-1} \to \text{O}_{1,n-1}$. Note that the $\text{Spin}(M)$ action is used to push forward $\mathcal{F}$, as in (18.15), and in addition the isomorphism (18.15) uses the $\text{Spin}(V)$-action on $\mathcal{S}$.

There are both classical and quantum relativistic field theories. A sequence of data/steps in the specification of and passage from a classical field theory to a quantum field theory is:

(A) A sheaf $\mathcal{F}$ of relativistic fields
(B) A subsheaf $\mathcal{M} \subset \mathcal{F}$ of “classical solutions”, often cut out by a wave equation
(C) A symplectic structure on $\mathcal{M}(M)$
(D) A positive energy polarization of the symplectic manifold $\mathcal{M}(M)$
(E) Quantize!

In §18.3 we stop short of (D) and (E), which we take up in the next lecture.

Remark 18.18.

(1) The data (B) and (C) is often derived from an action principle: a Lagrangian. It gives more than a symplectic structure (see §3.4). Namely, one constructs a complex line bundle $\mathcal{L} \to \mathcal{M}(M)$ with covariant derivative whose curvature is the symplectic form (up to a constant). In the purely symplectic theory this is called a “prequantization line bundle”.

This geometric prequantization is discussed in many sources, for example [Ko, SiWo].

(2) The positive energy condition on the polarization (D) is crucial; positive energy typically characterizes the polarization.

(3) When $\mathcal{M}(M)$ is an affine or linear symplectic space, then there is a good mathematical theory of quantization. Affine spaces of classical solutions are a signal of a “free” theory; see Definition 14.14(2). The theory of geometric quantization [BW] also illuminates quantization in many other situations, but the old adage still applies:

"Quantization is an art, not a functor"
18.3 The massive scalar field: classical theory

Fix the relativistic symmetry type

\[(18.19) \lambda: O^\uparrow(V) \times g_2 \longrightarrow O^\uparrow(V)\]

which is projection onto the first factor. The internal symmetry group \(K = g_2\) is cyclic of order 2. The corresponding affine symmetry group is

\[(18.20) \mathcal{G} = O^\uparrow(M) \times g_2.\]

The field \(\mathcal{F}\) is a real scalar field, defined as \(\mathcal{F}(U) = C^\infty(U; \mathbb{R})\). (We choose \(C^\infty\) functions for classical fields.) The internal symmetry acts on a scalar field \(\phi\) by \(\phi \mapsto -\phi\). For the real scalar field, \(\mathcal{F}\) is a sheaf of real vector spaces.

The inverse metric—the Lorentz inner product on the dual space \(V^*\)—is an \(O^\uparrow(V)\)-invariant vector in \(\text{Sym}^2 V\). View \(V\) as the space of translation-invariant vector fields on \(M\), and so the inverse metric defines an \(O^\uparrow(M)\)-invariant second order differential operator on \(M\): the wave operator \(\Box\). For standard Minkowski spacetime with coordinates \(x^0 = ct, x^1, \ldots, x^{n-1}\) and metric

\[(18.21) c^2 dt^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2,\]

we have

\[(18.22) \Box = \frac{1}{c^2} \partial_t^2 - \partial_1^2 - \cdots - \partial_{n-1}^2.\]

The key ingredient in the massive scalar field theory is the *Klein-Gordon equation* for mass \(m > 0\):

\[(18.23) \left(\Box + \frac{c^2}{\hbar^2} m^2\right) \phi = 0.\]

The equation is local, so defines a subsheaf \(\mathcal{M} \subset \mathcal{F}\): for an open \(U \subset M\), the subset \(\mathcal{M}(U) \subset \mathcal{F}(U)\) consists of \(C^\infty\) functions \(\phi: U \rightarrow \mathbb{R}\) that satisfy (18.23). Equation (18.23) is invariant under \(\phi \mapsto -\phi\), and so is compatible with the symmetry (18.19). Also, (18.23) is a linear equation, hence \(\mathcal{M}\) is a sheaf of real vector spaces.

The last ingredient of the classical theory is a symplectic structure on the real vector space \(\mathcal{M}(M)\) of global classical solutions. As mentioned in Remark 18.18(1), the symplectic structure can be derived from a Lagrangian; see [DF2, §3.2]. It turns out to be the 2-form

\[(18.24) \omega(\dot{\phi}_1, \dot{\phi}_2) = \int_N *d\dot{\phi}_1 \cdot \dot{\phi}_2 - *d\dot{\phi}_2 \cdot \dot{\phi}_1,\]

where \(N \subset M\) is a spacelike affine hyperplane and \(*\) is derived from the Lorentz inner product: it maps \(q\)-forms to orientation-twisted \((n - q)\)-forms, hence the integrand in (18.23) restricts to a density on \(N\). Also, the “variations” \(\dot{\phi}_1, \dot{\phi}_2\) are assumed to have compact spatial support.
Remark 18.25. This compact support condition reflects an important feature of field theory that we have not attended to. There is not a single classical field theory on $M$, but rather a family of classical field theories parametrized by the asymptotic values of the field at infinity in $M$, and furthermore the values at infinity are constrained to have finite energy. (Here again, energy is derived from the Lagrangian; see [DF2, §2.10].) Since the fields satisfy a wave equation (18.24), it suffices to specify the values at spatial infinity. These families of theories often encode symmetries.

It is convenient to view the global space of classical solutions $\mathcal{M}(M)$ via Fourier transform. For that we choose a basepoint $p_0 \in M$ and so identify

$$
V \longrightarrow M \\
\xi \longmapsto p_0 + \xi
$$

Finite energy imposes a decay condition at spatial infinity, which makes the Fourier transform well-behaved. First, there is an $O(V)$-invariant measure $d\xi$ on $V$ from the Lorentz metric: it assigns volume 1 to a parallelepiped spanned by an orthonormal basis. There is a dual measure $d\theta$ on $V^*$. Then the Fourier transform of $\phi = \phi(\xi)$ is a function $\hat{\phi}(\theta)$ on $V^*$:

$$
\hat{\phi}(\theta) = \frac{1}{(2\pi)^{n/2}} \int_V d\xi \, e^{i\theta(\xi)/\hbar} \phi(\xi)
$$

$$
\phi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{V^*} d\theta \, e^{-i\theta(\xi)/\hbar} \hat{\phi}(\theta)
$$

The Fourier transform of the second-order ordinary differential operator in (18.23) is multiplication by a quadratic polynomial; the Fourier transformed Klein-Gordon equation is

$$
\left(-\frac{|\theta|^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2}\right) \hat{\phi}(\theta) = 0.
$$

Also, the Fourier transform of a real-valued function $\phi$ satisfies the reality constraint

$$
\hat{\phi}(-\theta) = \overline{\phi(\theta)}.
$$

If $\hat{\phi}$ satisfies (18.29), then its support lies in $\mathcal{O}_m \cup -\mathcal{O}_m$, where the mass shell $\mathcal{O}_m \subset P^*$ is defined in (13.19); see Figure 54. In this way we identify $\mathcal{M}(M)$ as the real points in the complex vector space of suitable functions $\hat{\phi}: \mathcal{O}_m \cup -\mathcal{O}_m \rightarrow \mathbb{C}$ with real structure $\hat{\phi} \mapsto \iota^*\hat{\phi}$, where $\iota: V^* \rightarrow V^*$ is reflection in the origin: multiplication by $-1$.

The generator of the internal symmetry group $\mathfrak{g}_{12}$ acts on $\mathcal{M}(M)$ by $\phi \mapsto -\phi$ and so too $\hat{\phi} \mapsto -\hat{\phi}$. Also, under Fourier transform the multiplication in the symplectic form (18.24) becomes convolution, and the set of $\hat{\phi}$ with support on $\mathcal{O}_m$ is a complex Lagrangian subspace, as is the set of $\hat{\phi}$ with support on $-\mathcal{O}_m$.

\[54\] The signs are chosen so that (18.28) expresses $\phi$ as a linear combination of characters of $V$: recall that $\theta \in V^*$ labels the character $\xi \mapsto e^{-i\theta(\xi)/\hbar}$.
Lecture 19: Quantization of free fields

We discuss the quantization of free field theories. This is the passage from classical waves to quantum particles.

We begin with a general discussion of the quantization of a symplectic vector space, a topic we have already treated in the finite dimensional setting. The main point is the introduction of a polarization. Most relevant for free fields are the Kähler polarizations. There is also a quantization theory for odd symplectic vector spaces, but here we restrict to the even case.

In §19.2 we resume our discussion of the massive scalar field, now taking up its quantization. We briefly comment on the massless scalar field as well. The electromagnetic field is another example of a free field theory; the classical version is based on Maxwell equations. The new point here is gauge symmetry. Nonabelian gauge theories are mentioned at the end of the lecture as an example of a non-free quantum theory.

19.1 Quantization of a symplectic vector space

We recount the algebraic construction of a Heisenberg representation, which we apply in the next section of the massive scalar field. The finite dimensional version of this construction appears in Lecture 6: see (6.17).

Let $W$ be a real symplectic vector space with symplectic form $\omega$. If $W$ is infinite dimensional, then we assume that it is equipped with a suitable topology and that $\omega$ is continuous. Define the Weyl algebra (see Remark 6.21(2)) on the dual space $W^*$:

\[(19.1) \quad W(W^*) = \otimes^*(W^*)/I,\]

where $I$ is the 2-sided ideal in the tensor algebra $\otimes^*(W^*)$ generated by

\[(19.2) \quad \lambda_1 \otimes \lambda_2 - \lambda_2 \otimes \lambda_1 - \omega(\lambda_1, \lambda_2) \cdot 1, \quad \lambda_1, \lambda_2 \in W^*.\]
Here we use the induced symplectic form on the dual space. Replace $\omega$ by $t\omega$ for $t \in \mathbb{R} \neq 0$ to obtain a family of algebras $W_t(W^*)$. There is a limiting algebra as $t \to 0$: the symmetric algebra $\text{Sym}^\bullet(W^*)$. This is the algebra of polynomial, or algebraic, functions on $W$. Hence the Weyl algebra is a noncommutative deformation of the commutative algebra of functions on $W$. Recall from (1.18) that functions on $W$ are the observables of the classical system derived from $W$. The Weyl algebra plays the role of algebraic observables in a deformation quantization of this classical system. A complete quantization includes states as well as observables. For a $C^*$-algebra of observables we defined states in Definition 11.27. Here we have a nontopological algebra $W(W^*)$ of observables, and the first step is to construct an irreducible module over this algebra. Then we introduce a hermitian inner product and complete the module to a Hilbert space $H$ and construct states from $H$ as usual (§1.3).

The key ingredient is a polarization.

**Definition 19.3.** Let $(W,\omega)$ be a real symplectic vector space. An isotropic subspace is a subspace $L \subset W$ on which the symplectic form vanishes. If $L$ is maximal among isotropic subspaces, then it is a Lagrangian subspace. A polarization of $W$ is a choice of Lagrangian subspace. A complex polarization is a (complex) Lagrangian subspace $L \subset W_C = W \otimes_{\mathbb{R}} \mathbb{C}$. A complex polarization is real if $\overline{L} = L$. A complex polarization is of Kähler type if $L \cap \overline{L} = 0$, in which case $W_C = L \oplus \overline{L}$.

A complex polarization of Kähler type induces a complex structure on $W$ compatible with the symplectic structure, i.e., a symplectic automorphism $I$ of $W$ such that $I^2 = -\text{id}_W$. Conversely, such a complex structure induces a Lagrangian decomposition of the complexification of $W$ according to the eigenvalues of $I \otimes \mathbb{C}$:

$$
W_C = W \otimes_{\mathbb{R}} \mathbb{C} = W_+ \oplus W_-, \quad W_- = \overline{W}_+.
$$

(In the notation of Definition 19.3, $W_+ = L$ and $W_- = \overline{L}$.) There results a Hermitian form

$$
\langle \bar{w}_1, w_2 \rangle = 2i\omega(\bar{w}_1, w_2), \quad w_1, w_2 \in W_+,
$$

on $W_+$, and we say the complex structure is positive if this form is positive definite. Assume so.

Next, construct a complex module $D$ over $W(W^*)$. Namely, define

$$
D = \text{Sym}^\bullet(W_+^*), \quad \Omega = 1 \in \text{Sym}^0(W_+^*).
$$

The vector $\Omega$ is called the vacuum. The module structure is the algebra homomorphism

$$
W(W^*) \otimes \mathbb{C} \longrightarrow \text{End} D
$$

$$
\lambda_+ \mapsto M_{\lambda_+}, \quad \lambda_- \mapsto \iota_{\lambda_-}
$$

Some elements of $W(W^*)$ act on $H$ as unbounded operators. The problem is cleanest in the finite dimensional odd case. Then $W(W^*)$ is replaced by a Clifford algebra, and the problem is to write its complexification as endomorphisms of a super vector space. If the Clifford algebra has an odd number of generators, its commutant in the algebra is isomorphic to $\mathbb{C}L^2$ and the resulting $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space is viewed as a module over this commutant.
where \( \lambda_+ \in W_+^* \), \( \lambda_- \in W_-^* \), the operator \( M_{\lambda_+} \) is multiplication by \( \lambda_+ \in \text{Sym}^1(W_+^*) \), and \( \iota_{\lambda_-} \) is the contraction operator of degree \(-1\) defined by

\[
\iota_{\lambda_-}(\lambda_+^{(1)} \cdots \lambda_+^{(k)}) = \sum_{i=1}^k \omega(\lambda_- , \lambda_+^{(i)}) \lambda_+^{(1)} \cdots \lambda_+^{(i-1)} \lambda_+^{(i+1)} \cdots \lambda_+^{(k)} .
\]  

We call \( M_{\lambda_+} \) a creation operator and \( \iota_{\lambda_-} \) an annihilation operator. The annihilation operators annihilate the vacuum \((W_+^* \cdot \Omega = 0)\) and the creation operators generate the vector space \( D \) \((W_+^* \cdot \Omega = D)\).

Use (19.5) to induce a Hermitian form on \( D \), and then complete to obtain a Hilbert space \( \mathcal{H} \). It is this Hilbert space which is the quantization, but the dense algebraic subspace is useful for computations. We can also complete \( W_+^* = \text{Sym}^1(W_+^*) \) to a Hilbert space. In the context of quantum field theory this is the 1-particle state space. We can complete its symmetric powers to obtain multi-particle state spaces. The direct sum of these multi-particle state spaces is dense in \( \mathcal{H} \); it is called the Fock space. It is an intermediate space between \( D \) and \( \mathcal{H} \).

Remark 19.9. There are analogous constructions for an odd symplectic vector space that is used to construct free fermionic theories. In the finite dimensional case it appears at the end of Lecture 10. See [KoSt], [Kaz, §3] for the infinite dimensional case.

19.2 The quantum massive scalar field

Resume our relativistic setup, with its Minkowski spacetime \( M \), vector space \( V \) of translations, forward timelike cone \( P \subset V \), and speed of light \( c \). Recall the classical massive scalar field, as discussed in §18.3. Solutions to the classical linear wave equation (18.23) form a real symplectic vector space, denoted \( \mathcal{M}(M) \) in our exposition, which is identified by Fourier transform as a real subspace of a vector space of complex-valued functions on the union \( O_m \cup -O_m \) of a mass shell and its negative; see Figure 54.

Now apply §19.1 to the symplectic vector space \( \mathcal{M}(M) \). The key point is that we choose a polarization (19.4) in which \( W_+ \) consists of positive energy fields. First, observe that the complexification of \( \mathcal{M}(M) \) consists of complex-valued functions on \( O_m \cup -O_m \) without any reality condition (18.30). Choose \( W_+ \) to be the subspace of functions with support on the positive energy mass shell \( O_m \). This is where the crucial positive energy condition enters. Then set \( \mathcal{H}_1 \) to be the Hilbert space completion of \( W_+^* \). By Theorem 17.16 it carries an irreducible representation of \( \text{SO}^\uparrow(M) \), the representation of the massive scalar particle of mass \( m \). This is precisely the representation obtained in Lecture 14 by quantizing the particle of mass \( m \); see (15.2) and the surrounding text. As mentioned above, \( \mathcal{H}_1 \) is the 1-particle state space.

The Fock space is the symmetric algebra \( \text{Sym}^* \mathcal{H}_1 \). The Hilbert space completion of the subspace \( \text{Sym}^k \mathcal{H}_1 \) is the \( k \)-particle state space. The Hilbert space completion

\[
\mathcal{H} = \overline{\text{Sym}^* \mathcal{H}_1}
\]

of the symmetric algebra is the Hilbert space of the relativistic quantum mechanical system that represents the massive scalar field. It carries a representation of the symmetry group (18.20). The
generator of the internal symmetry group $\mu_2$ acts as $+1$ on $\text{Sym}^{\text{even}} H_1$ and as $-1$ on $\text{Sym}^{\text{odd}} H_1$. In other words, it gives a $\mathbb{Z}/2\mathbb{Z}$-grading by parity of particle number. Observe that the integer particle number is defined on the Fock space by the grading of the symmetric algebra, but it does not extend to the Hilbert space completion (19.10), whereas the mod 2 particle number does extend. The spectrum of the theory is depicted in Figure 55. There is a unique vacuum state $\text{Sym}^0 H_1 \subset H$, which sits discretely in the spectrum. There is a spectral gap. Next in the spectrum, ordered by mass, is the irreducible scalar representation of mass $m$, which sits discretely in the spectrum of representations of $O^\uparrow(M)$. Above that is continuous spectrum: the convex hull of the mass shell $O_{2m}$. Each representation $\text{Sym}^k H_1$, $k \in \mathbb{Z}^{\geq 0}$ is a closed subspace of $H$. Each representation $\text{Sym}^k H_1$, $k \in \mathbb{Z}^{\geq 0}$ is a closed subspace of $H$.

**Remark 19.11.** The quantization of a field—in this case the massive scalar field—leads to particle representations of the symmetry group. This “duality” between fields and particles is a basic feature of quantum field theory. In more complicated quantum field theories the 1-particle subspace may not be reducible, but it is a sum of a finite number of irreducible representations. Those irreducible representations comprise the particle spectrum of the theory.

**19.3 The massless scalar field**

This is ostensibly similar to the massive case—simply set $m = 0$ in (18.23)—but the result is quite different. Assume the dimension $n \geq 3$; there are special features in $n = 1$ and $n = 2$. We simply report on the positive energy unitary representation of $O^\uparrow(M) \times \mu_2$.

The 1-particle space $H_1$ is an irreducible representation of $O^\uparrow(M)$, but now a massless representation: it sits in the fiber over 0 in (17.17). This is the massless scalar particle: the representation of the little group is trivial. The Hilbert space $H$ is the completion of the Fock space, as in (19.10). The $k$-particle state space $\text{Sym}^k H_1 \subset H$ is a closed subspace. There is a unique vacuum $\text{Sym}^0 H_1$, but the system is gapless—there is no spectral gap. The spectrum of the 1-particle space is $N(V^*)^+$, which contains $\{0\}$ in its closure. The spectrum of the entire theory $H$ is the closed cone $P^*$ of positive energy vectors.
19.4 Maxwell theory: classical electromagnetism

We work in any dimension \( n \), though of course the electromagnetism of our world is \( n = 4 \), and we initially go beyond the setting of Minkowski spacetime. Let \( (N, g_N) \) be a Riemannian manifold. Define the split spacetime

\[
M = M^1 \times N
\]

with Lorentz metric \((dx^0)^2 - g_N\), where \( x^0 = ct \) is the standard coordinate on the time line \( M^1 \). The classical electric and magnetic fields are differential forms on \( N \) that depend on time:

\[
E: M^1 \to \Omega^1_N \\
B: M^1 \to \Omega^2_N
\]

The electromagnetic field is the 2-form on \( M \) that combines the electric and magnetic fields:

\[
F = B - dt \wedge E.
\]

Let \( *_N \) be the Hodge star operator for the positive definite metric on \( N \). Then if \( * \) is the Hodge \( * \) operator on \( M^1 \times N \) relative to the Lorentz metric, we compute

\[
* F = \frac{1}{c} *_N E + c dt \wedge *_N B.
\]

Maxwell’s equations in empty space are:

\[
\begin{align*}
  dB &= 0 \\
  dE &= -\frac{\partial B}{\partial t} \\
  d*_{N} E &= 0 \\
  c^2 d*_{N} B &= *_{N} \frac{\partial E}{\partial t}
\end{align*}
\]
They combine to the simple equations

\begin{align}
\text{(19.17)} \quad dF &= 0 \\
\quad d\ast F &= 0
\end{align}

for the 2-form $F$ on spacetime $M$.

In fact, we demand a stronger form of the first equation in (19.17), which states that $F$ is closed: we demand that $F$ be exact. In other words, we postulate a 1-form $A \in \Omega^1_M$ such that

\begin{align}
\text{(19.18)} \quad F &= F_A = dA.
\end{align}

The 1-form $A$ is determined up to a closed 1-form, so lives in the quotient

\begin{align}
\text{(19.19)} \quad A &\in \Omega^1_M / \Omega^1_{M, \text{closed}}.
\end{align}

We take (19.19) to be the fundamental field in Maxwell theory, and use the decomposition (19.14) to define the electric field $E = E_A$ and magnetic field $B = B_A$. The second equation in (19.17) is a wave equation for $A$:

\begin{align}
\text{(19.20)} \quad d \ast dA &= 0.
\end{align}

**Remark** 19.21. The first Maxwell equation in (19.17) follows from (19.18). The second is the Euler-Lagrange equation for the Lagrangian density

\begin{align}
\text{(19.22)} \quad L &= -\frac{1}{2} < F_A, F_A > \mu_M = \frac{1}{2} \left( \frac{\|E_A\|^2}{c^2} - \|B_A\|^2 \right) \mu_M,
\end{align}

where the norms of the forms $E_A, B_A$ are computed in the positive definite metric on $N$. A multiplicative constant is usually inserted in $L$, depending on the units used; we omit it.\textsuperscript{56}

Now resume the relativistic setup on a Minkowski spacetime $M$, but without the splitting (19.12). Since $M$ is contractible, closed 1-forms are exact. Hence the space of solutions to (19.20) in the quotient (19.19) is the first cohomology group of the complex

\begin{align}
\text{(19.23)} \quad \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{\ast d \ast d} \Omega^1_M
\end{align}

As in §18.3 rewrite the complex after Fourier transform: for $\theta \in V^*$, $f \in \Omega^0_V$, and $A \in \Omega^1_V$ we have

\begin{align}
\text{(19.24)} \quad (df)^\vee(\theta) &= \sqrt{-1} \hat{f}(\theta) \theta \\
\quad (\ast d \ast d\alpha)^\vee(\theta) &= \|\theta\|^2 \hat{A}(\theta) - \langle \hat{A}(\theta), \theta \rangle \theta.
\end{align}

\textsuperscript{56}In the “mks” system of units, the constant is written $\varepsilon_0 c^2 = 10^7 / 4\pi$. 
Then one can show that the first cohomology of (19.23) is isomorphic to the space of real functions \( \hat{A} : N(V^*) \to V^* \otimes \mathbb{C} \) on the lightcone that satisfy \( \langle \hat{A}(\theta), \theta \rangle = 0 \) modulo the space of functions \( \hat{A}(\theta) = \hat{f}(\theta)\theta \), where \( \hat{f} \) ranges over the complex functions on the lightcone. The reality condition is \( \hat{A}(-\theta) = \hat{A}(\theta) \). Polarize by positive energy. The resulting complex Lagrangian \( W_+^* \) is the space of sections of the complex vector bundle \( E \to N(V^*)^+ \) whose fiber at \( \theta \in N(V^*)^+ \) is \( N^\perp / N \otimes \mathbb{C} \), where \( N^* = \mathbb{R} \cdot \theta \). It follows from Proposition 16.46(2) that the 1-particle space—the Hilbert space of \( L^2 \) sections of \( E \to N(V^*)^+ \)—is the irreducible vector representation of spin 1. This is the “vector particle” that corresponds to the Maxwell field.

**Remark 19.25.** If \( n = 3 \), then the little group is \( \text{Spin}_1 \cong \mathfrak{u}_2 \), and the representation is the trivial representation. So in this dimension the vector particle representation reduces to the scalar particle representation. This is a manifestation of electromagnetic duality.

We can rewrite Maxwell theory as a gauge theory with gauge group \( \mathbb{R}^>0 \), the connected multiplicative group of real numbers. (One can also impose Dirac’s quantization of charge, which is encoded by choosing the compact gauge group \( T \), but we do not do so here.) For \( \mathcal{U} \subset \text{Open}(M) \) put \( \mathcal{F}(\mathcal{U}) = \{ \text{groupoid of } \mathbb{R}^>0\text{-connections on } \mathcal{U} \} \). Note that any principal bundle over the contractible space \( M \) admits a section, and that section pulls back a connection to a real 1-form \( A \in \Omega^1_M \). A change of section is given by a function \( g : M \to \mathbb{R}^>0 \), and it induces the gauge transformation \( A \mapsto A + dg/g \). Any such function is uniquely \( g = e^f \) for some \( f \in \Omega^0_M \), and then \( A \mapsto A + df \).

The subsheaf \( \mathcal{M} \subset \mathcal{F} \) of classical solutions is cut out by the equation

\[
(19.26) \quad d * F_A = 0,
\]

where \( * \) is the Lorentz signature star operator and \( F_A \in \Omega^2_M \) is the curvature of the connection \( A \).

### 19.5 A word about nonabelian gauge theories

Let \( G \) be a compact Lie group and let \( \langle -, - \rangle \) be a bi-invariant inner product on the Lie algebra \( \mathfrak{g} \). In a pure \( G \)-gauge theory the fields are \( \mathcal{F}(\mathcal{U}) = \{ \text{groupoid of } G\text{-connections on } \mathcal{U} \} \). If \( A \) is a connection with curvature \( F_A \), then the Bianchi identity and Yang-Mills equations of the pure gauge theory—the equation that cuts out classical solutions—are

\[
(19.27) \quad d_A F_A = 0 \quad d_A * F_A = 0
\]

This classical theory is conjectured to have a quantum counterpart.

**Conjecture 19.28 (Clay Problem).** There is a quantum field theory that corresponds to classical Yang-Mills theory, and it is gapped.

The task is to construct a quantum field theory, not just a relativistic quantum mechanical system, and to prove that it is gapped. We refer to [JW] for background and a precise formulation of the conjecture.
Remark 19.29. The classical theory is not free. Namely, the curvature

\[ F_A = F_{A_0} + d_{A_0} \alpha + \frac{1}{2} [\alpha \wedge \alpha] \]

of a perturbation \( A = A_0 + \alpha \) of a reference connection has quadratic terms, and so the Yang-Mills equation has cubic terms—it is not a linear equation. This nonlinearity, or non-freeness, means that the free theory is not necessarily a good guide, even qualitatively, to the quantum theory. In this case the quantum theory is expected to acquire a mass gap.

Lecture 20: Local observables

A symplectic manifold \( N \) determines the data of a classical system; a Hilbert space \( \mathcal{H} \), or a projective Hilbert space, determines the data of a quantum system. That is, from \( N \) or \( \mathcal{H} \) we define a theory of states and observables. For theories on spacetime we should think of these as global: \( N \) or \( \mathcal{H} \) is constructed from global data. We can also contemplate a theory of observables which is local on spacetime. Such local theories of observables are the subject of this lecture. In the previous lecture we defined fields as local objects in spacetime, so it is not a surprise that in field theory local observables can be constructed from local fields, an idea we develop in the next lecture. Locality of states is more subtle in quantum field theory; in discrete quantum mechanical systems it is more evident, as we see in (20.6). However, in general locality of states is more subtle; see Remark 20.8(2). The locality of states in Wick-rotated field theory is a subject for next semester.

The starting point in this lecture is classical field theory on Minkowski spacetime, where observables on an open subset form a commutative algebra. We then move on to a discrete quantum model—the toric code is used as a concrete illustration—where there are noncommutative algebras of observables that are local in space. Basic observables in this theory may be regarded as point operators in the sense that their support is adjacent to a single vertex or face. These observables do not separate vacua, however; there are line observables that do so.

Finally, we present a version of an axiom system for local observables in quantum field theory due to Haag-Kastler \[\text{[HK, Ha]}\]. We note that this approach—based as it is on bounded operators—follows in the tradition of Irving Segal \[\text{[S]}\]. We only give highlights and defer to the literature for details. In the next lectures we construct local observables in a field theory. The signal feature of local observables in Minkowski spacetime is causality: observables with spacelike separated support commute. There is an analogy between commutators in algebras and causal complements in spacetime.

We work mostly in a Minkowski spacetime \( M \) over a Lorentzian vector space \( V \), equipped with the usual accouterments: a cone \( P \) of forward timelike vectors, a speed of light \( c \), and a relativistic symmetry type \( \lambda: G \rightarrow O^\uparrow(V) \). Recall that \( \lambda \) induces a homomorphism \( \tilde{\lambda}: \mathcal{G} \rightarrow O^\uparrow(M) \).

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57 The words ‘observable’, ‘operator’, and also ‘defect’ are often used interchangeably in this context. In this lecture we stick to ‘observable’.

58 A geometer’s note. Affine subspaces in affine geometry: point, line, plane. Submanifolds in differential geometry: point, curve, surface. Standard physics terminology for the support of observables: point, line, surface.
20.1 Local observables in classical field theory

Suppose that \( \mathcal{F} \) is a collection of relativistic fields (Definition 18.14), which is a sheaf on \( M \). In a classical field theory, a local equation—for example, a wave equation—cuts out a subsheaf \( \mathcal{M} \subset \mathcal{F} \), and there is a symplectic structure on \( N = \mathcal{M}(M) \). The subsheaf \( \mathcal{M} \) is assumed \( \mathcal{G} \)-invariant (in the sense of (18.15)). From the symplectic manifold \( N \) one constructs global states and global observables, as usual.

For \( U \subset M \) an open set, let

\[
\mathcal{A}^\text{cl}(U) = C^\infty(\mathcal{M}(U), \mathbb{C}).
\]

Complex conjugation provides a real structure on \( \mathcal{A}^\text{cl}(U) \), and the real points are the classical observables on \( U \). If \( U_1 \subset U_2 \), then there is a restriction map \( \mathcal{M}(U_2) \to \mathcal{M}(U_1) \), and so an inclusion map \( \mathcal{A}^\text{cl}(U_1) \to \mathcal{A}^\text{cl}(U_2) \): an observable on \( U_1 \) is also an observable on the larger set \( U_2 \). This structure is dual to a presheaf; recall Definition 18.2 and Definition 18.11.

Definition 20.2. Let \( C \) be a category, let \( \text{Set} \) be the category of sets.

(1) A precosheaf of sets on \( C \) is a functor \( \mathcal{A}: C \to \text{Set} \).

(2) Let \( M \) be a topological space, and let \( \text{Open}(M) \) be the category of open subsets and inclusions. A precosheaf \( \mathcal{A}: \text{Open}(M) \to \text{Vect} \) of vector spaces is a cosheaf if for every open set \( U \subset M \) and every open cover \( \{U_\alpha\}_{\alpha \in A} \) of \( U \) the diagram

\[
\mathcal{A}(U) \leftarrow \bigoplus_{\alpha_0} \mathcal{A}(U_{\alpha_0}) \leftarrow \bigoplus_{\alpha_0,\alpha_1} \mathcal{A}(U_{\alpha_0} \cap U_{\alpha_1})
\]

is a coequalizer diagram.

The cosheaf condition (2) also applies to precosheaves of abelian groups.

Example 20.4. There is a cosheaf \( U \mapsto \mathcal{C}^\text{c}(U; \mathbb{C}) \) that assigns to each open subset \( U \subset M \) the space of compactly supported continuous functions on \( U \). Also, singular cochains on \( U \) form a cosheaf. If \( M \) is a smooth manifold, then distributions also form a cosheaf.

The precosheaf \( \mathcal{A}^\text{cl} \) satisfies the gluing condition (20.3): a coherent finite set of observables glues to a single observable. Furthermore, the cosheaf \( \mathcal{A}^\text{cl} \) is \( \mathcal{G} \)-invariant: if \( g \in \mathcal{G} \), then there are coherent isomorphisms \( \mathcal{A}^\text{cl}(U) \xrightarrow{\sim} \mathcal{A}^\text{cl}(\lambda(g)(U)) \), \( U \in \text{Open}(M) \). Finally, \( \mathcal{A}^\text{cl} \) is a cosheaf of commutative algebras. For convenience, here is an enumeration of these properties:

(A) \( \mathcal{A}^\text{cl}(U) \) is a commutative algebra

(B) \( \mathcal{A}^\text{cl} \) is a cosheaf

(C) \( \mathcal{A}^\text{cl} \) is \( \mathcal{G} \)-invariant

We turn now to an example of local observables in a discrete quantum system.

\[\text{As in (1.18) we could take } \mathcal{A}^\text{cl} \text{ to consist of Borel functions and define a dense subset } (\mathcal{A}^\text{cl})^\infty \text{ of } C^\infty \text{ functions.}\]
20.2 Local observables in the toric code

Kitaev’s toric code [Ki] was introduced in Section 5.5. It is formulated on a closed surface \( Y \) equipped with an embedded finite graph \( \Lambda \), as in Figure 11. There is a finite set \( \mathcal{D}(\Lambda, \Lambda^0) \) of isomorphism classes of double covers of \( \Lambda \) equipped with trivializations over the vertices, and the finite dimensional Hilbert space \( \mathcal{H} \) of toric code is the space (5.21) of complex functions on \( \mathcal{D}(\Lambda, \Lambda^0) \).

The Hilbert space of this theory has a local description. Recast \( \mathcal{D}(\Lambda, \Lambda^0) \) as the set of \( \mu_2 \)-valued functions on the edges of \( \Lambda \), as in (5.20), and set

\[
(20.5) \quad \mathcal{H}_e = \text{Map}(\mu_2, \mathbb{C}), \quad e \in \text{Edge}(\Lambda).
\]

Then there is a natural isomorphism

\[
(20.6) \quad \mathcal{H} \cong \bigotimes_e \mathcal{H}_e.
\]

This expresses the global state space \( \mathcal{H} \) as a finite tensor product of local state spaces \( \mathcal{H}_e \). As for the algebra \( \text{End} \mathcal{H} \), whose real points (self-adjoint operators) are observables, we have

\[
(20.7) \quad \text{End} \mathcal{H} \cong \bigotimes_e \text{End} \mathcal{H}_e.
\]

Remark 20.8.

1. There is an infinite version of the toric code on a noncompact surface, for example on an affine plane. This beautiful theory is developed in [Naa]. Our remark here is that an infinite analog of (20.7) is well-defined—one takes products of operators in \( \text{End} \mathcal{H}_e \) such that for all but finitely many \( e \) the operator equals \( \text{id} \). Then one can complete to a \( C^* \)-algebra and construct a good theory of observables. But there is no infinite analog of (20.6) since there is no distinguished vector in \( \mathcal{H}_e \). Therefore, one does not construct states directly from a Hilbert space, as we do in the finite model, but rather one constructs states as linear functionals on the algebra of complex observables, as in Definition 11.27.

2. The tensor product decomposition (20.6) expresses a locality of states in space. There is an analog in quantum field theory, but one does not have empty space between the sites of the local states and so the decomposition is more subtle. Locality of states is cleanest in topological field theories, as we shall see next semester, but even there the state space is not a simple tensor product as in (20.6).

Definition 20.9. Let \( U \subset Y \) be an open set in \( Y \). An operator in (20.7) has support in \( U \) if it lives in the subalgebra

\[
(20.10) \quad \mathcal{A}(U) = \bigotimes_e \text{End} \mathcal{H}_e.
\]

The embedding \( \mathcal{A}(U) \subset \text{End} \mathcal{H} \) is achieved by tensoring with \( \text{id} \in \text{End} \mathcal{H}_e \) for all \( e' \not\subset U \). The algebra \( \mathcal{A}(U) \) is the complex algebra of observables on the open set \( U \).

The assignment \( U \mapsto \mathcal{A}(U) \) has the following properties:
(A) \( \mathcal{A}(\mathcal{U}) \) is an algebra
(B) \( \mathcal{A} \) is a cosheaf
(C) If \( \mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset \), then \( \mathcal{A}(\mathcal{U}_1 \cup \mathcal{U}_2) = \mathcal{A}(\mathcal{U}_1) \otimes \mathcal{A}(\mathcal{U}_2) \); in particular, \( \mathcal{A}(\mathcal{U}_1) \) and \( \mathcal{A}(\mathcal{U}_2) \) commute

**Figure 57.** The support of point observables \( H_v, H_f \) and line observables \( H_C \)

The toric code has fundamental observables \( H_v, H_f \) associated to a vertex \( v \) and a face \( f \); they are defined in (5.24). Their supports are depicted in Figure 57. From a distance—if the lattice \( \Lambda \subset Y \) is “thick”—these supports appear pointlike: the support consists of edges adjacent to a single vertex or a single face. The vacuum subspace \( \mathcal{H}_0 \subset \mathcal{H} \), which is the kernel of the toric code Hamiltonian (5.25), is the simultaneous kernel of the operators \( H_v, H_f \) over all vertices and faces. It is isomorphic to the vector space of complex-valued functions on \( \pi_0 \mathcal{D}(Y) \), the set of isomorphism classes of double covers of the surface \( Y \). Typically it has dimension > 1—the vacuum is not unique—and clearly the point observables or point observables \( H_v, H_f \) do not separate vacua. Therefore, we introduce line observables: observables whose support is a 1-dimensional submanifold.\(^60\) Let \( C \subset Y \) be a closed loop, and define

\[
(20.11) \quad h_C: \mathcal{D}(\Lambda, \Lambda^0) \rightarrow \{0, 1\}
\]

to be the function that returns the isomorphism type of the restriction of a double cover to \( C \). Multiplication by \( h_C \) is a self-adjoint operator \( H_C \). These observables, taken over all \( C \subset Y \), separate vacua in the toric code.

**Remark 20.12.** The line observables \( H_C \) are topological when restricted to vacua. More precisely, they are invariant under deformation of \( C \), in fact under homologies of \( C \). This is another indication that the low energy, or vacuum, “sector” of the toric code is a topological theory; see Remark 5.28(2).

**20.3 Causal structure of spacetime**

In Minkowski spacetime \( M \) the analog of property (C) above depends on the causal structure of spacetime. Introduce the notation \( p \perp q \) for spacelike separated points \( p, q \in M \).

\(^{60}\)In general, observables can have support on a stratified manifold.
Definition 20.13. Let $\mathcal{U} \subset M$ be a subset.

1. The \textit{causal complement} to $\mathcal{U}$ is

$$U' = \{ q \in M : p \perp q \text{ for all } p \in \mathcal{U} \}. \quad (20.14)$$

2. The \textit{causal completion} of $\mathcal{U}$ is $\mathcal{U}''$.
3. $\mathcal{U}$ is \textit{causally complete} if $\mathcal{U}'' = \mathcal{U}$.
4. Subsets $\mathcal{U}_1, \mathcal{U}_2 \in M$ are spacelike separated if $\mathcal{U}_1 \subset \mathcal{U}_2'$ or equivalently $\mathcal{U}_2 \subset \mathcal{U}_1'$. In that case we write $\mathcal{U}_1 \perp \mathcal{U}_2$.

It follows immediately from (20.14) that $\mathcal{U} \subset \mathcal{U}''$ and that $\mathcal{U}' = \mathcal{U}'''$, i.e., $\mathcal{U}'$ is causally complete.

There are parallel definitions for algebras. Let $\mathcal{A}$ be an algebra. Introduce the notation $a \perp b$ for commuting elements $a, b \in \mathcal{A}$.

Definition 20.15. Let $\mathcal{A}$ be an algebra and $S \subset \mathcal{A}$ a subset.

1. The \textit{commutant} of $S$ is

$$S' = \{ b \in \mathcal{A} : a \perp b \text{ for all } a \in S \}. \quad (20.16)$$

2. Subsets $S_1, S_2 \in \mathcal{A}$ \textit{commute} if $S_1 \subset S_2'$ or equivalently $S_2 \subset S_1'$. In that case we write $S_1 \perp S_2$.

There are analogs of Definition 20.13(2,3) as well, though I do not know a term for the double commutant $S''$ of a subset $S \subset \mathcal{A}$. Note that $S \subset S''$ and $S' = S'''$. If $\mathcal{A} = \text{End } \mathcal{H}$ is the $*$-algebra of bounded operators in a Hilbert space $\mathcal{H}$, then the commutant $S'$ of a subset $S \subset \text{End } \mathcal{H}$ is a \textit{von Neumann algebra}. The algebra $S''$ is the \textit{von Neumann closure} of $S$. Recall (§11.4) that a $C^*$-algebra is closed in the norm topology. A von Neumann algebra satisfies a stronger condition: it is closed in the weak topology. For basics on von Neumann algebras, including the von Neumann double commutant theorem used here implicitly, see [Jo].
20.4 Local observables in Minkowski spacetime

The perspective here is due to Haag-Kastler; it is often called *algebraic quantum field theory*. The original reference is [HK] and there are many since. The book [Ha] provides a thorough account. The recent paper [Ded] contains a summary as well as extensive references. Our account makes no pretensions to completeness. Also, we begin with a relativistic quantum mechanical system—a positive energy unitary representation—and take local observables to be subalgebras of global observables, whereas typically the net of local observables takes values in abstract $C^*$-algebras.

Resume the setup at the beginning of the lecture: a Minkowski spacetime $M$, a relativistic symmetry type $\lambda: G \to \text{O}^+(V)$, and the rest. In the following definition the subsets $U \subset M$ are sometimes required to be causally closed; one can use axiom (E) to reduce to that case.

**Definition 20.17.** Suppose $\rho: \mathcal{G} \to \text{Aut} \mathcal{H}$ is a relativistic quantum mechanical system on $M$. A net of local observables for $\rho$ is an assignment

$$U \mapsto \mathcal{A}(U)$$

(20.18)

to each bounded open subset $U \subset M$ of a subalgebra $\mathcal{A}(U) \subset \text{End} \mathcal{H}$ such that:

- (A) $\mathcal{A}(U)$ is a $C^*$-algebra
- (B) If $U_1 \subset U_2$, then $\mathcal{A}(U_1) \subset \mathcal{A}(U_2)$
- (C) If $U_1 \perp U_2$, then $\mathcal{A}(U_1) \perp \mathcal{A}(U_2)$
- (D) $\mathcal{A}$ is $\mathcal{G}$-invariant: $\mathcal{A}(\lambda(g)U) = \rho(g)\mathcal{A}(U)\rho(g)^{-1}$, $g \in \mathcal{G}$
- (E) $\mathcal{A}(U'') = \mathcal{A}(U)$

Note that (C) is equivalent to $\mathcal{A}(U') \subset \mathcal{A}(U)'$. The net satisfies *Haag duality* if

$$\mathcal{A}(U') = \mathcal{A}(U)'$$

(20.19)

perhaps for some restricted class of $U$. For those opens $U$ that satisfy (20.19), the $C^*$-algebra $\mathcal{A}(U) = \mathcal{A}(U'') = \mathcal{A}(U)''$ is a von Neumann algebra.

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**Lecture 21: Fields and local observables: the Wightman picture**

We began our study of relativistic quantum systems in Lecture 14 with a straightforward generalization (Definition 14.16) of a quantum mechanical system, namely a positive energy unitary representation of the Poincaré group (or of a more general symmetry group). In Lecture 20 we introduced (Definition 20.17) local theories of observables à Haag-Kastler. Recall that we introduced a general concept of fields in Lecture 18, and sketched there how to construct a relativistic quantum mechanical system by quantizing a classical theory of fields. In this lecture we complete our survey of relativistic quantum systems by constructing local observables from fields, à la Wightman.

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61The paper surveys various attempts to axiomatize quantum field theory.
We begin by reprising the massive scalar field. First, in the classical theory we introduce commutative algebras of local observables defined as functions of fields. There are classical correlation functions of these observables. Free field quantization promotes the classical observables to quantum observables, but to get well-defined operators we smear against test functions. These field operators satisfy causality: they commute if the smearing functions have spacelike separated supports.

We defer to references, such as [SW, Ha, Kaz] for detailed accounts of the Wightman theory. In this lecture we sketch the crucial consequence of the positive energy hypothesis: correlation functions in Minkowski spacetime are boundary values of holomorphic functions in some complex domain. Furthermore, in that domain we can restrict to correlation functions on Euclidean space. Wick rotation is the process of passing from Minkowski spacetime to Euclidean space, here for correlation functions. In the next lecture Wick rotation of the representation of the Poincaré group leads us to the Segal axioms for quantum field theory. We conclude this lecture by commenting on Wick rotation of relativistic symmetry types and the reflection positivity property that is the Wick rotation of unitarity.

There is much more to say about relativistic quantum field theory, of course, both from the more mathematical point of view—results such as the Reeh-Schlieder theorem, explained in [W2], for example—and from real physics, for which there are many standard texts, e.g. [Wei2, Wei3, Wei4]. There is also a whole literature of constructions of quantum field theories in the Wightman framework; see [GJ, Ja] for an entrée. A major challenge is to construct 4-dimensional Yang-Mills theory (Conjecture 19.28); see [Ch] for a recent status update and survey of the literature.

21.1 Classical massive scalar field redux

This theory was discussed in Lecture 18. We work on a Minkowski spacetime $M$ over a Lorentzian vector space $V$ of dimension $n$. Real scalar fields form a sheaf

\[ \mathcal{F} : \text{Open}(M)^{\text{op}} \rightarrow \text{Set} \]

\[ \mathcal{U} \mapsto C^\infty(\mathcal{U}, \mathbb{R}) \]

and classical solutions comprise the subsheaf $\mathcal{M} \subset \mathcal{F}$ of real-valued functions that satisfy the Klein-Gordon equation

\[ \left( \Box + \frac{c^2}{\hbar^2} m^2 \right) \phi = 0. \]

The global solutions $\mathcal{M}(M)$ to the Klein-Gordon equation form a symplectic vector space, from which we construct the complex algebra

\[ \mathcal{A}^{\text{cl}}(M) = C^\infty(\mathcal{M}(M), \mathbb{C}). \]

This is a commutative $*$-algebra; its real points are the global observables. Following (20.1) we define local observables on an open subset $\mathcal{U} \subset M$:

\[ \mathcal{A}^{\text{cl}}(\mathcal{U}) = C^\infty(\mathcal{M}(\mathcal{U}), \mathbb{C}). \]
The definitions (21.3) and (21.4) apply to any classical field theory. For the real scalar field, a typical local observable is $O_p^{cl}$ for $p \in U \subset M$:

(21.5) \[ O_p^{cl}(\phi) = \phi(p), \quad \phi \in \mathcal{M}(U). \]

A “smeared out” variant attaches to a real-valued function $f \in C^\infty_c(U)$ of compact support in $U$ the local observable

(21.6) \[ O_f^{cl}(\phi) = \int_U f(p) \phi(p) \, d\mu_M(p), \quad \phi \in \mathcal{M}(U), \]

where $\mu_M$ is the $O(M)$-invariant measure constructed from the Lorentz metric. Both the “sharp” observable $O_p^{cl}$ and the smeared out observable $O_f^{cl}$ are well-defined in the classical theory.

Recall that a state in the classical theory is a probability measure $\nu$ on $\mathcal{M}(M)$. (A pure state is a point measure supported at some $\phi_0 \in \mathcal{M}(M).$) Fix a state $\nu$ and fix distinct points $p_1, \ldots, p_k \in U \subset M$. Then the classical correlation function of the local observables $O_{p_1}^{cl}, \ldots, O_{p_k}^{cl}$ is

(21.7) \[ \langle O_{p_1}^{cl} \cdots O_{p_k}^{cl} \rangle_{\nu} = \int_{\mathcal{M}(M)} O_{p_1}^{cl}(\phi) \cdots O_{p_k}^{cl}(\phi) \, d\nu(\phi). \]

For fixed $\nu$ this is a $C^\infty$ function of $p_1, \ldots, p_k \in U$. If $\nu$ is the pure state at $\phi_0 \in \mathcal{M}(M)$, then the correlation function reduces to

(21.8) \[ \langle O_{p_1}^{cl} \cdots O_{p_k}^{cl} \rangle_{\phi_0} = \phi_0(p_1) \cdots \phi_0(p_k). \]

Remark 21.9. Our comment that “for fixed $\nu$ this is a $C^\infty$ function of $p_1, \ldots, p_k \in U$” indicates that we must consider correlation functions as defined in families, and we should axiomatize the smoothness in parameters. This is clear in the Wightman theory, but is not usually formulated explicitly in other axiom systems, such as the Segal axioms we come to soon.

21.2 Field operator of the quantum massive scalar field

Recall the quantization (19.10) of the massive scalar field. The procedure is to fix a basepoint $p_0 \in M$, identify $M \approx V$, and then Fourier transform to identify $\mathcal{M}(M) \otimes \mathbb{C}$ as a vector space $W_\mathbb{C}$ of complex-valued functions on $\mathcal{O}_m \cup -\mathcal{O}_m \subset V^*$, as in Figure 59. There is a natural polarization (19.4) by positive energy: let $W_+ \subset W_\mathbb{C}$ be the subspace of functions supported on $\mathcal{O}_m$. The 1-particle Hilbert space $\mathcal{H}_1$ is the Hilbert space completion of $W_+^*$, which is the space of continuous linear functionals on $L^2(\mathcal{O}_m; \mathbb{C})$. The state space $\mathcal{H}$ of the theory is the Hilbert space completion of $\text{Sym}^\infty \mathcal{H}_1$. Furthermore, under quantization (19.7), continuous linear functionals on $L^2(\mathcal{O}_m; \mathbb{C})$ act on $\mathcal{H}$ as multiplication operators, whereas continuous linear functionals on $L^2(-\mathcal{O}_m; \mathbb{C})$ act on $\mathcal{H}$ as contraction operators.
Can one quantize $O_{cl}^p$ to an operator $O_p$ on $\text{Sym}^\bullet \mathcal{H}_1$? To investigate, rewrite the functional $\phi \mapsto \phi(p)$ in (21.5) as a functional of the Fourier transform $\hat{\phi}$ using the inverse Fourier transform (18.28):

\[
O_{cl}^p : \hat{\phi} \mapsto \frac{1}{(2\pi)^{n/2}} \int_{\mathcal{V}^*} d\theta e^{-i\theta(\xi)/\hbar} \hat{\phi}(\theta),
\]

where $p = p_0 + \xi$ for $\xi \in \mathcal{V}$. The integral reduces to an integral over $\mathcal{O}_m \cup -\mathcal{O}_m$ since $\hat{\phi}$ is supported there. But there is already an issue in the definition (21.5), even before passing to the Fourier transform: $O_{cl}^p$ is not well-defined on $L^2$ functions. Neither, then, is (21.10). So $O_{cl}^p$ has no good quantization. Consider instead the smeared operator (21.6), where $f$ has compact support. Since an $L^2$ function $f\phi$ with compact support is also $L^1$, the integral in (21.6) is well-defined. As a functional of the Fourier transform, the classical observable $O_{cl}^f$ is

\[
O_{cl}^f : \hat{\phi} \mapsto \int_{\mathcal{V}^*} d\theta \hat{f}(\theta) \hat{\phi}(\theta).
\]

Decompose $\int_{\mathcal{V}^*} = \int_{\mathcal{O}_m} + \int_{-\mathcal{O}_m}$ and so decompose $O_{cl}^f = O_{cl,+}^f + O_{cl,-}^f$. Then $O_{cl,+}^f$ is a continuous linear functional on $L^2(\mathcal{O}_m; \mathbb{C})$, i.e., $O_{cl,+}^f$ lies in the 1-particle Hilbert space $\mathcal{H}_1$. The corresponding quantum observable $O_{+}^f$ acts on the Fock space $\text{Sym}^\bullet \mathcal{H}_1$ as a multiplication operator. It extends to an unbounded operator on the Hilbert space completion $\mathcal{H}$. Similarly, $O_{cl,-}^f$ is a continuous linear functional on $L^2(-\mathcal{O}_m; \mathbb{C})$, i.e., $O_{cl,-}^f$ lies in the complex conjugate space $\overline{\mathcal{H}_1}$. The corresponding quantum observable $O_{-}^f$ acts on the Fock space $\text{Sym}^\bullet \mathcal{H}_1$ as a contraction operator, and it too extends to an unbounded operator on the Hilbert space completion $\mathcal{H}$. The sum $O_f = O_{+}^f + O_{-}^f$ is the quantum field operator that corresponds to the classical observable $O_{cl}^f$.

**Remark 21.12.**

1. The smearing by compactly supported functions in $\mathcal{U}$ is a general feature of field operators in quantum field theory. The operators obtained are *unbounded*, so are only defined on a dense
domain. Hence they do not fit directly into the Haag-Kastler framework (Definition 20.17) of bounded operators. Rather, apply the spectral theorem to construct bounded operators from the unbounded field operators.

(2) The quantum field is an operator-valued distribution: the function $f$ is a “test function”.

### 21.3 Wightman functions and Wick rotation

Continue with a Minkowski spacetime $M$ over a Lorentzian vector space $V$ of dimension $n$, and assume given as usual a cone $P$ of forward timelike vectors, a speed of light $c$, and a relativistic symmetry type $\lambda: G \to O^+(V)$; the latter induces an affine symmetry type $\lambda: \mathcal{G} \to O^+(M)$. The basic data of a Wightman quantum field theory is:

- a positive energy unitary representation $U: \mathcal{G} \to U(H)$
- $\mathcal{D} \subset H$ a $\mathcal{G}$-invariant dense subspace
- A $\mathcal{G}$-invariant vector $\Omega \in \mathcal{D}$
- an End $\mathcal{D}$-valued distribution on $M$

The vector $\Omega$ is called the vacuum vector, and the operator-valued distribution on $M$ is the field. It is defined in terms of a real linear representation $R$ of Spin($V$) and $R^*$-valued test functions; see [Kaz] for details. Here, for ease of notation, take $R = \mathbb{R}$; this choice applies to the scalar field. Let $\mathcal{O}_f$ denote the observable attached to the test function $f$ (of compact support). The Wightman “function” is a distribution on $M^{\times k}$:

\begin{equation}
W(f_1 \times \cdots \times f_k) = \langle \Omega, \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_k} \Omega \rangle.
\end{equation}

It has a Schwartz kernel $W_k(p_1, \ldots, p_k)$:

\begin{equation}
W(f_1 \times \cdots \times f_k) = \int_{M^{\times k}} d\mu_M(p_1) \cdots d\mu_M(p_k) W_k(p_1, \ldots, p_k) f_1(p_1) \cdots f_k(p_k).
\end{equation}

Invariance under $V$ acting on $M^{\times k}$ acting diagonally implies that $W_k$ descends to a distribution $\tilde{W}_k$ on the quotient affine space $M^{\times k}/V$, which—after a choice—is identified with the vector space $V^{\times (k-1)}$. The Fourier transform $\tilde{W}_k$ is a distribution on $(V^*)^{\times (k-1)}$. Define the backward tube as the open subset

\begin{equation}
\mathcal{I} = V - iP \subset V_C
\end{equation}

of the complexification $V_C = V \otimes \mathbb{R} \mathbb{C}$ of $V$.

**Lemma 21.16.** $\operatorname{supp} \tilde{W}_k \subset (\mathcal{I}^*)^{\times (k-1)}$.

**Corollary 21.17.** $W_k$ is the boundary value of a homomorphic function on $\mathcal{I}^{\times (k-1)}$.

---

62 One should take $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ to allow for fermionic states.

63 The vacuum is a state, the line $\mathbb{C}(\Omega)$. 
The support condition on $W_k$ is a consequence of the positivity of energy of the representation $U$, as we now sketch. The corollary is standard from there.

To give an idea of the proof, consider $k = 2$ for convenience of notation. Then for any $p \in M$ and $\xi \in V$ we compute (heuristically with distributions—use test functions!)

$$W_2(\xi) = \langle \Omega, \phi(p)\phi(p + \xi)\Omega \rangle$$
$$= \langle \phi(p)\Omega, \phi(p + \xi)\Omega \rangle$$
$$= \langle \phi(p)\Omega, U(\xi)\phi(p)U(\xi)^{-1}\Omega \rangle$$
$$= \langle \phi(p)\Omega, U(\xi)\phi(p)\Omega \rangle$$
$$= \langle \Psi, U(\xi)\Psi \rangle,$$

(21.18)

where $\Psi = \phi(p)\Omega \in \mathcal{H}$. To pass to the second line use the self-adjointness of the real operator $\phi(p)$, and to pass to the fourth line use the $V$-invariance of the vacuum vector $\Omega$. The representation $U$, restricted to the translation subgroup $V \subset G$, has a spectral representation

$$U(\xi) = \int_{V^*} e^{-i\theta(\xi)/\hbar} d\pi(\theta),$$

(21.19)

where $\pi$ is a self-adjoint projection-valued measure on $V^*$ with support on $P^*$. Hence

$$W_2(\xi) = \int_{V^*} e^{-i\theta(\xi)/\hbar} \langle \Psi, d\pi(\theta)\Psi \rangle.$$

(21.20)

In other words, the measure $\langle \Psi, d\pi(\theta)\Psi \rangle$ is the Fourier transform of $W_2$ at $\theta \in V^*$, and it vanishes unless $\theta \in P^*$ since $\text{supp} \pi \subset P^*$.

For the corollary, observe that the character $\xi \mapsto e^{-i\theta(\xi)/\hbar}$ of $V$ is the boundary value of a holomorphic function on $\mathcal{T} = V - iP$ if $\theta \in P^*$. (The crucial estimate is (11.12).)

The holomorphic function $W_2$ extends to the larger domain

$$D_2 = \text{SO}_n(\mathbb{C})(\mathcal{T}) \cup -\text{SO}_n(\mathbb{C})(\mathcal{T}) \subset V^*_e.$$

The domain $D_2$ in the case $n = 1$ (quantum mechanics) is illustrated in Figure 60. There are analogous domains $D_k$ for all $k \in \mathbb{Z}^{\geq 2}$ and in all dimensions. They are defined in [SW, §3-3] [Kaz, §2.1], for example, and the following lemma and its extension to higher $k$ is proved. Part (1) of the lemma is due to Jost.

**Lemma 21.22.** If $n \geq 3$, then

1. any real spacelike $\xi \in V$ lies in $D_2$, and
2. $D_2$ is connected.

Note from Figure 60 that both parts of Lemma 21.22 fail for $n = 1$. There is a variant for $n = 2$ in [SW, §3-3].

Here is a brief enumeration of some important theorems in the Wightman theory. Two of the most important have already been stated in §15.2. A thorough treatment may be found in [SW, Kaz].
A Wightman quantum field theory can be reconstructed from the Wightman functions \( \{ \mathcal{W}_k \} \).

2. \( \mathcal{W}_k \) extends to a holomorphic function \( \mathcal{W}^C_k \) on a domain \( D_k \subset V^\times(k-1)_E \). Both \( D_k \) and \( \mathcal{W}^C_k \) are \( G(\mathbb{C}) \times (k-1) \)-invariant, where \( G(\mathbb{C}) \) is the complexification of the Lie group \( G \).

3. Write \( V = U \oplus U^\perp \) for a timelike line \( U \subset V \), and set \( V_E = \sqrt{-1}U \oplus U^\perp \subset V_C \). The Lorentz inner product on \( V \) complexifies to a nondegenerate complex symmetric bilinear form on \( V_C \) which then restricts to a (negative) definite inner product on \( V_E \). The case \( n = 1 \) is illustrated in Figure 60. The theorem is that

\[
V^\times(k-1)_E \setminus \text{diagonals} \subset D_k.
\]

The restriction of \( \mathcal{W}^C_k \) to \( V^\times(k-1)_E \) minus diagonals is the Schwinger function, also called the Euclidean correlation function.

The vector space \( V_E \) contains imaginary time translation \( iU \). There is a Euclidean space \( E \) over \( V_E \) which is the Wick rotation of the Minkowski spacetime \( M \).\(^{64}\) The Euclidean correlation function on \( V^\times(k-1)_E \) minus diagonals lifts to \( E^\times k \) minus diagonals.

Osterwalder-Schrader [OS] axiomatized properties of the Euclidean correlation functions; see [Kaz, §2.2]. They proved an important reconstruction theorem. The Osterwalder-Schrader theorem recovers a Wightman field theory on Minkowski spacetime from a Euclidean field theory on Euclidean space. The passage from quantum field theory on Minkowski spacetime to quantum field theory on Euclidean space is Wick rotation. In Section 11.3 we discussed the Wick rotation of the positive energy unitary representation of time translation. Here we have Wick-rotated correlation functions. In doing so we pass from Minkowski spacetime to a complex domain to Euclidean space:

\[
M \leftrightarrow D_k \leftrightarrow E
\]

\(^{64}\)We leave the reader to use standard models to systematize this Wick rotation.
There is an additional passage $E \rightarrow X$ from Euclidean space to a Riemannian manifold that we take up soon.

**Remark 21.25.** Fields in Wightman theory are assumed to be linear; they are functions on Minkowski spacetime with values in a real vector space $R$, as defined before (21.13). But there fields in $\sigma$-models and in gauge theories are not of this type. We gave a very general definition of a field in Lecture 18, and it is this general notion that will be used in the axiom system for Wick-rotated field theory on compact manifolds. There are other limitations of the Wightman theory.

### 21.4 Relativistic symmetry types and Wick rotation

Recall (Definition 15.13) that a relativistic symmetry type is a homomorphism $\lambda: G_n \rightarrow O_{1,n-1}^\uparrow$ whose image contains the identity component $SO_{1,n-1}^\uparrow$. The kernel $K$ of $\lambda$ is the group of internal symmetries. Recall from (17.29) that there is a corresponding affine symmetry type $\tilde{\lambda}: \mathbb{G}_n \rightarrow T_{1,n-1}^\uparrow$.

Wick rotation of $\lambda$ is encoded in the diagram

\begin{equation}
\begin{array}{c}
1 \\
\downarrow \\
K \\
\rightarrow \\
G_R \\
\lambda \\
\rightarrow \\
O_{1,n-1}^\uparrow \\
\downarrow \\
G_n(C) \\
\rightarrow \\
K(C) \\
\rightarrow \\
G_n(C) \\
\rightarrow \\
O_n(C) \\
\downarrow \\
G_n^E \\
\lambda \\
\rightarrow \\
O_n \\
\end{array}
\end{equation}

(21.26)

The top row is the relativistic symmetry type; the bottom row is the Wick-rotated *Euclidean symmetry type*. Wick rotation passes through the complex Lie group $G_n(C)$, which acts as symmetries of the holomorphic correlation functions $W_k^C$. Also, the internal symmetry group $K$ does not change under Wick rotation: it is the same in the top and bottom rows.

If $K$ is a compact Lie group, then so too is $G_n^E$. The rigidity of Lie groups leads to a structure theory for these relativistic symmetry types and their Wick rotations [FH1, §2]. A few highlights are expressed in the following.

**Theorem 21.27.** Suppose $n \geq 3$.

1. The pullback of $G_n^E$ over $\text{Spin}_n$ splits into a direct product $\text{Spin}_n \times K$.
2. The Lie algebra $\mathfrak{g}$ of $G$ splits as a direct sum $\mathfrak{g} = \mathfrak{so}_{1,n-1} \times \mathfrak{k}$.
3. There exists $k_0 \in K$ of order dividing 2 such that the pullback of $G_n^E$ over $SO_n$ is

\begin{equation}
\text{Spin}_n \times K / \langle (\epsilon, k_0) \rangle.
\end{equation}

---

65 We have often used the orthogonal group of a specific Lorentz vector space, but as here we use the model space $\mathbb{R}^{1,n-1}$ for a symmetry type and then we can transport using spaces associated to right torsors; recall §7.4.
The pullback, say over Spin\(_n\), means pullback in the diagram

\[
\begin{array}{ccc}
?? & - & \Rightarrow \text{Spin}_n \\
\downarrow & & \downarrow \\
G^E_n & \stackrel{\lambda}{\rightarrow} & O_n
\end{array}
\]

Remark 21.30. The splittings in Theorem 21.27 are a version of the Coleman-Mandula theorem [CM], but from a different perspective.

There are generalizations of the spin-statistics theorem (Theorem 15.4) and the CRT theorem (Theorem 15.6) to general relativistic symmetry types. We assume both bosonic and fermionic states. The general spin-statistics theorem is as follows.

**Theorem 21.31.** Consider a Wightman quantum field theory which includes the positive energy unitary representation \( U : \mathcal{G} \rightarrow U(\mathcal{H}^0 \oplus \mathcal{H}^1) \). Let \( k_0 \in K \) be the special element in Theorem 21.27(3). Then \( U(k_0) \) is the grading operator

\[
(21.32) \quad \text{id}_{\mathcal{H}^0} - \text{id}_{\mathcal{H}^1}
\]

on \( \mathcal{H}^0 \oplus \mathcal{H}^1 \).

The statement of the general CRT theorem is more complicated, and we defer to [FH1, Appendix A] for a detailed discussion and proof.

### 21.5 Euclidean quantum field theory

This is the field theory of Osterwalder-Schrader mentioned earlier. We work on a Euclidean space \( E \) over an inner product space \( V_E \) of dimension \( n \). The Euclidean symmetry type is a group homomorphism \( \lambda : G^E_n \rightarrow O_n \) such that the image of \( \lambda \) is either SO\(_n\) or O\(_n\). We might also assume that \( G^E_n \) is a compact Lie group. (As in the relativistic case, it could be a super group, a homotopical group, etc.) There is a corresponding affine group \( \mathcal{G}^E_n \) which acts on Euclidean space:

\[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
\downarrow & & \downarrow \\
K & \rightarrow & K
\end{array}
\]

\[
\begin{array}{ccc}
1 & \rightarrow & V_E \\
\downarrow & & \downarrow \\
\mathcal{G}^E_n & \stackrel{\lambda}{\rightarrow} & G^E_n \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\]
there is a local theory of observables: a vector space \( \mathcal{A}(U) \) for open sets \( U \subset E \). One of the key properties—a Wick-rotated version of unitarity—is \textit{reflection positivity}. As in Figure 61, let \( \Pi \subset E \) be an affine hyperplane, and let \( U \subset E \) be an open set supported on one side of \( \Pi \). Define \( \overline{U} \subset E \) to be the reflection of \( U \) in \( \Pi \). If \( \mathcal{O} \in \mathcal{A}(U) \) is an observable in \( U \), then there is a reflected observable \( \overline{\mathcal{O}} \in \overline{U} \). Reflection positivity is the statement

\[
\langle \overline{\mathcal{O}} \mathcal{O} \rangle \geq 0.
\]

(21.34)

\textbf{Figure 61.} Reflection positivity

For a heuristic derivation of (21.34) from the path integral, see [Detal, Problem FP19].

\section*{Lecture 22: More Wick rotation; mathematical background}

Our first encounter with Wick rotation was in quantum mechanics in Lecture 11. We showed how positivity of energy leads to a 1-parameter holomorphic semigroup of contractive operators such that time evolution is a boundary limit. Then restriction to a real 1-parameter semigroup is the Wick-rotated representation (11.15) of time translation. In relativistic quantum field theory (Lecture 21) we Wick-rotated correlation functions rather than the unitary representation of the symmetry group. Here we begin by recounting Segal’s Wick rotation of the positive energy unitary representation of translations in dimension 2, but for quantum field theory in general, not just for conformal field theory. The first move is to compactify space, which is a step toward compactifying Wick-rotated time as well, a step we take in the next lecture.

Then we turn to mathematical background for the next lecture. The first topic is the passage from affine geometries to geometries on smooth manifolds, à la Cartan. This may be regarded as a continuation of the discussion in §7.4. The second topic is the notion of a symmetric monoidal structure on a category. In the next lecture we express a Wick-rotated theory as a homomorphism of such categories.
22.1 Wick rotation in 2 dimensions

Wick rotation of time evolution in quantum mechanics is summarized in Figure 25. Time translations form the vector space $V = \mathbb{R}$ and are represented unitarily on a Hilbert space $\mathcal{H}$ by a representation $t \mapsto e^{-itH/\hbar}$, where $H$ is the Hamiltonian. The positive energy assumption is that the spectrum of $H$ is contained in $P^* \subset V^* = \mathbb{R}^*$, where $P^* = (\mathbb{R}^*)^{<0}$. Then there is a contractive holomorphic semigroup on $V - iP = \mathbb{C}_-$ whose boundary value on $V$ is the unitary time evolution. Restriction to the real semigroup $-iP \simeq P$ is the Wick-rotated evolution $\tau \mapsto e^{-\tau H/\hbar}$.

Consider standard Minkowski spacetime $M = \mathbb{A}^2$ with coordinates $t, x$ and Lorentz metric

$$ds^2 = c^2 dt^2 - dx^2,$$

where $c$ is the speed of light. The translation group $V = \mathbb{R}^2$ acts, and we use lightcone coordinates $\xi^1, \xi^2$ relative to the lightlike basis of $V$ indicated in Figure 62. The positive timelike vectors $(\xi^1, \xi^2)$ satisfy $\xi^1, \xi^2 > 0$. The dual space $V^* = (\mathbb{R}^2)^*$ has a dual basis, and the dual cone $P^*$ is the set of vectors $(\theta_1, \theta_2)$ with $\theta_1, \theta_2 > 0$.

Compactify space: identify $x \sim x + 2\pi$. The quotient of $M$ by this discrete group of translations is diffeomorphic to $\mathbb{R} \times S^1$. It is a torsor over the group $T$ which is the quotient of $V$ defined by

$$1 \rightarrow 2\pi \mathbb{Z} \rightarrow V \rightarrow T \rightarrow 1$$

$$2\pi n \mapsto (\pi n, -\pi n)$$

The Pontrjagin dual group $T^\vee$ of characters of $T$ sits as the sub in the group extension

$$\begin{aligned}
1 &\rightarrow T^\vee \rightarrow V^* \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 1 \\
(\theta_1, \theta_2) &\mapsto \frac{\theta_1 - \theta_2}{2} \pmod{\mathbb{Z}}
\end{aligned}$$

$\footnote{Of course $\mathbb{R}^* = \mathbb{R}$, but we use the $*$ to emphasize that $\mathbb{R}^*$ is the space of smooth unitary characters of $\mathbb{R}$.}$
Figure 63. Translation group $T$, the forward timelike cone $Q$, and their duals

Figure 63 depicts $T$ and $T^\vee$. Each has a subsemigroup of positive elements, which in these coordinates are

$$Q = \{ (\xi^1, \xi^2) : \xi^1, \xi^2 > 0 \}$$
$$Q^* = \{ (\theta_1, \theta_2) : \theta_1, \theta_2 > 0 \}$$

Compactification of space leads to discretization of momentum. The crucial point is that a character in $Q^*$ is the boundary value of a holomorphic semicharacter of the complex semigroup $T - iQ$. The restriction to the real semigroup $-iQ \cong Q$ is the semicharacter

$$(\tau^1, \tau^2) \mapsto e^{-(\theta_1 \tau^1 + \theta_2 \tau^2)}/\hbar$$

A 2-dimensional positive energy unitary representation of $T$ has spectrum supported in $Q^*$, so it has a Wick rotation which is a contractive real semigroup of operators parametrized by $Q$.

22.2 Cartan: from symmetry types to structures on manifolds

For an introduction to Cartan’s theory of $G$-structures, see [Che, Sb]. We barely touch the surface of this beautiful theory.

**Definition 22.6.** An $n$-dimensional symmetry type is a pair $(G_n, \lambda_n)$ in which $G_n$ is a Lie group and $\lambda_n : G_n \to \text{GL}_n \mathbb{R}$ is a homomorphism of Lie groups. The kernel $\ker(\lambda_n)$ is called the internal symmetry group.

We neither assume that $\lambda_n$ is injective nor that $\lambda_n$ is surjective. (Traditionally, Cartan’s notion of $G$-structure requires that $\lambda_n$ be injective.) A symmetry type induces the action $\tilde{\lambda}_n$ of a group of
affine symmetries of the model space $\mathbb{A}^n$ via pullback from the group $\text{Aff}_n$ of affine symmetries:

$$
\begin{array}{cccc}
1 & \longrightarrow & \mathbb{R}^n & \longrightarrow \mathcal{G}_n \longrightarrow G_n & \longrightarrow 1 \\
\downarrow & & \downarrow & \downarrow \lambda_n & \\
1 & \longrightarrow & \mathbb{R}^n & \longrightarrow \text{Aff}_n & \longrightarrow \text{GL}_n \mathbb{R} & \longrightarrow 1
\end{array}
$$

(22.7)

In terms of the Erlangen program (Definition 7.45), $(G_n, \lambda_n)$-geometry is the geometric type $\mathcal{G}_n \subset \mathbb{A}^n$. In other words, $(G_n, \lambda_n)$-geometry is the study of structures/properties/quantities in the model affine space $\mathbb{A}^n$ that are invariant under the $G_n$-action. We leave the reader to think through basic examples: Euclidean geometry ($G_n = O_n$), affine symplectic geometry ($G_{2m} = \text{Sp}_{2m} \mathbb{R}$), affine geometry with a volume ($G_n \subset \text{GL}_n \mathbb{R}$ the group of matrices with determinant $\pm 1$), affine spin geometry ($G_n = \text{Spin}_n$), affine geometry with a translation-invariant foliation, etc. This formalism covers first-order geometric structures; there are also higher order geometric structures, such as conformal structures.

As a preliminary, we recall change of structure group in a principal bundle. Let $G$ be a Lie group, and recall that a principal $G$-bundle is a fiber bundle of right $G$-torsors. For a single $G$-torsor, we described the mixing construction in Definition 7.41. There is an analog for principal $G$-bundles, but we only need a special case.

**Definition 22.8.** Let $\lambda: G' \to G$ be a homomorphism of Lie groups, and suppose $\pi': P' \to X$ is a principal $G'$-bundle. The associated principal $G$-bundle $\pi: \lambda(P') \to X$ has total space

$$
P' \times_G G = (P' \times G) / G',
$$

(22.9)

where the right $G'$-action on $P' \times G$ is $(p', g) \cdot g' = (p' \cdot g', \lambda(g')^{-1}g)$.

**Definition 22.10.** Let $\lambda: G' \to G$ be a homomorphism of Lie groups, and suppose $\pi: P \to X$ is a principal $G$-bundle. A lift or reduction $(\pi', \theta)$ of $\pi$ along $\lambda$ is a principal $G'$-bundle $\pi': P' \to X$ and an isomorphism

$$
P \xrightarrow{\theta} \lambda(P')
$$

(22.11)

of principal $G$-bundles.

The following definition is due to Élie Cartan. (I do not know an exact reference, but surely this should be ascribed to him). A smooth $n$-dimensional manifold carries a canonical principal $\text{GL}_n \mathbb{R}$-bundle

$$
\pi: \text{GL}(X) \longrightarrow X,
$$

(22.12)

its frame bundle. A point of $\text{GL}(X)$ is a point $x \in X$ and a basis $\mathbb{R}^n \xrightarrow{\pi} T_x X$ of the tangent space.
Definition 22.13. Let \((G_n, \lambda_n)\) be an \(n\)-dimensional symmetry type, and suppose \(X\) is a smooth \(n\)-dimensional manifold with frame bundle \(\pi: \text{GL}(X) \rightarrow X\). A \((G_n, \lambda_n)\)-structure on \(X\) is a reduction of \(\pi\) along \(\lambda_n: G_n \rightarrow \text{GL}_n\mathbb{R}\).

We often use ‘\(G_n\)-structure’ when the homomorphism \(\lambda_n\) is an inclusion or is unambiguous.

Example 22.14.

1. An \(O_n\)-structure on a smooth \(n\)-manifold is equivalent to a Riemannian structure.
2. Let \(G_n \subset \text{GL}_n\mathbb{R}\) be the stabilizer subgroup of \(\mathbb{R}^k \subset \mathbb{R}^n\):

\[
G_n = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}
\]

A \(G_n\)-structure on a smooth \(n\)-manifold is a \(k\)-dimensional distribution.
3. Suppose \(n = 2m\) is even. Then a \(U_m\)-structure on a smooth \(n\)-manifold is usually called an almost complex structure.
4. Let \(K\) be a Lie group, and let \(\lambda_n: O_n \times K \rightarrow \text{GL}_n\mathbb{R}\) be the composition of projection onto the first factor and the inclusion \(O_n \rightarrow \text{GL}_n\mathbb{R}\). The corresponding structure on a smooth \(n\)-manifold is a Riemannian metric together with a principal \(K\)-bundle.
5. The reader should spell out spin and \(\text{spin}^c\) structures. What is the internal symmetry group in each case?

There is a category of \((G_n, \lambda_n)\)-manifolds, which we encourage the reader to define carefully. For example, an \(Sp_{4k}\)-manifold is a hyperkähler manifold, and a morphism, or map, of \(Sp_{4k}\)-manifolds is what is often called a ‘triholomorphic map’.

In the model affine space \(\mathbb{A}^n\), the group \(\mathbb{G}_n\) in (22.7) acts by global symmetries. On a \((G_n, \lambda_n)\)-manifold the linear symmetry group \(G_n\) becomes an infinitesimal structure. In view of the group extension (22.7), one can ask for the curved manifold manifestation of the translation subgroup \(\mathbb{R}^n\). It too becomes infinitesimal data. Infinitesimal translation on a smooth manifold is encoded in a linear connection, here required to be compatible with the \((G_n, \lambda_n)\)-structure.

Definition 22.16. Let \((G_n, \lambda_n)\) be an \(n\)-dimensional symmetry type, and suppose \(X\) is a smooth \(n\)-dimensional \((G_n, \lambda_n)\)-manifold. A differential \((G_n, \lambda_n)\)-structure on \(X\) is a connection on the principal \(G_n\)-bundle \(P \rightarrow X\) that defines the \((G_n, \lambda_n)\)-structure.

A differential \((G_n, \lambda_n)\)-structure induces a connection on the frame bundle, and one can further ask that this induced linear connection be torsionfree. There is an existence and uniqueness theory for torsionfree differential \((G_n, \lambda_n)\)-structures [Joy, §2.6]. The obstructions to torsionfree connections are familiar: Frobenius tensor of a distribution, Nijenhuis tensor of an almost complex structure, differential of an almost symplectic form, . . . . The existence of a torsionfree \((G_n, \lambda_n)\)-connection is an integrability condition on the \((G_n, \lambda_n)\)-structure.

22.3 Symmetric monoidal categories

For your convenience I reproduce part of [F4, Lecture 13] here. Those notes on bordism contain much relevant material for the next lecture and especially for next semester, particularly concerning topological field theories, and there is much more background about categories as well in those notes.
A category is an enhanced version of a set; a symmetric monoidal category is an enhanced version of a commutative monoid. Just as a commutative monoid has data (composition law, identity element) and conditions (associativity, commutativity, identity property), so too does a symmetric monoidal category have data and conditions. Only now the conditions of a commutative monoid become data for a symmetric monoidal category, and there are new and numerous conditions. We do not spell them all out, but defer to the references.

If \( C', C'' \) are categories, then there is a Cartesian product category \( C = C' \times C'' \). The set of objects is the Cartesian product \( C_0 = C'_0 \times C''_0 \) and the set of morphisms is likewise the Cartesian product \( C_1 = C'_1 \times C''_1 \). There is a transposition functor \( \tau: C' \times C'' \to C'' \times C' \).

**Definition 22.17.** Let \( C \) be a category. A symmetric monoidal structure on \( C \) consists of an object

\[
1_C \in C,
\]
a functor

\[
\otimes: C \times C \to C
\]
and natural isomorphisms

\[
\begin{align*}
\alpha &: C \times (C \times C) \cong (C \times C) \times C \\
\sigma &: C \times C \cong (C \times C) \times C
\end{align*}
\]
and

\[
\begin{align*}
\iota &: C \cong 1_C \otimes C \\
\end{align*}
\]

The quintuple \((1_C, \otimes, \alpha, \sigma, \iota)\) is required to satisfy the axioms indicated below.
As indicated above, the functor \( \tau \) in (22.20) is transposition:

\[
\tau : C \times C \longrightarrow C \times C \\
y_1, y_2 \longmapsto y_2, y_1
\]

(22.23)

A crucial axiom is that

\[
\sigma^2 = \text{id}.
\]

(22.24)

Thus for any \( y_1, y_2 \in C \), the composition

\[
y_1 \otimes y_2 \xrightarrow{\sigma} y_2 \otimes y_1 \xrightarrow{\sigma} y_1 \otimes y_2
\]

(22.25)

is \( \text{id}_{y_1 \otimes y_2} \). The other axioms express compatibility conditions among the extra data (22.18)–(22.22). For example, we require that for all \( y_1, y_2 \in C \) the diagram

\[
\begin{array}{ccc}
1_C \otimes (y_1 \otimes y_2) & \xleftarrow{\alpha} & (1_C \otimes y_1) \otimes y_2 \\
\downarrow & & \downarrow \\
1_C \otimes y_1 \otimes y_2 & \xrightarrow{\iota} & y_1 \otimes y_2
\end{array}
\]

commutes. We can state the axioms informally as asserting the equality of any two compositions of maps built by tensoring \( \alpha, \sigma, \iota \) with identity maps. These compositions have domain a tensor product of objects \( y_1, \ldots, y_n \) and any number of identity objects \( 1_C \)—ordered and parenthesized arbitrarily—to a tensor product of the same objects, again ordered and parenthesized arbitrarily. Coherence theorems show that there is a small set of conditions which needs to be verified; then arbitrary diagrams of the sort envisioned commute. You can find precise statements and proof in [Mac, JS]

A symmetric monoidal functor is a homomorphism between symmetric monoidal categories, but as is typical for categories the fact that the identity maps to the identity and tensor products to tensor products is expressed via data, not as a condition. Then there are higher order conditions.

**Definition 22.27.** Let \( C, D \) be symmetric monoidal categories. A *symmetric monoidal functor* \( F : C \to D \) is a functor with two additional pieces of data, namely an isomorphism

\[
1_D \longrightarrow F(1_C)
\]

(22.28)

and a natural isomorphism

\[
\begin{array}{ccc}
C \times C & \xrightarrow{F(- \otimes -)} & D \\
\downarrow & & \downarrow \psi \\
F(-) \otimes F(-) & \xrightarrow{F(- \otimes F(-))} & D
\end{array}
\]

(22.29)

There are many conditions on this data.
The first condition expresses compatibility with the associativity morphisms: for all \( y_1, y_2, y_3 \in C \) the diagram

\[
\begin{array}{ccc}
(F(y_1) \otimes F(y_2)) \otimes F(y_3) & \xrightarrow{\psi} & F(y_1 \otimes y_2) \otimes F(y_3) \\
\downarrow{\alpha_D} & & \downarrow{\psi} \\
F(y_1) \otimes (F(y_2) \otimes F(y_3)) & \xrightarrow{\psi} & F((y_1 \otimes y_2) \otimes y_3) \\
\downarrow{\psi} & & \downarrow{F(\alpha_C)} \\
F(y_1) \otimes F(y_2 \otimes y_3) & \xrightarrow{\psi} & F(y_1 \otimes (y_2 \otimes y_3))
\end{array}
\]

is required to commute. Next, there is compatibility with the identity data \( \iota \): for all \( y \in C \) we require that

\[
\begin{array}{ccc}
F(1_C) \otimes F(y) & \xrightarrow{F(\psi)} & F(1_C \otimes y) \\
\downarrow{(22.28)} & & \downarrow{F(\iota)} \\
1_D \otimes F(y) & \xrightarrow{\iota} & F(y)
\end{array}
\]

commute. The final condition expresses compatibility with the symmetry \( \sigma \): for all \( y_1, y_2 \in C \) the diagram

\[
\begin{array}{ccc}
F(y_1) \otimes F(y_2) & \xrightarrow{\sigma_D} & F(y_2) \otimes F(y_1) \\
\downarrow{\psi} & & \downarrow{\psi} \\
F(y_1 \otimes y_2) & \xrightarrow{F(\sigma_C)} & F(y_2 \otimes y_1)
\end{array}
\]

We leave the reader to define a symmetric monoidal natural transformation of symmetric monoidal functors.

Lecture 23: Segal axioms

In this lecture we discuss the axioms (Axiom System 23.14) for Wick-rotated field theory introduced by Graeme Segal [Se2] in the 1980s and elaborated recently in a joint paper with Maxim Kontsevich [KS]. (The 2011 Felix Klein lectures [Se3] by Segal remain also a valuable resource.) Segal’s original context is 2-dimensional conformal field theory. By the end of the 1980s, Atiyah [A1] adopted the Segal axioms to topological field theory, and he also made a strong connection with bordism theory. Over the ensuing decade it became apparent that the point of view in this axiom system—Wick-rotated field theory is a linear representation of a bordism category—applies to general quantum field theories. The Segal axioms provide a pathway into quantum field theory which
has been of enormous value and influence. At the same time they codify structures that have appeared in many mathematical contexts: low dimensional topology, symplectic geometry, geometric representation theory, higher algebra, number theory, etc. We remark that at the time Segal wrote his first paper, 2-dimensional conformal field theory was very much in the collective consciousness due to the first superstring revolution. Segal’s motivations—as stated in [Se2]—came not only from theoretical physics, but also from relations to concrete problems in mathematics: representations of the monster finite group, the theory of positive energy representations of loop groups, the geometry of moduli spaces of Riemann surfaces and relations to $\text{Diff}^+(S^1)$, and elliptic cohomology. Thus from the beginning of this axiom system, physics and mathematics were intertwined.

The route we have taken in these lectures to the Segal axiom system is through quantum mechanics and relativistic quantum field theory. As a codification of quantum field theory in general, the axioms should be treated as malleable, or at the very least incomplete. It is early days, still, and one glaring problem is the dearth of rigorously constructed examples. Nonetheless, this framework has led to interesting applications in quantum theory. And, apart from the physics, the axioms are a natural mathematical structure that has wide applicability, as alluded to above. We emphasize that the axioms go beyond what is strictly required by Wick rotating relativistic quantum field theories. The additional generalizations are part of their power in mathematics.

In the previous lecture (§22.1) we reviewed Wick rotation of the unitary representation of time translation in quantum mechanics. Then we made an analogous construction in 2-dimensional quantum field theory. The first move in that Wick rotation, taken in the previous lecture—is compactification of space. (This is a step toward compactifying Wick-rotated time as well. Indeed, it is an important feature of the Segal axioms that theories are only evaluated on compact manifolds.) The Wick-rotated representation of translations in Minkowski spacetime is a contractive representation of a certain real semigroup. The second move is to identify this semigroup with a particular semigroup of Riemannian cylinders. That is the starting point in this lecture. The identification of these Wick rotations of semigroups of spacetime translations as semigroups of Riemannian manifolds is a crucial step. And the generalization to arbitrary Riemannian manifolds is yet a further step toward the Segal axiom. The third move is to allow the topology of space to change. This leads to a variant of Milnor’s bordism category [Mi2] that includes Riemannian metrics. (Technically, we obtain a semicategory: identity maps are not included, at least at first glance.)

At this point we have arrived at the concise Segal axioms (Axiom System 23.14): a field theory is a linear representation of a Riemannian bordism category. This is analogous to the linear representation of a group, so fits into a long line of mathematical structures.

The abstract Segal axioms seem far from their physics origins. To make the connection, we derive local observables, correlation functions, and Hamiltonians in the Segal picture. We illustrate the singular nature of local observables in the case of quantum mechanics. One can regard the Segal axioms a first step toward localization of states. (Recall that we defined local theories of observables in Lecture 20.) Namely, there is a state space assigned to every closed Riemannian $(n-1)$-manifold in an $n$-dimensional theory, and these $(n-1)$-manifolds undergo

---

67 This is one reason we use ‘Axiom System’ in place of ‘Definition’: the latter sounds much too definitive. Specifically, one needs to assume some good short range behavior if a quantum field theory is to have good analytic properties, much the same as one needs well-posedness in PDE: the backward heat equation satisfies the definition of a PDE, but one cannot evolve far with it.
topology change: Morse surgeries. The situation is not as local as it might be—in topological
theories we will localize states completely—but regardless of the depth of localization, states are
paramount in Segal’s approach. (This is in contrast to the primacy of observables in the Irving
Segal, Haag-Kastler, Costello-Gwilliam line.)

There are three further topics in this lecture: background fields, topological vector spaces, and
reflection positivity. The first two elaborate on the domain and codomain, respectively, of a field
theory. The third is the Wick rotation of unitarity.

We introduced the notion of a relativistic symmetry type in Definition 15.13. It controls whether
a theory has time-reversal symmetry, what the internal symmetries are, and how they interact with
spacetime symmetries. The Wick rotation to a Euclidean symmetry type for Euclidean field theory
is depicted in the diagram (21.26). Recall from Theorem 21.27 that Wick-rotated symmetry types
are highly constrained. In the previous lecture (§22.2) we reviewed the passage from a Euclidean
symmetry type to structure on curved manifolds. Here we morph the result and generalize to
background fields. This allows for local background structures not induced by symmetry, such as
scalar fields with values in nonconnected manifolds.

The codomain of a quantum field theory (23.15) is a category of topological vector spaces and
continuous linear maps. The linear maps need to be “small” to model the contractive nature of
Wick-rotated time evolution in quantum mechanics; see (11.15). The germ surrounding objects in
the bordism category (Remark 8.39) leads to two canonical systems of bordisms, and under limits
we obtain a canonical pair of nuclear topological vector spaces attached to the object. It is these
vector spaces that play a crucial role, and we give some indication of their properties.

Finally, unitarity is not part of Axiom System 23.14, just as the definition of a linear repre-
sentation of a Lie group (think $\text{SL}_2\mathbb{R}$) does not include unitarity. Indeed, for representations
of semisimple Lie groups, unitarizability is an interesting question; the subclass of unitary repre-
sentations has special properties. The Wick-rotated field theory analog of unitarity is reflection
positivity, as stated in (21.34). For field theory on curved manifolds—that is, as a function out
of a bordism category—reflection is an extra structure, both on the background fields and on the
symmetric monoidal functor that is the field theory. Then positivity is a condition on the reflection
structure.

We conclude with further mathematical background: basic definitions for topological vector
spaces and some material on duality in symmetric monoidal categories.

These notes have far more than we were able to include in the last lecture of the semester.

23.1 Semigroups of Riemannian manifolds

In 1-dimensional quantum mechanics, Wick rotation produces a semigroup of contractive oper-
ators parametrized by $\mathbb{R}^+$; in a 2-dimensional theory with space diffeomorphic to $S^1$, the relevant
semigroup is $Q$ as defined in (22.4). We now identify each as a semigroup of Riemannian manifolds,
with composition by gluing.

Remark 23.1. I know no analog of semigroups of Riemannian manifolds that models groups of
spacetime translations.

For the 1-dimensional semigroup $\mathbb{R}^+$, observe that a Riemannian metric on a manifold dif-
fefomorphic to $[0,1]$ has a single invariant: length. Furthermore, the length adds under gluing
of Riemannian intervals—see Figure 64. Therefore, the semigroup of Riemannian closed intervals under gluing—concatenation—is isomorphic to $\mathbb{R}^{\geq 0}$. Wick-rotated time evolution in quantum mechanics is a contractive representation of this semigroup.

Figure 64. Gluing of Riemannian intervals

Figure 65. A semigroup of Riemannian cylinders

For the 2-dimensional real semigroup $Q$ there is no pithy isomorphic semigroup of Riemannian manifolds. The underlying smooth manifold should be diffeomorphic to a cylinder $[0,1] \times S^1$, but the space of isometry classes of Riemannian metrics on the cylinder is infinite dimensional: the Gauss curvature is an invariant and already there is an infinite dimensional space of those. We must standardize space to be the circle $S^1(2\pi)$ of length $2\pi$, certainly for the initial and final spaces—the boundary of the cylinder—and so we demand an isometry of each boundary circle to a standard circle of length $2\pi$. To realize $Q$, then, restrict to cylinders isometric to $[0,\tau] \times S^1(2\pi)$, as depicted in Figure 65, together with an isometry of standard $S^1(2\pi)$ onto each boundary component, as indicated by the images of a basepoint in standard $S^1(2\pi)$.

Remark 23.2.

(1) This realization of $Q$ has a preferred splitting as a Cartesian product of Wick-rotated time and space. On general Riemannian manifolds, to which we generalize forthwith, there is no such splitting.

(2) The analog of the real semigroup $Q$ in 2-dimensional conformal field theory is infinite dimensional, and is a main ingredient in [Se2]. (It was introduced independently by Kontsevich and Neretin [Ne].) Note that every conformal cylinder, or annulus, is conformally equivalent to a standard annulus $\{ z \in \mathbb{C} : r \leq |z| \leq 1 \}$ for some $r \in (0,1)$ by uniformization; the semigroup in question also includes parametrizations of the two boundary circles. That 2-dimensional conformal field theory has a contractive representation of this infinite dimensional semigroup makes it more tractable than most other quantum field theories: there is more symmetry to deploy.
In the 2-dimensional case it is natural to consider not only representations of the real semigroup \( Q \) obtained by Wick rotation of the semigroups of spacetime translations, but rather representations of the much larger semigroup of all Riemannian cylinders. (In the 1-dimensional case we already have the semigroup of all Riemannian intervals.) Similarly, in arbitrary dimension \( n \), for any closed \((n - 1)\)-manifold \( Y \) we consider arbitrary Riemannian metrics on the cylinder \([0, 1] \times Y\).

**Remark 23.3.** In relativistic quantum field theory, one usually assumes the existence of an *energy-momentum* tensor. This may be interpreted as encoding an infinite order formal deformation of the flat Lorentz metric on Minkowski spacetime. A stronger form of that assumption is the formulation of the theory on arbitrary Lorentz manifolds. We make the corresponding assumption in the Wick-rotated setting.

### 23.2 Topology change

The final conceptual input is topology change. In dimension 2 one motivation comes from string theory, where topology change can be viewed as the splitting and joining of strings. In any dimension we can think that this splitting and gluing of space is an expression of the locality of state spaces in the quantum theory.

![Figure 66. Topology change of “space” in quantum mechanics](image)

**Remark 23.4.** In dimension 1—quantum mechanics—letting in topology change has the consequence that the “Wick-rotated processes” depicted in Figure 66 are allowed. In the first process, two copies of the system come together and annihilate; in the second, two copies are created out of nothing. Or, we can view the illustration as a single process from two copies to two copies. As we will see ‘copies’ means the conjunction (oft called ‘stacking’) of a system with itself without interaction: the state space of the conjunction is the tensor product of the individual state spaces, and the Hamiltonian of the conjunction is the sum of the individual Hamiltonians.

In his lectures on the h-cobordism theorem [Mi2], Milnor introduced the mathematical structure that tracks topology change: *bordism categories*. First, recall the classical definition of a bordism.

**Definition 23.5.** Fix \( d \in \mathbb{Z}^{\geq 0} \). Closed \( d \)-manifolds \( Y_0, Y_1 \) are *bordant* if there exists a compact \((d + 1)\)-manifold \( X \) with boundary such that \(^{68} Y_0 \sqcup Y_1 \approx \partial X\).

\(^{68}\)For smooth manifolds the symbol ‘\( \approx \)’ means ‘is diffeomorphic to’.
The manifold \( X \) is called a bordism. The condition in Definition 23.5 defines an equivalence relation on diffeomorphism classes of closed \( d \)-manifolds; the set of equivalence classes is denoted \( \Omega_d \). Furthermore, disjoint union passes through this equivalence relation, where it becomes a commutative, associative composition law on \( \Omega_d \). The class of the empty manifold \( \emptyset^d \) is a unit for this composition law. Finally, every element has order 2: the manifold \( X = [0,1] \times Y \) is a null bordism of \( Y \cup \emptyset^2 \). In summary, \( \Omega_d \) is an abelian group.

Remark 23.6.

1. There are variants in which the manifolds carry additional topological structure, such as an orientation. Inverses still exist, but manifolds may not be their own inverse. For example, the inverse of a closed oriented \( d \)-manifold is the oppositely oriented manifold. A standard reference on bordism theory is [Sto].

2. A homomorphism from a bordism group to an abelian group is a bordism invariant. For example, the signature of a manifold of dimension \( 4k \) is a homomorphism \( \Omega_{4k}(SO) \rightarrow \mathbb{Z} \) that is a crucial invariant in differential topology.

Bordism groups are classical; Milnor’s idea was to “categorify” by promoting the bordism in Definition 23.5 from a condition to data. In Wick-rotated field theory, the manifolds \( Y \) play the role of space and the manifolds \( X \) play the role of a Wick-rotated spacetime. Hence introduce \( n = d + 1 \) for the dimension of \( X \).

\[ \xymatrix{ \emptyset_0 \ar[r] & 0 \ar[r] & 1 \ar[r] & \emptyset_1 } \]

**Figure 67.** A bordism \( X : Y_0 \rightarrow Y_1 \)

**Definition 23.7.** Let \( Y_0, Y_1 \) be closed \((n-1)\)-manifolds. A bordism \((X, p, \theta_0, \theta_1)\) from \( Y_0 \) to \( Y_1 \) consists of a compact \( n \)-manifold \( X \) with boundary, a partition \( p: \partial X \rightarrow \{0,1\} \) of its boundary, and disjoint embeddings

\[
\begin{align*}
\theta_0 &: [0,1) \times Y_0 \rightarrow X \\
\theta_1 &: (-1,0] \times Y_1 \rightarrow X
\end{align*}
\]

such that \( \theta_i(0,Y_i) = (\partial X)_i, \ i = 0,1 \), where \( (\partial X)_i = p^{-1}(i) \).

A bordism is depicted in Figure 67. The function \( p \) is depicted by an arrow on each boundary component that points in (incoming=domain) or out (outgoing=codomain). One should include 2-sided collars to facilitate gluing, or even better germs of 2-sided collars.
Remark 23.10. In the application to Wick-rotated field theory, the arrow at a boundary component is a Wick-rotated time-orientation.

Definition 23.11 (Milnor). The bordism category $\text{Bord}_n$ is the category whose objects are closed $(n-1)$-manifolds and whose morphism sets are equivalence classes of bordisms:

$$\text{Hom}_{\text{Bord}_n}(Y_0, Y_1) = \{(X, p, \theta_0, \theta_1)\} / \sim$$

An equivalence of bordisms is a diffeomorphism of manifolds $X$ that commutes with $p, \theta_0, \theta_1$. As in Remark 23.6(1) there are variants for manifolds equipped with tangential structures, such as spin structures, framings, etc. Observe that the identity map in $\text{Hom}_{\text{Bord}_n}(Y,Y)$ is represented by a cylinder $[0,1] \times Y$ with identity boundary identifications. A diffeomorphism of $Y$ is represented by the same cylinder, but with the boundary identification $\theta_1: Y \rightarrow \{1\} \times Y$ equal to the inverse of the diffeomorphism. Two diffeomorphisms are equivalent if they are pseudoisotopic; see [Mi2, §1].

The composition law in the abelian bordism group—disjoint union—categorifies to a symmetric monoidal structure on the category $\text{Bord}_n$. We provide background material on symmetric monoidal structures in Section 22.3. A monoidal structure on a category is an associative composition law on objects and morphisms, but now associativity on objects is data rather than a condition and there is another layer that brings in a condition. A symmetry of the monoidal product is data that expresses commutativity.

![Figure 68. A closed $(n-1)$-manifold $Y$ embedded in a germ $U$ with coorientation](image)

Now, motivated by our previous discussions, we augment Milnor’s bordism category $\text{Bord}_n$ to include Riemannian metrics. An object in $\text{Bord}_n(\text{Riem})$ is a closed $(n-1)$-manifold $Y$, a germ $U$ of an $n$-manifold into which $Y$ is embedded as a submanifold, a coorientation of $Y \subset U$, and a Riemannian metric on $U$. See Figure 68 for an illustration. One can take $U$ to be an $n$-manifold diffeomorphic to $(-1, 1) \times Y$. The germ is the inverse limit of an inverse system of such Riemannian manifolds $U$ containing $Y$ as a cooriented codimension one submanifold. We overload ‘$Y$’ to mean the manifold $Y$ together with the germ, coorientation, and Riemannian metric. Morphisms in $\text{Bord}_n(\text{Riem})$ are equivalence classes of bordisms, only now the bordisms are also equipped with a Riemannian metric and the diffeomorphisms $\theta_i$ are isometries of germs of $n$-manifolds containing closed cooriented codimension one submanifolds. (So now the shading in Figure 67 is 2-sided, as in Figure 68.)
Remark 23.13.

1. As remarked in [KS, §3], the endomorphism sets in $\text{Bord}_n(\text{Riem})$ do not contain identities, and so $\text{Bord}_n(\text{Riem})$ is a semicategory rather than a category, in line with the semigroups $\mathbb{R}^{>0}$ and $Q$ that appeared earlier. One can rectify this by adjoining identity maps, and as well one should adjoining isometries of objects to the category—they too should act in a field theory. (They are Wick-rotated remnants of isometries of space in Minkowski spacetime, once one fixes a spacelike codimension one affine hyperplane.)

2. In the application to Wick-rotated field theory, a morphism $X$ in $\text{Bord}_n(\text{Riem})$ is not a “spacetime”: it carries a Riemannian metric, not a Lorentzian metric. We could call it a ‘Wick-rotated spacetime’ or, more simply, say that we study field theory on $X$.

23.3 Axiom system for field theory

Recall that the Wick rotation of the unitary positive energy representation of the translation group of Minkowski spacetime is a contractive linear representation of a real semigroup. Based on this, we now give a provisional definition of a field theory, provisional subject to more elaboration of the domain and codomain. (There are more substantial provisos: footnote 67, Remark 23.16, Remark 23.18, Remark 23.23(1), and Remark 23.31.) Let $t\text{Vect}$ be a “suitable” category of complex topological vector spaces and continuous linear maps, which we suppose has a suitable tensor product. Crucially, the linear maps are nuclear, which is a class of linear maps analogous to contractive maps between Banach spaces. The following is what we have been calling the ‘Segal axioms’. The first occurrence in print that I am aware of is [KS, §3].

**Axiom System 23.14.** A (Wick-rotated) field theory is a homomorphism

\[ F: \text{Bord}_n(\text{Riem}) \rightarrow t\text{Vect}. \]

The symmetric monoidal structure on $\text{Bord}_n(\text{Riem})$ is disjoint union; the symmetric monoidal structure on $t\text{Vect}$ is tensor product. A homomorphism in this context is oft called a symmetric monoidal functor. It is a map that preserves all algebraic structures, but as is characteristic in category theory that “preservation” includes data as well as conditions. So $F$ maps compositions of bordisms to compositions of linear maps, it maps disjoint unions to tensor products, and it maps the empty $(n-1)$-manifold to the vector space $\mathbb{C}$. An endomorphism of $0^{n-1}$ in $\text{Bord}_n(\text{Riem})$ is a closed Riemannian $n$-manifold $X$, and $F(X)$ is a complex number, the partition function of $X$.

**Remark 23.16.** In Lecture 8 we emphasized that quantum theory is a projective system, not a linear system, and we illustrated that principle in quantum mechanics. (Recall the slogan (8.1).) Hence we should replace Axiom System 23.14 by a projective representation of the bordism category, i.e., by a homomorphism into a symmetric monoidal category of projective spaces. The precise construction does not have this exact description.

---

69 The empty set is a manifold of any dimension and it is the unit for disjoint union. The vector space of scalars $\mathbb{C}$ is the unit for tensor product.

70 The nomenclature derives from statistical mechanics and reflects a relationship between Wick-rotated quantum field theory and statistical mechanics.

71 The precise construction does not have this exact description.
type of invertible field theory, which in this context is called the anomaly. In the meantime, we persist with the linear representation (23.15).

A field theory assigns a vector space $F(Y)$ to each closed $(n-1)$-manifold $Y$ (with germ, coorientation, and Riemannian metric). This can be regarded as the state space of a quantum mechanical system; states and maps of states have center stage in the Segal axioms. Observables are derived quantities. In the remainder of this lecture we explain how to tease out observables and correlation functions from Axiom System 23.14. We refer to [KS] for more exposition on these matters.

23.4 Hamiltonians

In the Hamiltonian setting of relativistic mechanics, spacetime is a Cartesian product of space and time. So too in Wick-rotated field theory we only expect time evolution in the presence of a product structure. Hence let $Y$ be an object in $\text{Bord}_n(\text{Riem})$—a closed $(n-1)$-manifold—which is embedded in the Cartesian product $(-1,1) \times Y$ equipped with a Riemannian metric that is the product of the standard metric on $(-1,1)$ and a Riemannian metric on $Y$. A field theory $F$ assigns to this data a topological vector space $F(Y)$. For $\tau \in \mathbb{R}^>0$ consider the bordism

$$X_\tau = [0,\tau] \times Y : Y \to Y$$

with the product Riemannian metric. Then $F(X_\tau) : F(Y) \to F(Y)$ is a 1-parameter semigroup of (nuclear) linear maps. This is the Wick rotation of time evolution. One can hope to construct a continuous linear operator $H_Y : F(Y) \to F(Y)$ such that $F(X_\tau) = e^{-\tau H_Y/\hbar}$ if one can give meaning to the exponential. For example, in a unitary theory—unitarity is discussed in the next lecture—there is a Hilbert space associated to $F(Y)$, and then it is reasonable to carry this out. The operator $H_Y$ is the Hamiltonian.

Remark 23.18. In this argument we evaluate $F$ on a family $\{X_\tau\}$ of bordisms; recall the discussion in the bonus lecture. Clearly Axiom System 23.14 should be expanded to include smoothness in families. See [ST] for an implementation in Euclidean field theory.

23.5 Local observables

Next, we construct a Wick-rotated analog of a local net of observables (Definition 20.17). Let $F$ be an $n$-dimensional field theory, and suppose $X$ is a closed Riemannian $n$-manifold. Let $U \subset X$ be an open subset whose closure $\overline{U}$ is a smooth $n$-manifold with boundary. To each such smooth open subset assign the topological vector space

$$(23.19) \quad \mathcal{A}(U) = F(\partial \overline{U}).$$

If, as in Figure 69, $U_1, \ldots, U_k \subset V \subset X$ is an inclusion of disjoint smooth open subsets $U_i$ in an open subset $V$, then

$$(23.20) \quad V \setminus (U_1 \sqcup \cdots \sqcup U_k) : \partial U_1 \sqcup \cdots \sqcup \partial U_k \to \partial V$$

There is also a Hamiltonian if $Y$ is embedded in a germ with a product metric, as we show below. Smooth open subset is not a standard term; I use it only here and only for convenience.
is a morphism in $\text{Bord}_n(\text{Riem})$, and the value of $F$ is a continuous linear map

(23.21)  \[ A(U_1) \otimes \cdots \otimes A(U_k) \rightarrow A(V). \]

This satisfies an operadic property: if for each $i \in \{1, \ldots, k\}$ we have given disjoint smooth open subsets $U_1^i, \ldots, U_j^i \subset U_i$, then there is an identity between (23.21) and the maps

(23.22)  \[ A(U_1^i) \otimes \cdots \otimes A(U_j^i) \rightarrow A(U_i) \]

that expresses associativity. An assignment $A$ with these properties is called a \textit{prefactorization algebra} (on the category $\text{sOpen}(X)$ of smooth open subsets of $X$ and inclusion maps).

Remark 23.23.

1. A \textit{factorization algebra} is a prefactorization algebra that satisfies a local-to-global condition analogous to a sheaf condition. I do not know if it follows from the Segal axioms as stated; presumably one needs additional conditions. Factorization algebras generalize the \textit{chiral algebras} of Beilinson-Drinfeld [BD], which were inspired by 2-dimensional conformal field theory. The theory of factorization algebras is developed by Costello-Gwilliam in [CG1].

2. In relativistic field theory, a net of observables (Definition 20.17) assigns an algebra to each open set, and the algebras of spacelike separated sets commute. In a prefactorization algebra, by contrast, to each smooth open set is attached a topological vector space; the algebra structure is replaced by the maps (23.21). Furthermore, the analog of commutation holds for every pair of disjoint open sets: the tensor product in (23.21) is symmetric monoidal. Observe that in Euclidean space—and on Riemannian manifolds—two distinct points are spacelike separated: the displacement vector between distinct points in Euclidean space is spacelike since the inner product is negative definite.

23.6 Correlation functions

Next, we illustrate how to recover correlation functions from \textbf{Axiom System 23.14}, following discussions in [KS]. Given a field theory $F$ and a closed Riemannian manifold $X$, we seek:
Figure 70. Correlation function of point observables

(1) a topological vector space $\mathcal{O}_x$ of point observables for each $x \in X$, and
(2) a continuous multilinear function

\[(23.24) \quad \mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_k} \longrightarrow \mathbb{C} \]

for each subset $\{x_1, \ldots, x_k\} \subset X$ of distinct points.

The main idea is an inverse limit construction.

Figure 71. Inverse limit construction of $\mathcal{O}_x$

Let $D = D^n$ be the standard closed $n$-disk, and let $\hat{D}$ denote its interior. For $x \in X$ set

\[(23.25) \quad \mathcal{D}_x = \{ f : D \hookrightarrow X : f \text{ is an embedding with } x \in f(\hat{D}) \} . \]

If $f_0, f_1 \in \mathcal{D}_x$ satisfy $f_0(D) \subset f_1(\hat{D})$, as depicted in Figure 71, then there is a bordism

\[(23.26) \quad f_1(D) \setminus f_0(\hat{D}) : f_0(\partial D) \longrightarrow f_1(\partial D) . \]

These bordisms correspond to morphisms in an inverse system whose set of objects is $\mathcal{D}_x$. Define the point observables as an inverse limit in $t\text{Vect}$:

\[(23.27) \quad \mathcal{O}_x = \lim_{\mathcal{D}_x} F(f(\partial D)) . \]
This construction gives a continuous linear map

\[(23.28) \quad O_x \rightarrow F(\partial D)\]

for each \(f \in \mathcal{D}_x\): an observable at \(x\) induces an observable in the smooth open set \(f(\bar{D})\).

To construct the correlation function \((23.24)\), choose \(f^{(i)} \in \mathcal{D}_x\) with disjoint images. Then apply \(F\) to the bordism depicted in Figure 72,

\[(23.29) \quad X \setminus \bigcup_i f^{(i)}(\bar{D}); \bigcup_i f^{(i)}(\partial D) \rightarrow \emptyset^{n-1},\]

and compose with the maps \((23.28)\).

Finally, recall from Remark 21.12(2) that in a relativistic field theory a quantum field is an operator-valued distribution. An analog in our Wick-rotated setting is as follows. Suppose \(X: Y_0 \rightarrow Y_1\) is a morphism in \(\text{Bord}_n(\text{Riem})\). Let \(x \in \bar{X}\) be an interior point, as in Figure 73. Apply \(F\) to obtain a continuous linear map

\[(23.30) \quad O_x \rightarrow \text{Hom}(F(Y_0), F(Y_1))\]

from observables at \(x\) to continuous linear maps.
**Remark 23.31.** As in Remark 23.18 we should evaluate \( F \) on the family of bordisms parametrized by \( \hat{X} \). Smoothness of that evaluation should include a vector bundle structure on the family of vector spaces \( \{O_x\} \to \hat{X} \), and furthermore that vector bundle should carry a flat covariant derivative. That given, one can hope to make sense of operator-valued distributions. We refer to [KS] for further discussion.

23.7 Wick-rotated quantum mechanics in the axiom system

This material is from [Se3, Lecture 2].

Quantum mechanics—in Wick-rotated form—is the case \( n = 1 \) of Axiom System 23.14. An object in \( \text{Bord}_1(\text{Riem}) \) is a finite set of points embedded in a germ of a Riemannian 1-manifold, and equipped with a coorientation. Consider a single point. Using parametrization by arclength for Riemannian 1-manifolds, we see that the germ is isometric to the germ of \( \{0\} \subset (-1,1) \) with the standard metric and standard coorientation. A theory \( F \) maps this to a topological vector space which, assuming unitarity (as will be discussed below), we can replace with a Hilbert space \( \mathcal{H} \).

Then, as in (23.17), the bordisms \( [0,\tau] : \{0\} \to \{\tau\}, \tau \in \mathbb{R}^>0 \), map under \( F \) to a 1-parameter semigroup of contractive operators \( e^{-\tau H/\hbar} \) on \( \mathcal{H} \).

**Figure 74.** The inverse limit that computes observables

The topological vector space of observables is defined in (23.27). Let \( x \in (0,\tau) \). Then the boundary of an embedded disk containing \( x \) in its interior is \( \{x - \delta_1, x + \delta_2\} \) for some \( \delta_1, \delta_2 > 0 \). A bordism (23.26) is depicted in Figure 74. Under \( F \) it evaluates to the map

\[
T \mapsto e^{-\tau_2 H/\hbar} \circ T \circ e^{-\tau_1 H/\hbar}
\]

on endomorphisms of \( \mathcal{H} \). The inverse limit of End \( \mathcal{H} \) under these maps consists of singular operators.

**Example 23.33.** Consider a particle on a real analytic Riemannian manifold \( M \) with state space \( \mathcal{H} = L^2(M;\mathbb{C}) \) and Hamiltonian the Hodge laplacian \( H = \Delta \), as in Lecture 5. The heat operator \( e^{-\tau \Delta/\hbar} \) maps distributions (in the dual space to analytic functions) to analytic functions: it is a smoothing operator. So the inverse limit under (23.32) contains operators that map analytic functions to distributions. These are unbounded operators on \( L^2 \) functions, which is what one expects for observables in quantum mechanics.

23.8 Euclidean symmetry types and background fields

Recall Wick rotation of symmetry types from §21.4. A *relativistic symmetry type* is a homomorphism of Lie groups \( \lambda : G_n \to O_{1,n-1}^1 \) whose image contains the identity component \( SO_{1,n-1}^1 \). (The latter ensures relativistic invariance: the symmetry group cannot be too small.) The kernel \( K \) of \( \lambda \) is the group of internal symmetries. The Wick rotation of \( (G_n, \lambda) \) appears in (21.26). For convenience, we formalize the definition.
Definition 23.34. An $n$-dimensional Euclidean symmetry type is a pair $(G^E_n, \lambda)$ in which $G^E_n$ is a Lie group and $\lambda: G^E_n \to O_n$ is a homomorphism of Lie groups whose image contains the identity component $SO_n$. The kernel $K$ of $\lambda$ is the group of internal symmetries.

We might require that $K$, and hence $G^E_n$, be compact. As commented earlier, we can also loosen the requirement that $K$ be a Lie group and allow super Lie groups as well as homotopical generalizations of groups.

Notice that a $(G^E_n, \lambda)$-structure on a smooth manifold $X$ induces a Riemannian structure on $X$, and so a principal $O_n$-bundle

$$\pi: B_O(X) \to X$$

of orthonormal frames. Then (23.35) carries a unique torsionfree connection, the Levi-Civita connection. We require that the connection on a differential $G^E_n$-manifold (Definition 22.16) induce the Levi-Civita connection on the orthonormal frame bundle (23.35).

So far, in Axiom System 23.14, we defined a field theory as a homomorphism out of a bordism category of Riemannian manifolds. Given a Euclidean symmetry type $(G^E_n, \lambda)$, we can form a bordism category $\text{Bord}_n(G^E_n)$ of differential $G^E_n$-manifolds. Objects are closed $(n-1)$-manifolds embedded in a germ of a differential $G^E_n$-manifold of dimension $n$. Bordisms are compact $n$-manifolds equipped with a differential $G^E_n$-structure. Then we can modify (23.15) to be a homomorphism

$$F: \text{Bord}_n(G^E_n) \to t\text{Vect}.$$ 

This is sufficient for many field theories.

Example 23.37. Consider the Euclidean symmetry type $G^E_n = SO_n \times K$ with $\lambda: G^E_n \to O_n$ given as projection to $SO_n$ followed by inclusion into $O_n$. Then $\text{Bord}_n(G^E_n)$ is the domain of Wick-rotated $n$-dimensional theories with no time-reversal symmetry and an internal symmetry group $K$ which, in its relativistic incarnation, does not interact with symmetries that move points of spacetime. Objects and morphisms in the bordism category come equipped with a principal $K$-bundle with connection, which is called a “background gauge field” for the symmetry in the physics literature.

However, this does not cover all background fields that occur in field theories, both in their physical incarnation as Wick rotations of relativistic quantum field theories and in their mathematical applications. For example, we might have a background field that is a smooth map to the two-point set $\{A, B\}$. This amounts to having two theories—Type A and Type B—and allowing different components of a manifold to have different theories on them. (A physical example is the Type II superstring.) It is also useful to allow “$\sigma$-model fields” into an arbitrary smooth manifold that may have more than two points.

We already introduced a general notion of ‘field’ in Lecture 18. Recall that $\text{Man}_n$ is the category of smooth $n$-dimensional manifolds and local diffeomorphisms, and $\text{Set}_\Delta$ is the category of simplicial sets and simplicial maps between them.

Definition 23.38. A (set of) $n$-dimensional background fields is a sheaf

$$\mathcal{F}: \text{Man}_n \to \text{Set}_\Delta.$$
For example, the Wick-rotated quantum theory of a free spinor field has

\[(23.40)\quad \mathcal{F} = \{\text{Riemannian metrics, spin structures}\}.\]

In other words, for \(M\) a smooth \(n\)-manifold an object of the groupoid, or simplicial set, \(\mathcal{F}(M)\) is a pair of a Riemannian metric on \(M\) and a spin structure on \(M\). This is a differential \(G^E_n\)-structure on \(M\) for the Euclidean symmetry type \(\text{Spin}_n \to O_n\). There are examples not coming from Euclidean symmetry types. A Wick-rotated classical \(\sigma\)-model into a fixed Riemannian manifold \(P\) might have \(\mathcal{F} = P\), meaning \(\mathcal{F}(M) = C^\infty(M, P)\) is the set of smooth maps \(M \to P\). There are also background fields of a more topological nature. For example, the sheaf of \(n\)-framings has \(\mathcal{F}(M)\) equal to the set of trivializations of the tangent bundle \(TM \to M\), i.e., the set of sections of the frame bundle (22.12). It corresponds to the affine symmetry type \(\{e\} \hookrightarrow O_n\), which does not fit Definition 23.34.

**Remark 23.41.** Replacing Euclidean symmetry types by more general affine symmetry types, and even more generally by background fields, is one way in which the axiom system goes beyond Wick-rotated relativistic field theory.

The domain of a Wick-rotated field theory is a bordism category of manifolds equipped with background fields from an arbitrary sheaf on \(\text{Man}_n\). This leads to the following variation of **Axiom System 23.14**.

**Axiom System 23.42.** Fix a positive integer \(n\) and a simplicial sheaf \(\mathcal{F}\) on \(\text{Man}_n\). A (Wick-rotated) \(n\)-dimensional field theory over \(\mathcal{F}\) is a symmetric monoidal functor

\[(23.43)\quad F: \text{Bord}_n(\mathcal{F}) \to t\text{Vect}.\]

We sometimes use the language ‘a field theory on \(\mathcal{F}\).’

**Remark 23.44.** The dimension \(n\) and the background fields \(\mathcal{F}\) tell what “type” of field theory we have. It is useful to always articulate explicitly what \(n\) and \(\mathcal{F}\) are for a given field theory.

### 23.9 Topological vector spaces

Now we turn to the codomain \(t\text{Vect}\) of a Wick-rotated field theory. It is a symmetric monoidal category of topological vector spaces, and we need to specify it precisely. I will make some remarks in this direction following [KS, §3]. The PhD thesis [Wed] of Richard Wedeen has a more complete development of these ideas. Some basic definitions can be found in §23.13.1.

A topological vector space is the marriage of a linear space and a topological space. We require (Definition 23.66) that the vector space operations be continuous and that the topology be Hausdorff. The simplest class of topological vector spaces is the collection of Hilbert spaces. On a smooth manifold \(M\) these include spaces of \(L^2\) functions and Sobolev spaces. Many theorems about finite dimensional inner product spaces have generalizations to Hilbert spaces, and the study of linear operators is rich and detailed, including for example the spectral theorem for self-adjoint operators (Theorem 1.46). The next simplest class of topological vector spaces is the collection of
Banach spaces. A Banach space is a complete normed linear spaces. Already here the geometry
can be quite varied; Banach spaces provide a sufficiently rich playground for deep mathematical in-
quiry. The bounded continuous functions on any Hausdorff locally compact topological space form
a Banach space, and on a smooth manifold so too do the $C^k$ functions for any $k \geq 0$. Perhaps the
next simplest class of topological vector spaces is the collection of Fréchet spaces (Definition 23.67).
They are locally convex, metrizable, and complete, but in general their topology is not given by
a norm. (It is given by a countable family of seminorms; see Definition 23.68.) The canonical
example of a Fréchet space is the space of smooth functions on a smooth manifold.

Remark 23.45. One can develop calculus on affine spaces over these topological vector spaces, and
then define infinite dimensional manifolds locally modeled on them. Calculus on affine spaces over
Banach spaces behaves much like ordinary calculus. In particular, the implicit function theorem
holds. But, say, spaces of smooth maps between smooth manifolds are not Banach manifolds—they
are Fréchet manifolds—and for these the implicit function theorem is much more delicate. Even
on Banach spaces one may not have bump functions, so partitions of unity exist only on certain
Banach manifolds, e.g., Hilbert manifolds.

The key property of Wick-rotated time evolution in quantum mechanics (11.15) is that it is by
contractive operators on a Hilbert space. A contractive operator is “small”, and there are different
kinds of smallness for an operator on Hilbert space: compact, Hilbert-Schmidt, trace class, etc.
The Hilbert space structure is tied to unitarity, and in the general Axiom System 23.42 we do not
assume unitarity. So the topological vector spaces in the image of a Wick-rotated field theory are
not assumed to be Hilbert spaces, and we need a notion of smallness for linear operators between
more general topological vector spaces. This was a notion introduced by Grothendieck in his first
major work [Gr]: nuclear operators. He also introduced the concept of a nuclear space, which
is a special type of topological vector space, and also a good theory of tensor products of nuclear
spaces. In fact, there is a symmetric monoidal category whose objects are nuclear vector spaces
and whose morphisms are continuous linear maps; the monoidal structure is the tensor product. The
definitions are complicated, and we do not tackle them in this version of the notes. An excellent
summary of nuclear spaces is [Co, Appendix 2] to which we refer the reader.

Example 23.46. Let $M$ be a smooth manifold. Then the space $C^\infty(M)$ of smooth functions on $M$
is nuclear, as is its dual $C^{-\infty}(M)$, which is a space of distributions. If $M$ is real analytic, then the
space $C^\omega(M)$ of real analytic functions is nuclear, as is its dual space. These spaces behave well
under the tensor product of nuclear spaces. For example, if $M_1, M_2$ are smooth manifolds, then
there is an isomorphism

\[ C^\infty(M_1 \times M_2) \cong C^\infty(M_1) \otimes C^\infty(M_2). \] 

Of particular importance to us are nuclear Fréchet spaces and their duals, dual nuclear Fréchet
spaces. (The dual of a Fréchet space is not necessarily metrizable, hence is not necessarily Fréchet.)
The basic properties we need are as follows:

1. The dual of a nuclear Fréchet space is dual nuclear Fréchet, and the dual of a dual nuclear
   Fréchet space is a nuclear Fréchet space.
(2) The inverse limit of a countable sequence of nuclear maps of Fréchet spaces is a nuclear
Fréchet space.
(3) The direct limit (colimit) of a countable sequence of nuclear maps of Fréchet spaces is a
dual nuclear Fréchet space.
(4) A continuous map from a dual nuclear Fréchet space to a nuclear Fréchet space is nuclear.

Example 23.48. For a smooth manifold $M$, the space $C^\infty(M)$ is nuclear Fréchet and the dual
space $C^{-\infty}(M)$ is dual nuclear Fréchet. For a real analytic manifold $M$, the space $C^\omega(M)$ of real
analytic functions is dual nuclear Fréchet and the dual space $C^{-\omega}(M)$ is nuclear Fréchet.

23.10 The codomain of a field theory

Now we apply the foregoing in the context of Wick-rotated field theory. Again, I view this
material as provisional.

The first task is to define a symmetric monoidal category $t\text{Vect}$ which serves as the codomain
of a Wick-rotated field theory, as in Axiom System 23.42. In [KS] they take the codomain to be
the category of Fréchet spaces and nuclear maps. That seems reasonable, as it gives latitude in
quantum mechanics, say, to take $C^\infty$ functions, $L^2$ functions, Sobolev spaces of functions, etc.
However, there is not a good theory of tensor products of general Fréchet spaces. See [Wed] for
much more in this direction.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure75.png}
\caption{A direct system and an inverse system centered on $Y$}
\end{figure}

Assume given a functor (not symmetric monoidal) $F : \text{Bord}_n(F) \rightarrow t\text{Vect}$ with values in this
category. Let $Y$ be a closed $(n-1)$-manifold equipped with a germ $U$ of an $n$-manifold, a coorien-
tation of $Y \subset U$, and a background field $\phi \in F(U)$. Represent the germ by an actual $n$-manifold $U$.
Since $Y \subset U$ is cooriented and has codimension one, its normal bundle is trivializable. The tubular
neighborhood theorem gives an embedding of the tubular neighborhood in $U$ and so a foliation of
a neighborhood of $Y \subset U$ by “parallel” copies of $Y$; see Figure 75. Choose a double sequence

\begin{equation}
\ldots Y_{-1}, Y_{-1/2}, Y_{-1/3}, \ldots Y_{1/3}, Y_{1/2}, Y_{1} \ldots
\end{equation}
of these parallel copies using the coorientation to determine the ordering.\(^7^4\) There is a direct system

\[
\begin{array}{c}
Y_{-1} \rightarrow Y_{-1/2} \rightarrow Y_{-1/3} \rightarrow \cdots \\
\end{array}
\]

(23.50)

with a map of each element to \(Y\); the arrows are bordisms embedded in \(U\). Apply \(F\) to obtain a direct system of Fréchet spaces and nuclear maps

\[
\begin{array}{c}
F(Y)_{-1} \rightarrow F(Y)_{-1/2} \rightarrow F(Y)_{-1/3} \rightarrow \cdots \\
\end{array}
\]

(23.51)

The direct limit of the top row of (23.51) is a dual nuclear Fréchet space \(\tilde{F}(Y)\) equipped with a continuous linear map \(\tilde{F}(Y) \rightarrow F(Y)\). Similarly, there is an inverse system

\[
\begin{array}{c}
Y \rightarrow \cdots \rightarrow Y_{1/3} \rightarrow Y_{1/2} \rightarrow Y_1 \\
\end{array}
\]

(23.52)

and so an inverse system of Fréchet spaces and nuclear maps

\[
\begin{array}{c}
F(Y) \rightarrow \cdots \rightarrow F(Y)_{1/3} \rightarrow F(Y)_{1/2} \rightarrow F(Y)_1 \\
\end{array}
\]

(23.53)

The inverse limit of the bottom row of (23.53) is a nuclear Fréchet space \(\hat{F}(Y)\) equipped with a continuous linear map \(F(Y) \rightarrow \hat{F}(Y)\). Altogether, then, we obtain a “rigged” space

\[
\begin{array}{c}
\tilde{F}(Y) \rightarrow F(Y) \rightarrow \hat{F}(Y) \\
\end{array}
\]

(23.54)

in which \(\tilde{F}(Y)\) is dual nuclear Fréchet and \(\hat{F}(Y)\) is nuclear Fréchet. One should assume that the composite is an injection with dense image.

**Example 23.55.** For a quantum mechanical particle moving on a smooth Riemannian manifold \(M\), as in **Example 23.33,\(^7^5\)** if \(Y = \text{pt}\) then \(\tilde{F}(Y)\) is a space of real analytic functions (a notion determined by the Hodge Laplacian) and \(\hat{F}(Y)\) is its dual space. One has flexibility in choosing the Fréchet space \(F(Y)\) in between.
Consider now a bordism $X: Y_0 \to Y_1$, as in Figure 76. Not depicted is the background field $\phi \in \mathcal{F}(X)$, which restricts on the germ of the boundaries $Y_0, Y_1$ to the background fields that are part of those objects. (More precisely, there is a specified isomorphism.) The functor $F$ assigns to $X$ a continuous linear map $F(X): F(Y_0) \to F(Y_1)$. The construction of the direct and inverse limits, and the use of bordisms embedding in $X$ between the various parallel copies of the boundary components, gives a sequence of continuous linear maps

$$\tilde{F}(Y_0) \to F(Y_0) \to \tilde{F}(Y_0) \to \tilde{F}(Y_1) \to F(Y_1) \to \tilde{F}(Y_1)$$

In particular, we have a commutative diagram

$$\begin{array}{ccc}
\tilde{F}(Y_0) & \xrightarrow{\tilde{F}(X)} & \tilde{F}(Y_1) \\
\downarrow & & \downarrow \\
\tilde{F}(Y_0) & \xrightarrow{\tilde{F}(X)} & \tilde{F}(Y_1)
\end{array}$$

in which the columns are injections with dense image. The rows are nuclear by virtue of each factoring through a map from a dual nuclear Fréchet space to a nuclear Fréchet space as in (23.56).

In this way we are led to perhaps a better choice for $t\text{Vect}$. Namely, its objects are dense inclusions $\tilde{Y} \to \hat{Y}$ from a dual nuclear Fréchet space to a nuclear Fréchet space and its morphisms are diagrams (23.57). We would also need to assume that the colimit of (23.51) and the limit of (23.53) are the spaces given by the functor $F$ into this choice of $t\text{Vect}$. This is the approach taken in [Wed].

\[74\] The negatively indexed spaces are “upstream” and the positively indexed spaces are “downstream”, which explains the upwards and downwards pointing carrots in (23.54) below.

\[75\] As written in the notes, we assumed that $M$ is real analytic. However, we should only assume that $M$ is smooth and use the Laplace operator to define a notion of “analytic” functions.

\[76\] assumed nuclear by our choice of $t\text{Vect}$, but we do not use that assumption in this argument.
23.11 Duality

There is an involution $\ast\colon \text{Bord}_n(\mathcal{F}) \to \text{Bord}_n(\mathcal{F})^{\text{op}}$ as follows. If $Y$ is an object in $\text{Bord}_n(\mathcal{F})$, then $Y^\ast$ is the same closed $(n-1)$-manifold, the same germ of an $n$-manifold, the same background field, but the opposite coorientation of $Y$ inside the germ. Similarly, if $X\colon Y_0 \to Y_1$ is a bordism, then $X^\ast\colon Y_1^\ast \to Y_0^\ast$ reverses the decomposition of $\partial X$ into incoming and outgoing submanifolds.

Remark 23.58. This involution plays the role of duality in a symmetric monoidal category, as in Definition 23.78, but since $\text{Bord}_n(\mathcal{F})$ is only a semicategory (Remark 23.13(1)) one needs to give more care to this point.

Kontsevich-Segal prove in [KS, §3] that there are isomorphisms

\begin{equation}
\begin{aligned}
\tilde{F}(Y^\ast) &\cong \tilde{F}(Y)^\ast \\
\tilde{F}(Y^\ast) &\cong \tilde{F}(Y)^\ast
\end{aligned}
\end{equation}

Furthermore, for a bordism $X\colon Y_0 \to Y_1$ the image of $X^\ast\colon Y_1^\ast \to Y_0^\ast$ is dual to (23.57):

\begin{equation}
\begin{aligned}
\tilde{F}(Y_0^\ast) &\leftarrow \tilde{F}(X^\ast) \tilde{F}(Y_1^\ast) \\
\tilde{F}(Y_0^\ast) &\leftarrow \tilde{F}(X^\ast) \tilde{F}(Y_1^\ast)
\end{aligned}
\end{equation}

23.12 Reflection positivity

As stated many times, unitarity is not built into Axiom System 23.42. Here we tell the additional structure that encodes unitarity in Wick-rotated form, which is reflection positivity. We emphasize that reflection is a structure and positivity is a condition. We rely on the material in §23.13.2.

Definition 23.61. A reflection structure on $\text{Bord}_n(\mathcal{F})$ is an involution $-\colon \text{Bord}_n(\mathcal{F}) \to \text{Bord}_n(\mathcal{F})$.

We denote the involution as $Y \mapsto Y$ and $X \mapsto X$. For the Wick rotation of a relativistic symmetry type there is a canonical choice; see [FH1, §4].

Example 23.62. If $\mathcal{F} = \{\text{Riemannian metrics}\}$, then the involution is the identity. If $\mathcal{F} = \{\text{orientations, Riemannian metrics}\}$, then the involution is orientation reversal. There are canonical choices of involution for spin structures, pin structures, etc.

Definition 23.63. Suppose $F\colon \text{Bord}_n(\mathcal{F}) \to t\text{Vect}$ is a field theory and $\text{Bord}_n(\mathcal{F})$ is equipped with a reflection structure. Then a reflection structure on $F$ is equivariance data, where $t\text{Vect}$ is equipped with the involution of complex conjugation.

Recall from Definition 23.84 that a hermitian structure on an object $Y$ in $\text{Bord}_n(\mathcal{F})$ is an isomorphism $h\colon Y \to Y^\ast$ that satisfies a symmetry condition. An object $Y$ may not admit a hermitian structure. For example, if $\mathcal{F}$ includes a Riemannian metric, then a hermitian structure on an object $Y$ is an isometry of the metric in the germ of $Y$ that reverses the coorientation, and not every Riemannian metric admits such a reflection symmetry. If $Y$ has a hermitian structure, then so too does $F(Y)$, assuming that $F$ has a reflection structure. A hermitian structure on $F(Y)$ is a hermitian metric on the underlying vector space.
Definition 23.64. A reflection structure on a theory $F$ is *positive* if the hermitian inner product on $F(Y)$ is positive definite for all hermitian objects $Y$ in $\text{Bord}_n(\mathcal{F})$.

If this is the case, then for any hermitian object $Y$ we can complete $F(Y)$ in this inner product to form a Hilbert space $F^{\text{Hilb}}(Y)$. It sits in a rigging

$$F(Y) \hookrightarrow F^{\text{Hilb}}(Y) \hookrightarrow \hat{F}(Y)$$

in which each map is a dense injection.

23.13 Mathematical background

23.13.1 Topological vector spaces. The following are some basic definitions. There are many texts that cover this material, among them [RS, Ru, Tr]. We take the ground field to be $\mathbb{C}$.

Definition 23.66. A topological vector space is a vector space $V$ equipped with a topology such that every point of $V$ is closed and the operations of vector addition $V \times V \rightarrow V$ and scalar multiplication $\mathbb{C} \times V \rightarrow V$ are continuous.

A finite dimensional vector space has a unique compatible topology. One need not include the closedness of points in the definition. It is a theorem that with this definition every topological vector space is Hausdorff.

Definition 23.67. Let $V$ be a topological vector space.

1. $V$ is *locally convex* if there exists a local basis for the topology at $0 \in V$ consisting of convex open sets.
2. $V$ is *metrizable* if it is the underlying topological space of a metric space.
3. $V$ is a Fréchet space if it is locally convex, metrizable, and complete.
4. $V$ is *normable* if its topology is given by a norm.

We review the definition of a norm presently.

The topology of a locally convex topological vector space can be specified by seminorms.

Definition 23.68. Let $V$ be a vector space. A seminorm $\rho: V \rightarrow [0, \infty)$ is a function that satisfies

$$\rho(\xi_1 + \xi_2) \leq \rho(\xi_1) + \rho(\xi_2), \quad \xi_1, \xi_2 \in V;$$

$$\rho(\lambda \xi) = |\lambda| \rho(\xi), \quad \xi \in V, \lambda \in \mathbb{C}. $$

Furthermore, $\rho$ is a norm if

$$\rho(\xi) = 0 \text{ iff } \xi = 0, \quad \xi \in V.$$ 

A family $\{\rho_\alpha\}_{\alpha \in A}$ of seminorms separates points if

$$\rho_\alpha(\xi) = 0 \text{ for all } \alpha \in A \text{ implies } \xi = 0, \quad \xi \in V.$$
A separating family of seminorms \( \{ \rho_\alpha \}_{\alpha \in A} \) defines a locally convex topology on \( V \), the weakest topology for which each \( \rho_\alpha \) is continuous. This topology satisfies the conditions of Definition 23.66; in particular, it is Hausdorff.

**Theorem 23.72.** Let \( V \) be a locally convex topological vector space. Then \( V \) is metrizable if and only if its topology is generated by a countable family of seminorms.

The natural maps \( T: V \to W \) between topological vector spaces are those which are linear and continuous. In particular, we have the following.

**Definition 23.73.** Let \( V \) be a topological vector space. The dual space \( V^* \) is the space of continuous linear functions \( V \to \mathbb{C} \).

The dual space has a natural strong topology. Quite generally, if \( V, W \) are topological vector spaces and \( \text{Hom}(V, W) \) is the vector space of continuous linear maps, then it has a topology generated by a basis of neighborhoods \( U(B, U) \) of zero, where \( B \subset V \) is bounded, \( U \subset W \) is open, and \( U(B, U) \) consists of continuous linear maps \( T: V \to W \) such that \( T(B) \subset U \).

### 23.13.2 Duality and hermitian objects

For the reader’s convenience we reprint some material from [FH1, Appendix B].

**Definition 23.74.** Let \( \mathcal{C} \) be a category.

1. An **involution** of \( \mathcal{C} \) is a pair \( (\tau, \eta) \) of a functor \( \tau: \mathcal{C} \to \mathcal{C} \) and a natural isomorphism \( \eta: \text{id}_\mathcal{C} \to \tau^2 \) such that for any \( x \in \mathcal{C} \) we have \( \tau \eta_x = \eta_{\tau x} \) as morphisms \( \tau x \to \tau^3 x \).

2. A **fixed point** of \( \tau \) is a pair \( (x, \theta) \) of an object \( x \in \mathcal{C} \) and an isomorphism \( x \theta \to \tau x \) such that \( \tau \theta \circ \theta = \eta_x \) as morphisms \( x \to \tau^2 x \).

If \( \mathcal{C} \) is a symmetric monoidal category, then the involution \( \tau \) is required to be a symmetric monoidal functor: for \( x, y \in \mathcal{C} \) there is given an isomorphism \( \tau x \otimes \tau y \cong \tau(x \otimes y) \) and these isomorphisms are compatible with the symmetry and with \( \eta \).

**Example 23.75.** Let \( \mathcal{C} = \text{Vect}_\mathbb{C} \) be the category of complex vector spaces and linear maps. Define \( \tau: \mathcal{C} \to \mathcal{C} \) to be the functor that takes complex vector spaces and linear maps to their complex conjugates. (The complex conjugate vector space is the same underlying real vector space with the sign of multiplication by \( \sqrt{-1} \in \mathbb{C} \) reversed; the complex conjugate of a linear map is the same map of sets.) Then there is a canonical identification of \( \tau^2 \) with \( \text{id}_\mathcal{C} \). A fixed point is a complex vector space with a real structure.

**Definition 23.76.** Let \( \mathcal{B}, \mathcal{C} \) be categories with involutions and \( F: \mathcal{B} \to \mathcal{C} \) a functor. Then **equivariance data** for \( F \) is an isomorphism \( \phi: F \tau_{\mathcal{B}} \cong \tau_{\mathcal{C}} F \) of functors \( \mathcal{B} \to \mathcal{C} \) such that for every object \( x \in \mathcal{B} \) the diagram

\[
\begin{array}{ccc}
Fx & \xrightarrow{F \eta_{\mathcal{B}}} & F \tau_{\mathcal{B}}^2 x \\
\downarrow \eta_{\mathcal{C}} & & \downarrow \phi^2 \\
\tau_{\mathcal{C}}^2 Fx & \end{array}
\]

(23.77)

commutes.
There are additional compatibilities for a symmetric monoidal functor between symmetric monoidal categories; we do not spell them out. We often loosely say that “F is an equivariant functor”, but it is important to remember that equivariance is data+condition, not simply a condition.

Next, we review duality in a symmetric monoidal category. Let \( \mathcal{C} \) be a symmetric monoidal category and \( x \in \mathcal{C} \). Denote the tensor unit by \( 1 \in \mathcal{C} \).

**Definition 23.78.** Let \( x \) be an object in a symmetric monoidal category \( \mathcal{C} \). Duality data for \( x \) is a triple \( (x^\vee, c, e) \) consisting of an object \( x^\vee \in \mathcal{C} \) together with morphisms \( c: 1 \to x \otimes x^\vee \) and \( e: x^\vee \otimes x \to 1 \) such that the compositions

\[
\begin{align*}
    x \xrightarrow{c \otimes \text{id}} & \to x \otimes x^\vee \otimes x \xrightarrow{\text{id} \otimes e} x \\
    x^\vee \xrightarrow{\text{id} \otimes c} & \to x^\vee \otimes x \otimes x^\vee \xrightarrow{e \otimes \text{id}} x^\vee
\end{align*}
\]

are identity maps. If \( x_0 \xrightarrow{f} x_1 \) is a morphism, then the dual morphism is the composition

\[
f^\vee: x_1^\vee \xrightarrow{\text{id} \otimes c \otimes \text{id}} x_1^\vee \otimes x_0 \otimes x_0^\vee \xrightarrow{\text{id} \otimes f \otimes \text{id}} x_1^\vee \otimes x_1 \otimes x_0^\vee \xrightarrow{e \otimes \text{id} \otimes \text{id}} x_0^\vee
\]

The morphism \( c \) is called coevaluation and \( e \) is called evaluation. We say that \( x^\vee \) is “the” dual to \( x \) since any two triples of duality data are uniquely isomorphic. Assuming all objects have duals, we can make choices of duality data for all objects at once and so obtain a duality involution \( \delta \) on \( \mathcal{C} \), but \( \delta \) does not satisfy Definition 23.74 since the direction of morphisms is reversed (23.80); in other words, \( \delta \) is a functor to the opposite category.

**Definition 23.81.** Let \( \mathcal{C} \) be a category.

1. A twisted involution of \( \mathcal{C} \) is a pair \((\delta, \eta)\) of a functor \( \delta: \mathcal{C} \to \mathcal{C}^{\text{op}} \) and a natural isomorphism \( \eta: \text{id}_{\mathcal{C}} \to \delta^{\text{op}} \circ \delta \) such that for any \( x \in \mathcal{C} \) we have \( \delta \eta_x \circ \eta \delta x = \text{id}_{\delta x} \).
2. A fixed point of \( \delta \) is a pair \((x, \theta)\) of an object \( x \in \mathcal{C} \) and an isomorphism \( x \xrightarrow{\theta} \delta x \) such that \( \delta \theta \circ \eta_x = \theta \) as morphisms \( x \to \delta x \).

**Example 23.82.** For \( \mathcal{C} = \text{fVect}_\mathbb{C} \) the category of finite dimensional complex vector spaces, the duality involution \( \delta: \mathcal{C} \to \mathcal{C}^{\text{op}} \) maps a vector space \( V \) to its dual \( V^* \) and a linear map \( f: V \to W \) to \( f^*: W^* \to V^* \). A fixed point of \( \delta \) is a vector space \( V \) equipped with a nondegenerate symmetric bilinear form. A fixed point for the composite of duality and complex conjugation (Example 23.75) is a complex vector space \( V \) with a nondegenerate hermitian form.

**Remark 23.83.** There is a higher categorical context for Definition 23.81. Let \( \text{Cat} \) denote the 2-category of categories. There is an involution \( \alpha: \text{Cat} \to \text{Cat}^{\text{op}} \) that sends a category \( \mathcal{C} \) to its opposite \( \mathcal{C}^{\text{op}} \). (There is an extra categorical layer over Definition 23.74: there is a triple \((\alpha, \eta_1, \eta_2)\) of data and a single condition.) A twisted involution in the sense of Definition 23.81 is fixed point data for \( \alpha \).

**Definition 23.84.** Let \((\tau, \eta)\) be an involution on a symmetric monoidal category \( \mathcal{C} \). A hermitian structure on an object \( x \in \mathcal{C} \) is an isomorphism \( h: \tau x \to x^\vee \) such that the composition

\[
\tau x \cong \tau ((x^\vee)^\vee) \xrightarrow{\tau (h^\vee)} \tau ((\tau x)^\vee) \cong \tau^2 (x^\vee) \xrightarrow{\eta^{-1}} x^\vee
\]

is equal to \( h \).
Observe that if $F: \mathcal{B} \to \mathcal{C}$ is an equivariant symmetric monoidal functor between symmetric monoidal categories with involution, as in Definition 23.76, then the image of a hermitian structure on an object $b \in \mathcal{B}$ is a hermitian structure on $Fb$.

References


Problem Set # 1

Math262a: Quantum theory from a geometric viewpoint I

These problems are supplementary to the lectures and the lecture notes. I hope you tackle many of them, even if you don’t complete them. Many are open-ended. They provide a gateway to engage with the material in the class. I suggest you form small groups to discuss the class material, including these problems. Those getting a grade in the course should regularly hand in some solutions. Please come to office hours and the discussion section to discuss these problems, and also take advantage of Discord as a platform for discussion.

Problems

1. In this problem you will derive the symplectic form on the space of classical trajectories of a particle, for simplicity a particle of mass $m$ moving on the standard Euclidean line $E^1$ with coordinate $x$. Let $V: E^1 \to \mathbb{R}$ be a smooth function, which is the potential energy of the particle. Let $\mathcal{F} = \text{Map}(\mathbb{R}, E^1)$ be the space of smooth possible particle trajectories.

   (a) What is the space $N \subset \mathcal{F}$ of classical trajectories, i.e., trajectories that satisfy Newton’s law? Consider the special cases $V = 0$ and $V(x) = kx^2/2$ for some $k > 0$.

   (b) Review the derivation of Newton’s law from the calculus of variations (Euler-Lagrange equations) applied to the action

   $$S(\gamma) = \int \left[ \frac{1}{2} m \dot{x}(t)^2 - V(x(t)) \right] dt$$

   where $x: \mathbb{R} \to E^1$ is a smooth motion.

   (c) Rephrase this computation using calculus on $\mathcal{F} \times \mathbb{R}$. Observe that the Euler-Lagrange equations only depend integrand in the action—the lagrangian—not its integral. Write $\delta$ for the de Rham differential on $\mathcal{F}$ and $d$ for the de Rham differential $d = dt \cdot \partial / \partial t$ on $\mathbb{R}$. Then the total de Rham differential on $\mathcal{F} \times \mathbb{R}$ is $\delta + d$. In particular, $(\delta + d)^2 = 0$. A differential form on $\mathcal{F} \times \mathbb{R}$ has type $(p, q)$, $p \in \mathbb{Z}_{\geq 0}$, $q \in \{0, 1\}$, if it is nonzero only when evaluated on $p$ vectors tangent to $\mathcal{F}$ and $q$ vectors tangent to $\mathbb{R}$. Calculus on the Cartesian product $\mathcal{F} \times \mathbb{R}$ is depicted in the diagram

   $\begin{array}{c|cc}\hline
   & 0 & 1 & \ldots \\ \hline
   1 & \uparrow & \delta & \rightarrow \\
   0 & \downarrow & \delta & \rightarrow \\
   \mathbb{R} & \delta & \rightarrow & \mathcal{F}
   \end{array}$

   (If one writes the lagrangian and action as a density rather than a differential form—use $|dt|$ in place of $dt$—which we should do since we should not use an orientation on $\mathbb{R}$ to define this
mechanical system, then the formatting of the square makes more sense: starting from the top, the vertical degrees are then written as '|0⟩' and '|1⟩'. Let

\[ e: \mathcal{F} \times \mathbb{R} \rightarrow \mathbb{R}^1 \]
\[ x, t \mapsto x(t) \]

be the evaluation function. Show that the integrand above can be written as

\[ L = \left[ \frac{1}{2} m \dot{e}^2 - V \circ e \right] dt \]

and that it is an element of \( \Omega^{0,1}(\mathcal{F} \times \mathbb{R}) \). (Be sure to define \( \dot{e} \), which is a directional derivative of \( e \) along a certain vector field on \( \mathcal{F} \times \mathbb{R} \).)

(d) Compute \( \delta L \in \Omega^{1,1}(\mathcal{F} \times \mathbb{R}) \). Carry out the integration by parts in the following form: find \( \theta \in \Omega^{1,0}(\mathcal{F} \times \mathbb{R}) \) so that \( \delta L + d\theta \) is a function times \( \delta e \wedge dt \), i.e., it has no \( \delta e \wedge dt \) term.

(e) Show that \( \delta L + d\theta \) vanishes on \( N \), and in fact that vanishing defines \( N \).

(f) Show that the restriction of \( \delta \theta \in \Omega^{2,0}(\mathcal{F} \times \mathbb{R}) \) to \( N \times \{t_0\} \) is independent of \( t_0 \). Show that this restriction is a symplectic form. Identify it with the symplectic form you may have run into before in this context.

2. In the previous problem specialize to \( V(x) = kx^2/2 \) for fixed \( k > 0 \).

(a) Fix \( t \in \mathbb{R} \) and consider the observables (functions on \( N \)) given by \( \mathcal{O}_1(x) = x(0) \) and \( \mathcal{O}_2(x) = x(t) \). Choose a pure state \( \sigma \), which is a point of \( N \). Compute the expectation values \( \langle \mathcal{O}_1 \rangle_\sigma \) and \( \langle \mathcal{O}_2 \rangle_\sigma \). Compute the correlation function \( \langle \mathcal{O}_2 \mathcal{O}_1 \rangle_\sigma \). Investigate the dependence on \( t \) and on \( \sigma \).

(b) Now consider the mixed Gibbs state, which is a probability measure on the space \( N \) of classical solutions. Its probability density is a constant times \( e^{-\beta E} \), where \( \beta > 0 \) is a constant and \( E: N \rightarrow \mathbb{R} \) is the energy function \( E = \frac{1}{2} \dot{x}^2 + \frac{k}{2} x^2/2 \). Compute the Gibbs state (a function times the standard measure on the \( (x, \dot{x}) \)-plane) and evaluate the expectation values in part (a) in the Gibbs state.

3. Verify the axioms in §1.2 of the lecture notes for the case of a classical mechanical system.

4. In the context of a general mechanical system, let \( \sigma_1, \sigma_2 \) be states and let \( A, A' \) be observables. Fix \( t \in [0, 1] \). Prove that for the convex combination \( \sigma = t\sigma_1 + (1 - t)\sigma_2 \) the standard deviations \( \Delta_\rho B \) of the probability distributions \( \rho_B \) obtained from the various states \( \rho \) and observables \( B \) satisfy

\[ \Delta_\sigma A \Delta_\sigma A' \geq t(\Delta_{\sigma_1} A) (\Delta_{\sigma_1} A') + (1 - t)(\Delta_{\sigma_2} A) (\Delta_{\sigma_2} A') \]

In particular, conclude that \( \Delta_\sigma A \geq \min(\Delta_{\sigma_1} A, \Delta_{\sigma_2} A) \).
(Recall that the standard deviation of a probability measure on \( \mathbb{R} \) is the square root of the integral of \((\lambda - \mu)^2\) times the probability measure on the real line \( \mathbb{R}_\lambda \) with coordinate \( \lambda \), where \( \mu \) is the mean or expected value, i.e., \( \mu \) is the integral of \( \lambda \) times the probability measure.)
Problem Set # 1
Math 262a: Quantum theory from a geometric viewpoint I
Due: September 20

Problems

1. Let $N$ be a smooth manifold of dimension $n = 2m$, and suppose $\omega \in \Omega^2_N$.

(a) Prove that $\omega$ is nondegenerate (at each point of $N$) iff $\omega^m \in \Omega^{2m}_N$ is everywhere nonzero.

(b) Now assume that $\omega$ is a symplectic form. Prove that the subspace $\mathfrak{X}^\omega_N \subset \mathfrak{X}_N$ of symplectic vector fields is closed under Lie bracket of vector fields.

(c) Define a Lie algebra structure on $\Omega^0_N$ by transport from the Lie bracket on $\mathfrak{X}^\omega_N$ via the symplectic gradient map $f \mapsto \xi_f$ on functions. Prove the formula

$$\{f, g\} = \xi_f \cdot g = \omega(\xi_f, \xi_g), \quad f, g \in \Omega^0_N.$$ 

(d) A Hamiltonian vector field is a symplectic gradient of a function. Is the subspace of Hamiltonian vector fields invariant under Lie bracket?

2. Let $V$ be a real symplectic vector space, and suppose $A$ is an affine space over $V$. Feel free to work with the model spaces.

(a) Define the space of affine polynomial functions on $A$ and prove that it is closed under the Poisson bracket on smooth functions.

(b) Prove that the space of affine linear functions is also invariant. Do you recognize the resulting Lie algebra?

(c) Prove that the space of affine quadratic functions is also invariant. Do you recognize the resulting Lie algebra?

(d) Does the pattern continue?

3. (a) Prove that an almost symplectic manifold admits an almost complex structure. (An almost complex structure on a smooth manifold $N$ is a section $I$ of $\text{End}(TN) \to TN$ such that $I^2 = -\text{id}$ holds at every point.)

(b) Prove that $S^4$ does not admit an almost symplectic structure. (Hint: Use characteristic classes)
4. Let $(M, g)$ be a Riemannian manifold, and suppose $V: M \to \mathbb{R}$ is a smooth (potential energy) function. Fix $m > 0$, which we take to be the mass of a particle moving on $M$. Let $\mathcal{F}$ be the space of smooth maps $x: \mathbb{R} \to M$. Treat $\mathcal{F}$ as an infinite dimensional manifold, at least formally, assuming differential calculus carries over. Consider the Lagrangian

$$L = \left\{ \frac{m}{2} \langle \dot{x}, \dot{x} \rangle - V(x) \right\} |dt|,$$

where $x: \mathcal{F} \times \mathbb{R} \to M$ is the evaluation map. (You may replace ‘$|dt|$’ with $dt$.) Compute the Euler-Lagrange equations, the symplectic form, and the Hamiltonian. (You will need some familiarity with Riemannian geometry.)

5. Define the notion of a symmetry of a general mechanical system. Does a symmetry have to preserve the direction (arrow) of time, or can a symmetry reverse time flow?
Problem Set # 3
Math 262a: Quantum theory from a geometric viewpoint I
Due: September 29

Problems

1. Consider a classical particle moving on the Euclidean line $\mathbb{E}^1$ with standard metric and coordinate $x$. Fix $a > 0$ and set $V(x) = \frac{1}{2}(x^2 - a^2)^2$. What are the classical trajectories of least energy? Answer the same question for a classical particle moving in the inverted potential $-V$. For $T > 0$ is there a classical trajectory $\gamma: [-T/2, T/2]$ with initial position $\gamma(-T/2) = -a$ and final position $\gamma(T/2) = a$, and if so give a formula or characterize it? What happens if $T \to \infty$?

2. (a) Prove the formula (5.14) in the notes.
(b) Generalize the measurement depicted in Figure 9 of the notes to arbitrary states.

3. This concerns the particle on a ring $M = \mathbb{E}^1/2\pi \mathbb{Z}$ of length $2\pi L$, both classical and quantum.
   (a) Work out the details of Example 3.57 in the notes.
   (b) The quantum theory has Hilbert space $\mathcal{H} = L^2(M; \mathbb{C})$ and the Hamiltonian is the unbounded self-adjoint operator
      
      $$H = \frac{1}{2} \left( i \frac{\partial}{\partial x} + \frac{1}{2\pi} \theta \right)^2.$$

      Diagonalize $H$, and in particular compute its spectrum. Is there periodicity in $\theta$? How does that compare to the classical system?

4. Let $\mathcal{H}$ be a separable complex Hilbert space. A continuous linear operator $T: \mathcal{H} \to \mathcal{H}$ is compact if the closure of the image $T(D) \subset \mathcal{H}$ of the closed unit disk $D \subset \mathcal{H}$ is compact. Prove the following.

   **Theorem.** Let $T: \mathcal{H} \to \mathcal{H}$ be a positive self-adjoint compact operator. Prove that there exists a complete orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and positive numbers $\mu_1 \geq \mu_2 \geq \cdots$ such that
   
   (i) $T \psi_n = \mu_n \psi_n$;
   (ii) For any $c > 0$ there is a finite number of $\mu_n > c$;
   (iii) $\lim_{n \to \infty} \mu_n = 0$.

   The completeness means that the closure of the span of the set $\{e_n\}$ equals $\mathcal{H}$. One approach is to consider the quadratic form $\psi \mapsto \langle \psi, T \psi \rangle$ on the unit ball $D$. Since $T$ is compact, this form achieves its maximum, say at a unit norm vector $e_1$. Show that $T$ preserves the orthogonal complement to $\mathbb{C} \cdot e_1$, and iterate the argument.
5. Consider the toric code, as defined in §5.5 of the notes, formulated on a 2-dimensional manifold $Y$ equipped with a triangulation $\Gamma \subset Y$ (or, better, a CW structure).

(a) Prove that any two elements in the set of operators $\{H_v, H_f\}_{v,f}$ commute. These operators are defined in (5.24).

(b) Prove that the kernel is as stated in the notes: see (5.27).

(c) Describe the eigenspace for the minimum nonzero eigenvalue.

(d) For any edge $e$ in $\Gamma$ consider the permutation of $\mathcal{D}(\Gamma, \Gamma^0)$ that modifies a double cover at a single interior point of $\Gamma$: cut open the edge and reglue the double cover after applying the deck transformation (swap sheets) at the point. This induces an observable $\mathcal{O}_e$. Fix a vacuum state $\sigma$, a positive time $T > 0$, times $t_1, t_2 \in (0, T)$, and edges $e_1, e_2$. You may suppose $\sigma$ is induced from the delta function at a fixed double cover of $Y$. What can you say about the correlation function of $\mathcal{O}_{e_1}(t_1)$ and $\mathcal{O}_{e_2}(t_2)$ over the interval $[0, T]$ with initial and final state $\sigma$? Investigate how your answer depends on $T$ and on whether the edges share a vertex and/or a face. (I have not tried this, so it may very well be the start of a project on the toric code model. Incidentally, it is fine if you want to collaborate with one other person on a class project.)
Problem Set # 4
Math 262a: Quantum theory from a geometric viewpoint I
Due: October 6

Problems

1. The construction of the principal $\mathbb{R}$-bundle with connection $T \to N$ in §3.4 of the notes is stated poorly; here is another attempt.

Quite generally, if $\pi: M \to N$ is a map and if $\mathcal{O}$ is a contravariant local mathematical object on $M$—a cocycle for a cohomology class, a differential form, fiber bundle over $M$, a connection on a principal bundle over $M$, etc.—then there are two general ways we can think of obtaining a mathematical object $\mathcal{O}$ on $N$. First, there might be a pushforward or umker map or integration that produces $\pi_*\mathcal{O}$. In general, the geometric nature of $\mathcal{O}$ is different from that of $\mathcal{O}$. For example, there might be a degree shift (down by the relative dimension of the map $\pi$). The second general construction is descent: an object $\mathcal{O}$ on $N$ and an isomorphism of $\mathcal{O}$ with the pullback $\pi^*\mathcal{O}$. The descended object $\mathcal{O}$ has the same geometric nature as $\mathcal{O}$. It is this second scenario that applies here. We put ourselves in the situation of §3.4.

(a) Consider the product principal $\mathbb{R}$-bundle $p: P = (N \times \mathbb{R}_t) \times \mathbb{R}_s \to N \times \mathbb{R}_t$

Prove that the real-valued 1-form

$$\Theta = ds + p^*\gamma \in \Omega^1_P$$

is a connection. Compute its curvature $\Omega \in \Omega^2_{N \times \mathbb{R}}$.

(b) Prove that the curvature descends under the projection

$$\pi: N \times \mathbb{R} \to N$$

In other words, construct a 2-form $\omega \in \Omega^2_{N \times \mathbb{R}}$ such that $\Omega = \pi^*\omega$.

(c) Now descend $p: P \to N \times \mathbb{R}$ and its connection $\Theta$ to a principal $\mathbb{R}$-bundle $T \to N$ with connection. Be sure to produce the requisite isomorphism that proves you have a descent. (Hint: the fiber $T_n$ over $n \in N$ is the $\mathbb{R}$-torsor of parallel sections of the restriction of $p: P \to N \times \mathbb{R}$ to $\pi^{-1}(n) = \{n\} \times \mathbb{R}$.) How does this relate to the construction in the notes?

(d) Consider the translation action of $\mathbb{R}_t$ on $N \times \mathbb{R}_t$. Define a lift to $P$ such that that (i) the connection 1-form $\Theta$ is invariant, and (ii) if $\xi = \partial/\partial t$ denotes the vector field that generates this lifted translation action, then the contraction $\iota_\xi \Theta$ vanishes. Conclude that the 1-form $\Theta$ descends to the quotient of $P$ by the translation action.

(e) More generally, suppose $\pi: Q \to N$ is a principal $G$-bundle for some Lie group $G$. What are the descent conditions on a differential form $\Omega \in \Omega^*_Q$? In other words, characterize the image of the injective linear map $\pi^*: \Omega^*_N \to \Omega^*_Q$. 

\[1\]
2. Consider the simple harmonic oscillator: a mass one particle on \( \mathbb{E}^1 \) with potential \( V(x) = x^2/2 \). The Hilbert space is \( \mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C}) \) and the Hamiltonian is

\[
H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 \right)
\]

(a) Compute eigenfunctions for the four smallest eigenvalues of \( H \).

(b) State and prove the spectral decomposition for \( H \), namely, \( \mathcal{H} \) is the Hilbert space completion of the direct sum of the eigenspaces.

(c) Recall that the classical phase space is \( N = \mathbb{E}^1 \times \mathbb{R}^1 \) with symplectic form \( \omega = dy \wedge dx \). Verify that the 3-dimensional real vector space of affine linear functions on \( N \) is closed under Poisson bracket. Verify the same for the 6-dimensional subspace of affine quadratic functions. What about the affine cubic functions?

(d) The affine functions under Poisson bracket form the Heisenberg Lie algebra. Construct a representation by unbounded operators on \( \mathcal{H} \) (which are skew-adjoint.) Can you extend the representation to the affine quadratic functions?

(e) Construct an antiunitary operator on \( \mathcal{H} \) which implements the time-reversal symmetry that preserves the operator \( x \) and sends \( \partial/\partial x \mapsto -\partial/\partial x \). Show that the induced operator on \( \mathbb{P}\mathcal{H} \) has order 2. What about the operator on \( \mathcal{H} \)?

3. Let \( V: \mathbb{E}^1 \to \mathbb{R} \) be a smooth function and consider a quantum mass \( m > 0 \) particle moving on \( \mathbb{E}^1 \) with potential energy \( V \). An eigenfunction with eigenvalue \( E \in \mathbb{R} \) is an \( L^2 \) function \( \psi: \mathbb{E}^1 \to \mathbb{C} \) that satisfies the Schrödinger equation

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + (V(x) - E)\psi(x) = 0.
\]

(a) Prove that if \( \psi_1, \psi_2 \) are nonzero eigenfunctions with the same eigenvalue \( E \), then \( \psi_2 \) is proportional to \( \psi_1 \). In other words, eigenspaces of the Hamiltonian are 1-dimensional. Observe that, depending on \( V \), there may be no eigenspaces, the eigenspaces may span the space of \( L^2 \) functions, or anything in between. (Hint: Consider the Wronskian \( \psi_1 \psi_2' - \psi_1' \psi_2 \). You may need to use some facts about elliptic second-order ordinary differential equations. For example, solutions are smooth functions and their zero set contains no accumulation points.)

(b) Prove that each eigenspace contains a real-valued function.

(c) Prove that if the Hamiltonian is gapped, then the unique up to a constant eigenfunction for the minimal eigenvalue is a nonvanishing function.
4. Let $W$ be a finite dimensional real vector space. Recall that in lecture we discussed the bosonic harmonic oscillator as a representation of an algebraic operator algebra generated by $W \oplus W^*$ on the complex vector space $\text{Sym}^n W_C \otimes W_C$. The goal now is to introduce inner products and complete to a Hilbert space. For this, suppose $W$ is endowed with a real inner product.

(a) Induce a hermitian inner product on $W_C^n$.
(b) Induce a hermitian inner product on $\text{Sym}^n W_C^n$.
(c) What is the adjoint of the operator $A(w), w \in W$, defined in lecture?
(d) Is the Hamiltonian (formally) self-adjoint?

5. The abelian group $\mathbb{T} \subset \mathbb{C}$ of unit norm complex numbers has an automorphism $\lambda \mapsto \lambda^{-1}$. Let $\mu_2 \subset \mathbb{T}$ be the group $\{\pm 1\}$ of square roots of unity.

(a) Classify group extensions (up to isomorphism)

$$1 \rightarrow \mathbb{T} \rightarrow G \rightarrow \mu_2 \rightarrow 1$$

in which $\mu_2$ acts on $\mathbb{T}$ by the nontrivial automorphism.

(b) Classify group extensions

$$1 \rightarrow \mathbb{T} \rightarrow G \rightarrow O_2 \rightarrow 1$$

in which $O_2$ acts nontrivially on $\mathbb{T}$ via the determinant $O_2 \rightarrow \mu_2$. 
Problem Set # 5
Math 262a: Quantum theory from a geometric viewpoint I
Due: October 13

Problems

1. Recall the 2-component Lie group PQ constructed in lecture (and in the lecture notes). I write ‘PQ’, but you are welcome to work this problem with PQ_{n} and a finite dimensional quantum mechanical system. Given a right PQ-torsor and a Hamiltonian \( H \) (where does it live?), construct the data of a quantum mechanical system, that is, the data in Axiom System 1.1 of the notes. There are many parts to this, so first make a list of the data you must produce and then knock them out one-by-one. Can you generalize to a parametrized family of data? To a piece of data with symmetry? Can you combine these?

2. For \( n \in \mathbb{Z}^{>0} \) the standard Clifford algebra \( \text{Cl}_{+n} \) is the unital algebra over \( \mathbb{R} \) generated by \( e_{1}, \ldots, e_{n} \) subject to the relation
\[
e_{i}e_{j} + e_{j}e_{i} = 2\delta_{ij}, \quad 1 \leq i, j \leq n.
\]
\( \text{Cl}_{-n} \) is similar, but the relation is \( e_{i}e_{j} + e_{j}e_{i} = -2\delta_{ij} \). The complex version, for which the sign is irrelevant but we take the plus sign for definiteness, is denoted \( \text{Cl}_{+n}^{\mathbb{C}} \). The abstract version for a vector space \( U \) (complex or real) with a symmetric bilinear form \( B \) is the free unital algebra \( \text{Cl}(U, B) \) generated by \( U \) subject to
\[
u_{1}v_{2} + v_{2}v_{1} = B(v_{1}, v_{2}), \quad v_{1}, v_{2} \in U.
\]
These Clifford algebras are superalgebras.

(a) Construct an isomorphism of superalgebras \( \text{Cl}_{2}^{\mathbb{C}} \overset{\cong}{\longrightarrow} \text{End}(\mathbb{C}^{1|1}) \).
(b) Define the tensor product \( A_{1} \otimes A_{2} \) of superalgebras \( A_{1}, A_{2} \). Mind your Koszul sign rule!
(c) Let \( (U_{i}, B_{i}) \) be real vector spaces and symmetric bilinear forms. Construct an isomorphism
\[\text{Cl}(U_{1}, B_{1}) \otimes \text{Cl}(U_{2}, B_{2}) \overset{\cong}{\longrightarrow} \text{Cl}(U_{1} \oplus U_{2}, B_{1} \oplus B_{2})\].
(d) Construct an isomorphism of superalgebras \( \text{Cl}_{+8} \overset{\cong}{\longrightarrow} \text{End}(\mathbb{R}^{8|8}) \). (You may want to apply (c).)
(e) Observe that \( \mathbb{R}^{n} \subset \text{Cl}_{\pm n} \). For \( \xi \in \mathbb{R}^{n} \) a vector of unit norm, check that the map \( \xi \mapsto -\xi\zeta\xi^{-1} \) reflects \( \zeta \in \mathbb{R}^{n} \) in the hyperplane perpendicular to \( \xi \).
(f) For \( \theta \in \mathbb{R} \) define \( g_{\theta} = \cos(\theta/2) + \sin(\theta/2)e_{1}e_{2} \). What transformation does \( g_{\theta} \) induce on \( \mathbb{R}^{n} \)?
(g) Show that “quadratic” elements \( \frac{1}{2}A^{ij}e_{i}e_{j} \) in the Clifford algebra are closed under commutation. Can you identify this Lie algebra? (Here the \( A^{ij} \) are scalars and the expression is summed over the repeated indices \( i, j \).)
3. For computations in superalgebra, say over \( \mathbb{R} \), it is wise to extend scalars from \( \mathbb{R} \) to the commutative superalgebra \( \mathbb{R}[\eta_1, \ldots, \eta_N] \) for arbitrary \( N \). Multiply odd elements by a variable \( \eta_i \) to obtain an even element, and then use the even elements to compute, thereby eliminating signs until the end of the computation. For example, if \( Q_1, Q_2 \) are odd elements of a super Lie algebra, then compute the sign rule for the Lie bracket as follows:

\[
[\eta_1 Q_1, \eta_2 Q_2] = -[\eta_2 Q_2, \eta_1 Q_1] = +\eta_2 \eta_1 [Q_2, Q_1] = -\eta_1 \eta_2 [Q_2, Q_1].
\]

On the other hand,

\[
[\eta_1 Q_1, \eta_2 Q_2] = -\eta_1 \eta_2 [Q_1, Q_2],
\]

and comparing we deduce the correct rule for the Lie bracket:

\[
[Q_1, Q_2] = [Q_2, Q_1].
\]

(a) Revisit part (b) of the previous problem and apply these “even rules”.

(b) Deduce the proper signs for commutativity in the symmetric algebra \( \text{Sym}^*(V) \) of a super vector space \( V \).

(c) Deduce the Jacobi identity for a super Lie algebra \( \mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \). Verify that (i) \( \mathfrak{g}^0 \) is an (ungraded) Lie algebra, (ii) \( \mathfrak{g}^1 \) is a \( \mathfrak{g}^0 \)-module, (iii) Lie bracket induces a symmetric pairing \( \mathfrak{g}^1 \times \mathfrak{g}^1 \to \mathfrak{g}^0 \) of \( \mathfrak{g}^0 \)-modules, and (iv) for \( x \in \mathfrak{g}^1 \) we have \( [x, [x, x]] = 0 \). Conversely, show that the data (i)–(iv) determine a super Lie algebra.

(d) Define the opposite superalgebra to a superalgebra. What is the opposite superalgebra to \( \text{Cl}(U, B) \)?

4. This problem identifies the double cover of low dimensional orthogonal groups. Many parts do not require background in Lie group theory.

Let \( V \) be a 4-dimensional complex vector space and fix a nonzero volume form \( \mu \in \bigwedge^4 V^* \). Define a \( \mathbb{C} \)-valued bilinear form \( b \) on \( \bigwedge^2 V \) by \( b(\alpha, \beta) = \langle \mu, \alpha \wedge \beta \rangle \) for \( \alpha, \beta \in \bigwedge^2 V \).

(a) Choose a basis \( \{e_1, e_2, e_3, e_4\} \) of \( V \) and the induced basis \( \{e_i \wedge e_j : i < j\} \) of \( \bigwedge^2 V \). Then choose \( \mu = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \), where \( \{e^1, e^2, e^3, e^4\} \) is the dual basis. Show that \( b \) is nondegenerate.

(b) Define the homomorphism

\[
\pi: \text{Aut}(V, \mu) \longrightarrow \text{Aut}(\bigwedge^2 V, b)
\]

which maps a volume-preserving automorphism of \( V \) to a bilinear form-preserving automorphism of \( \bigwedge^2 V \). Write the corresponding map of Lie algebras. Prove that the latter is an isomorphism.

(c) Deduce that the image of \( \pi \) is open and a subgroup, whence \( \pi \) maps onto the identity component of \( \text{Aut}(\bigwedge^2 V, b) \). (Show that the latter group has two components.)
(d) Prove that the kernel of $\pi$ is $\pm \text{id}$ so that $\pi$ is a 2:1 covering. Deduce that $\text{SL}_4(\mathbb{C})$ is a double covering of $\text{SO}_6(\mathbb{C})$.

(e) Now choose $\omega \in \Lambda^2 V^*$ with $(\omega \wedge \omega)/2 = \mu$. For example, take $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$. Let $W \subset \Lambda^2 V$ be the annihilator of $\omega$. Then use the restriction of the map $\pi$ above to define

$$\pi: \text{Aut}(V, \omega) \longrightarrow \text{Aut}(W, b).$$

As before, prove that $\pi$ is a 2:1 covering map. Deduce that $\text{Sp}_4(\mathbb{C})$ is a double covering of $\text{SO}_5(\mathbb{C})$.

(f) Write $V = U_1 \oplus U_2$ as the direct sum of 2-dimensional subspaces. Choose nonzero $\omega_1 \in \Lambda^2 U_1^*$ and $\omega_2 \in \Lambda^2 U_2^*$. Construct a decomposition $\Lambda^2 V^* \cong \Lambda^2 U_1^* \oplus \Lambda^2 U_2^* \oplus U_1^* \otimes U_2^*$. Then, following the ideas in previous parts, construct a 2:1 covering

$$\pi: \text{Aut}(U_1, \omega_1) \times \text{Aut}(U_2, \omega_2) \longrightarrow \text{Aut}(U_1^* \otimes U_2^*, b).$$

(In the definition of $b$ take $\mu = \omega_1 \wedge \omega_2$.) Deduce that $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ is a double covering of $\text{SO}_4(\mathbb{C})$.

(g) Construct a double covering $\pi: \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_3(\mathbb{C})$.

(h) From the beginning introduce a hermitian inner product on $V$ and restrict to the subgroups that preserve this inner product. What do you learn?
Problem Set # 6
Math 262a: Quantum theory from a geometric viewpoint I
Due: October 27

Problems

1. In this problem you can work over any field, for example the complex numbers.
   (a) Let
   \[ 0 \to V' \to V \to V'' \to 0 \]
   be a short exact sequence of vector spaces. Show that the space of splittings has a natural structure as an affine space over \( \text{Hom}(V'', V') \). Generalize to short exact sequences of groups, Lie algebras, …
   (b) Let \( V \) be a finite dimensional vector space and \( W \subset V \) a subspace. Let \( A_W \) denote the set (or, better, space) of subspaces \( W' \subset V \) such that \( V = W \oplus W' \). Show that \( A_W \) has a natural structure of an affine space over \( \text{Hom}(V/W, W) \). Give a geometric construction for the sum of an element of \( A_W \) and a linear map \( V/W \to W \).
   (c) Use (b) to construct an atlas of charts on the Grassmannian \( \text{Gr}_k(V) \) of \( k \)-dimensional subspaces of \( V \). (Index the charts by subspaces of dimension \( n - k \), where \( \dim V = n \). What are the transition functions?)
   (d) Let \( M \) be a Galilean spacetime with translation group \( V \) equipped with a codimension one subspace \( W \). Let \( C, C' \subset M \) be affine lines that are transverse to the simultaneity foliation, i.e., \( C, C' \) are affine worldlines. Produce a linear map \( V/W \to W \) and interpret it as the relative velocity of the affine worldlines (in order). Define the acceleration of single affine worldline \( C \subset M \) as a linear map \( (V/W)^{\otimes 2} \to W \).

2. (a) Let \( \mathcal{O} \) be a C*-algebra. Prove that the space of states is convex. Let \( S \) be a compact Hausdorff space and \( \mathcal{O} = C^0(S; \mathbb{C}) \) the algebra of continuous functions on \( S \). What are the states on \( \mathcal{O} \)? What are the pure states?
   (b) Let \( (\mathcal{H}, U_t) \) be the (linear) data of a QM system. Identify a C*-algebra of “observables”. (I use scare quotes since ‘observables’ are the real points.) What are its states in the sense of C*-algebras? Does this agree with the QM states as defined in §1.4 of the notes?

3. Investigate the existence of time-reversal symmetries in the toric code (§5.5). In the particle on a ring (§7.3). Consider both a single system and families of systems.

4. In lecture we (will) construct the symplectic manifold that is the phase space of a classical relativistic particle. Do the same for a classical Galilean particle.
5. Take up Problem 4 on Problem Set #5. Deduce the chart of low dimensional spin groups (dimensions $\leq 6$) that appeared in lecture and (will appear) in the notes. You previously did the Euclidean case; now do the Lorentz case.

6. Consider Maxwell’s equations in whatever form is familiar to you. Define an action of the group of isometries of Minkowski spacetime on the variables (fields) in those equations. Are the equations invariant? If not, is there a subgroup under which they are invariant?
Problem Set # 7
Math 262a: Quantum theory from a geometric viewpoint I
Due: November 3

Problems

1. (a) Let $M$ be an affine space over a real vector space $V$, and suppose $C \subset M$ is a submanifold of dimension $k$. Let $\text{Gr}_k(V)$ denote the Grassmannian of $k$-dimensional subspaces of $V$. The Gauss map

$$\Gamma: C \longrightarrow \text{Gr}_k(V)$$

assigns to each $p \in C$ its tangent space $T_pC \subset V$. Prove that for $p \in C$ the differential

$$d\Gamma_p: T_pC \longrightarrow \text{Hom}(T_pC, V/T_pC)$$

can be interpreted as a symmetric bilinear form

$$T_pC \times T_pC \longrightarrow V/T_pC$$

(b) Suppose $\dim M = 2$ and $k = 1$. How is the differential of the Gauss map related to the classical curvature of a (cooriented) plane curve.

2. (a) Recall that the phase space $M$ of the classical relativistic particle in a Minkowski spacetime $M$ is the manifold of all timelike affine lines $C \subset M$. The isometry group $\text{O}(M)$ acts on $M$. Prove that the subgroup $\text{O}^+(M)$ acts preserving the symplectic form and its complement in $\text{O}(M)$ acts by reversing the symplectic form (to its negative).

(b) Repeat the analysis of the classical relativistic particle for the classical Gallilean particle.

3. (a) Let $M$ be a smooth manifold equipped with a transitive action of a Lie group $G$. Fix $p \in M$ and let $G_p \subset G$ be the stabilizer subgroup at $p$. Define the notion of a $G$-invariant smooth measure on $M$. Prove that $G$-invariant smooth measures form either an $\mathbb{R}^{>0}$-torsor or form an empty set. Show that the former possibility prevails if $G_p$ is compact.

(b) Let $G$ be a Lie group. Prove that there exist left-invariant smooth measures on $G$.

(c) Investigate the existence of bi-invariant smooth measures on $G$. What condition(s) on $G$ guarantee their existence? Give an example of $G$ and a proof of non-existence for that $G$.

4. Consider standard Minkowski spacetime $\mathbb{M}^n$ with coordinates $t, x^1, \ldots, x^{n-1}$ and Lorentz metric

$$c^2dt^2 -(dx^1)^2 - \cdots -(dx^{n-1})^2.$$ 

For $m > 0$ consider the mass shell $\mathcal{O}_m$ in $(\mathbb{R}^n)^*$ defined as the manifold of vectors $E dt + p_idx^i$ that satisfy $E^2/c^2 - \sum(p_i)^2 = m^2c^2$ and $E > 0$. Write a formula for a smooth density on $\mathcal{O}_m$ that is invariant under the group $\text{O}^+(\mathbb{M}^n)$. Is it invariant under $\text{O}(\mathbb{M}^n)$?
I’m giving fewer problems this week in the hope that you will be able to catch up if you’ve fallen behind. Please come see me if you’d like to discuss the class.

Problems

1. (a) Construct a finite dimensional representation of the Lie group $\text{SL}_2\mathbb{R}$.

(b) Prove that a unitary irreducible representation of $\text{SL}_2\mathbb{R}$ is either the trivial representation or is infinite dimensional.

(c) Conclude the same for a representation of the Lorentz group $\text{Spin}_{1,n-1}$ for $n \geq 3$.

2. A spin group $\text{Spin}_{p,q}$ ($p, q \in \mathbb{Z}_{\geq 0}$) has a vector representation on $V = \mathbb{R}^{p,q}$ that acts via the quotient $\text{SO}_{p,q}$. (The vector space $\mathbb{R}^{p,q}$ is $\mathbb{R}^{p+q}$ with a nondegenerate symmetric bilinear form of the indicated signature.) Recall the chart (12.24) of low dimensional spin groups for $p = n$, $q = 0$ (positive definite) and $p = 1$, $q = n - 1$ (Lorentz). Let $S$ be a real spin representation, which in dimensions 3–6 are the real representations underlying the defining representations. That is a bit vague, but enough for you to find a few examples in the chart to play with, and you can think about dimensions 1,2 as well. The question: Is there a $\text{Spin}_{p,q}$-invariant nonzero symmetric bilinear map

$$\Gamma : S \times S \longrightarrow V$$

Investigate in examples. Can you make a conjecture based on your work?
Problem Set # 9
Math 262a: Quantum theory from a geometric viewpoint I
Due: November 17

The first exercise makes precise why when one considers the presheaf of double covers on manifolds, one must use groupoids (and not equivalence classes of double covers) to achieve locality. The same principle applies to connections for a fixed gauge group, also known as “gauge fields”. We use the notation \( A = (A_0, A_1) \) for (small) groupoids, where \( A_0 \) is the set of objects and \( A_1 \) is the set of morphisms.

Problems

1. Let \( A, A', B \) be groupoids.
   (a) Suppose
   \[
   \begin{array}{ccc}
   A' & \xrightarrow{\alpha'} & B \\
   \downarrow{\alpha} & & \\
   A & \xrightarrow{} & B
   \end{array}
   \]
   is a diagram of functors. Define the pullback
   \[
   \begin{array}{ccc}
   C & \xrightarrow{\gamma'} & A' \\
   \downarrow{\gamma} & & \downarrow{\alpha'} \\
   A & \xrightarrow{\alpha} & B
   \end{array}
   \]
   as follows. Objects of \( C \) are triples \((a, a', \theta) \in A_0 \times A'_0 \times B_1 \) where \( \theta: \alpha(a) \rightarrow \alpha'(a') \). A morphism \((a, a', \theta) \rightarrow (b, b', \phi)\) in \( C \) is a pair \((f, f') \in A_1 \times A'_1 \) such that \( f: a \rightarrow b \), \( f': a' \rightarrow b' \), and the diagram
   \[
   \begin{array}{ccc}
   \alpha(a) & \xrightarrow{\theta} & \alpha'(a') \\
   \downarrow{\alpha(f)} & & \downarrow{\alpha(f')} \\
   \alpha(b) & \xrightarrow{\phi} & g a'(b')
   \end{array}
   \]
   commutes.
   (b) Suppose \( \beta: A \rightarrow B \) is a map of groupoids and \( b \in B_0 \) an object. Compute the pullback of
   \[
   \begin{array}{ccc}
   A & \xrightarrow{\beta} & B \\
   \downarrow{\beta} & & \\
   * & \xrightarrow{b} & B
   \end{array}
   \]
   This is the fiber of \( \beta \) over \( b \).
(c) Suppose that

\[ D \xrightarrow{\delta'} A' \xleftarrow{\alpha'} A \xrightarrow{\alpha} B \]

is a commutative diagram in the sense that we are given a natural isomorphism \( \varphi : \alpha \delta \rightarrow \alpha' \delta' \). Prove that the morphisms \( \delta, \delta' \) factor through the pullback:

\[ D \xrightarrow{\delta'} A' \xleftarrow{\alpha'} A \xrightarrow{\alpha} B \]

What precisely do we mean by the commutativity of the diagram? What kind of uniqueness do we have?

(d) Formulate the universal property satisfied by the pullback in this context. What uniqueness is satisfied?

(e) Carefully define the groupoid double covers as a presheaf on the category of smooth manifolds and smooth maps. Formulate the local \( \rightarrow \) global condition for this to be a sheaf and verify it. If you don’t want to tackle the general case, then formulate the gluing for the cover of \( S^1 \) consisting of two overlapping open intervals; the intersection is the disjoint union of two open intervals.

2. Let \( M^1 \) be standard 1-dimensional Minkowski spacetime: the affine line \( \mathbb{A}^1 \) with metric \( c^2 dt^2 \), where \( c > 0 \) is the speed of light. Consider the Klein-Gordon equation for mass \( m > 0 \):

\[
\left( \frac{1}{c^2} \partial_t^2 + \frac{c^2}{\hbar^2} m^2 \right) \phi = 0,
\]

where \( \phi : M^1 \rightarrow \mathbb{R} \) is a real-valued function. What is the space \( M \) of global solutions? Compute the symplectic form

\[
\omega(\dot{\phi}_1, \dot{\phi}_2) = *d\dot{\phi}_1 \cdot \dot{\phi}_2 - *d\dot{\phi}_2 \cdot \dot{\phi}_1
\]

evaluated at some \( t \in M^1 \). Here \( \dot{\phi}_1, \dot{\phi}_2 \) are tangents to \( M \). Prove that the form is independent of \( t \). What happens under Fourier transform? Formulate a dictionary to the simple harmonic oscillator.
The first exercise makes precise why when one considers the presheaf of double covers on manifolds, one must use groupoids (and not equivalence classes of double covers) to achieve locality. The same principle applies to connections for a fixed gauge group, also known as “gauge fields”. We use the notation $A = (A_0, A_1)$ for (small) groupoids, where $A_0$ is the set of objects and $A_1$ is the set of morphisms.

**Problems**

1. Let $A, A', B$ be groupoids.
   (a) Suppose
   
   $\begin{align*}
   &A' \\
   \downarrow^{\alpha'} \\
   A \rightarrow^\alpha B
   \end{align*}$

   is a diagram of functors. Define the pullback
   
   $\begin{align*}
   C & \cdots \rightarrow A' \\
   \gamma & \downarrow \alpha' \\
   \gamma' & \downarrow \alpha \\
   A & \rightarrow B
   \end{align*}$

   as follows. Objects of $C$ are triples $(a, a', \theta) \in A_0 \times A_0' \times B_1$ where $\theta: \alpha(a) \rightarrow \alpha'(a')$. A morphism $(a, a', \theta) \rightarrow (b, b', \phi)$ in $C$ is a pair $(f, f') \in A_1 \times A_1'$ such that $f: a \rightarrow B$, $f': a' \rightarrow b'$, and the diagram

   $\begin{align*}
   \alpha(a) \rightarrow^\theta \alpha'(a') \\
   \alpha(f) \downarrow \alpha(f') \\
   \alpha(b) \rightarrow^{\phi} ga'(b')
   \end{align*}$

   commutes.

   (b) Suppose $\beta: A \rightarrow B$ is a map of groupoids and $b \in B_0$ an object. Compute the pullback of

   $\begin{align*}
   \star \rightarrow^b B
   \end{align*}$

   This is the fiber of $\beta$ over $b$. 

(c) Suppose that

\[
\begin{array}{ccc}
D & \xrightarrow{\delta'} & A' \\
\downarrow{\delta} & & \downarrow{\alpha'} \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

is a commutative diagram in the sense that we are given a natural isomorphism \( \varphi: \alpha \delta \to \alpha' \delta' \). Prove that the morphisms \( \delta, \delta' \) factor through the pullback:

\[
\begin{array}{ccc}
D & \xrightarrow{\delta'} & A' \\
\downarrow{\delta} & & \downarrow{\alpha'} \\
C & \xrightarrow{\gamma} & A' \\
\downarrow{\gamma} & & \downarrow{\alpha'} \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

What precisely do we mean by the commutativity of the diagram? What kind of uniqueness do we have?

(d) Formulate the universal property satisfied by the pullback in this context. What uniqueness is satisfied?

(e) Carefully define the groupoid double covers as a presheaf on the category of smooth manifolds and smooth maps. Formulate the local \( \to \) global condition for this to be a sheaf and verify it. If you don’t want to tackle the general case, then formulate the gluing for the cover of \( S^1 \) consisting of two overlapping open intervals; the intersection is the disjoint union of two open intervals.

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