Teichmüller dynamics and unique ergodicity via currents and Hodge theory

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Abstract

We present a cohomological proof that recurrence of suitable Teichmüller geodesics implies unique ergodicity of their terminal foliations. This approach also yields concrete estimates for periodic foliations and new results for polygonal billiards.

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1 Introduction

Let $\mathcal{M}_g$ denote the moduli space of compact Riemann surfaces X of genus g, and let $\Omega\mathcal{M}_g \to \mathcal{M}_g$ denote the bundle of nonzero holomorphic 1–forms (X, ω). Any 1–form determines a horizontal foliation $\mathcal{F}(\omega)$ of X together with a transverse invariant measure. If this measure is unique up to scale, we say $\mathcal{F}(\omega)$ is *uniquely ergodic.*

The purpose of this note is to present a cohomological proof of the following important result of Masur:

Theorem 1.1 *Suppose the Teichmüller geodesic ray generated by shrinking the leaves of $\mathcal{F}(\omega)$ is recurrent in moduli space $\mathcal{M}_g$. Then the foliation $\mathcal{F}(\omega)$ is uniquely ergodic.*

The perspective we adopt is based on currents and Hodge theory. First, we introduce the convex cone $P(\omega)$ of closed, positive currents carried by $\mathcal{F}(\omega)$. These are the 1–forms ξ on X, with distributional coefficients, satisfying

$$d\xi = 0,\ \xi \wedge \beta = 0 \quad \text{and} \quad \alpha \wedge \xi \geq 0,$$

where $\omega = \alpha + i\beta$. As we will see in §3, there is a natural bijection between such currents and transverse invariant measures for $\mathcal{F}(\omega)$.

The language of currents provides a useful bridge between foliations, differential forms and Hodge theory. Moreover, the closed currents $P(\omega)$ map injectively into $H^1(X, \mathbb{R})$ when $\mathcal{F}(\omega)$ has a dense leaf, so unique ergodicity can be addressed at the level of cohomology.

In this language, our main result is:

Theorem 1.2 *Suppose X lies in a compact subset $K \subset \mathcal{M}_g$, and the geodesic ray generated by (X, ω) spends at least time T in K. Then the closed, positive currents carried by $\mathcal{F}(\omega)$ determine a convex cone*

$$[P(\omega)] \subset H^1(X, \mathbb{R})$$

which meets the unit sphere in a set of diameter $O(e^{-\lambda(K)T})$.

Here the unit sphere and diameter are defined using the Hodge norm on $H^1(X, \mathbb{R})$, and $\lambda(K) > 0$ depends only on K.

One can regard Theorem 1.2 as a quantitative refinement of Theorem 1.1. In the recurrent case we can take $T = \infty$, $[P(\omega)]$ reduces to a single ray, and we obtain unique ergodicity (see §5).

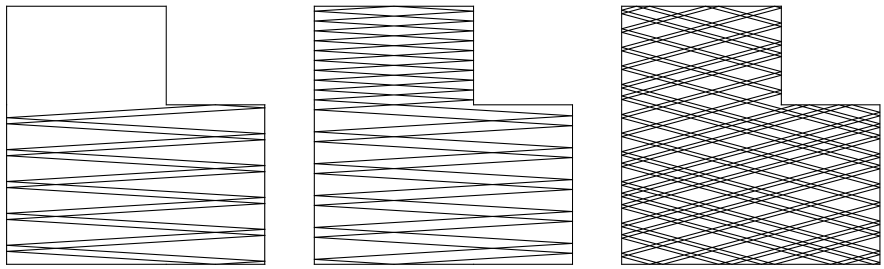

Figure 1. Periodic billiard trajectories with slopes tending to zero.

Billiards. Theorem 1.2 also sheds light on the distribution of closed geodesics on $(X, |\omega|)$, and leads to new results on billiards in polygons. To illustrate this connection, in §6 we will show:

Theorem 1.3 *Consider a sequence of periodic billiard trajectories of slope $s_n \to s$ on the golden L–shaped table. If the lengths of the golden continued fractions for s_n tend to infinity, then the trajectories become uniformly distributed as $n \to \infty$.*

Three examples with $s_n \to 0$ are shown in Figure 1. Only the last sequence of trajectories is uniformly distributed.

An analogous statement holds for any lattice polygon and any 1–form generating a Teichmüller curve $V \subset \mathcal{M}_g$. These applications were our original motivation for proving Theorem 1.2. A more complete development will appear in a sequel [Mc5].

The cone of positive currents. Here is a sketch of the proof of Theorem 1.2.

Let $\xi \in P(\omega)$ be a closed, positive current carried by $\mathcal{F}(\omega)$ as above, with $\xi \neq 0$. The standard transverse measure for $\mathcal{F}(\omega)$ corresponds to the smooth current $\beta = \mathrm{Im}(\omega)$. To compare the two, we first scale ξ so it has the form

$$\xi = \beta + \delta,$$

where $\int_X \delta \wedge \omega = 0$. Let X_t, $t \geq 0$, denote the Teichmüller geodesic ray generated by ω with $X_0 = X$. The natural flat connection on cohomology groups allows us to transport the Hodge norm from $H^1(X_t, \mathbb{R})$ to a varying family of norms on $H^1(X, \mathbb{R})$, which we denote by $\| \cdot \|_{X_t}$.

In §3 we show that a *cone condition* of the form

$$\frac{\|\delta\|_{X_t}}{\|\beta\|_{X_t}} \leq C(K)$$

holds whenever $X_t \in K$. On the other hand, in §4 we show that as $t \to \infty$, the Hodge norm of β shrinks more rapidly than that of δ: there is a $\lambda = \lambda(K) > 0$ such that

$$\frac{\|\delta\|_{X_t}}{\|\beta\|_{X_t}} \geq e^{\lambda T} \cdot \frac{\|\delta\|_X}{\|\beta\|_X}, \tag{1.1}$$

where T is the amount of time X_s spends in K for $s \in [0, t]$.

Combining these bounds gives the stronger cone condition

$$\frac{\|\delta\|_X}{\|\beta\|_X} \leq e^{-\lambda T} C(K)$$

whenever $X_t \in K$. This inequality says that the line through $[\xi] = [\beta + \delta]$ is exponentially close to the line through $[\beta]$ in $H^1(X, \mathbb{R})$, and Theorem 1.2 follows. (For more details, see §5.)

Conceptually, equation (1.1) follows from uniform contraction (over K) of the *complementary period mapping*

$$\sigma : \mathbb{H} \to \mathfrak{H}_{g-1},$$

which records the Hodge structure on the part of $H^1(X, \mathbb{R})$ orthogonal to ω, as X moves along a complex geodesic (see §4).

Notes and references. Many of the ideas presented in §4 below were developed independently and earlier by Forni and others, with somewhat different aims and formulations. In particular, a version of Theorem 4.1 for strata is given in [AF, Thm. 4.2], and a variant of equation (A.1) is derived, by different means, in [Fo, Lemma 2.1′]. The strategy to prove Theorem 1.1 is similar to the proof that $\mathcal{F}(\omega)$ is ergodic for almost every $\omega \in \Omega\mathcal{M}_g$ sketched in [FM, Remark 60]. Here we use currents and the Hodge norm throughout, and exploit the cone condition given in Theorem 3.1.

Masur's original proof of Theorem 1.1, which also applies to quadratic differentials, is given in [Mas, Thm 1.1]; see also [Mc2]. The original argument works directly with dynamics and Anosov properties of the foliation of X. A strengthening of Theorem 1.1 is given in [Tr, Thm. 4].

A discussion of currents and foliations on general manifolds can be found in [Sul]; another instance of their use in the present setting is given in [Mc4, §2]. For more on the interaction between Hodge theory and Teichmüller theory, see e.g. [Ah], [Roy], [Fo], [Mc1], [Mo1], [EKZ], [FM] and [FMZ].

I would like to thank J. Chaika and G. Forni for useful discussions and references.

2 Background

We begin by recalling some basic results regarding the Hodge theory, foliations, geodesics in Teichmüller space, and the action of $\mathrm{SL}_2(\mathbb{R})$ on the moduli space of holomorphic 1–forms. For more details, see e.g. [FLP], [GH], [Ga], [IT], [Nag], [MT] and [Mo2].

The Hodge norm. Let X be a Riemann surface of genus g. The spaces of holomorphic and real harmonic 1–forms on X will be denoted by $\Omega(X)$ and $\mathcal{H}^1(X)$ respectively.

By Hodge theory, the map sending a cohomology class to its harmonic representative provides an isomorphism $H^1(X,\mathbb{R}) \cong \mathcal{H}^1(X)$. These representatives, together with the Hodge star, give a rise to a natural inner product

$$\langle \alpha, \beta \rangle_X = \int_X \alpha \wedge *\beta \tag{2.1}$$

on $H^1(X,\mathbb{R})$; and the associated *Hodge norm* is defined by

$$\|\alpha\|_X^2 = \langle \alpha, \alpha \rangle_X.$$

Similarly, the space $\Omega(X)$ carries a natural Hermitian form defined by

$$\langle \omega_1, \omega_2 \rangle_X = \frac{i}{2} \int_X \omega_1 \wedge \overline{\omega}_2,$$

whose associated norm is given by

$$\|\omega\|_X^2 = \int_X |\omega|^2.$$

These norms are compatible in the sense that the map $\omega \mapsto \mathrm{Re}\,\omega$ gives a norm–preserving, real linear isomorphism

$$\Omega(X) \cong H^1(X,\mathbb{R}).$$

Foliations and measures. Every nonzero $\omega \in \Omega(X)$ determines a natural *horizontal foliation* $\mathcal{F}(\omega)$ of X. To describe this foliation, recall that

$$\omega = \alpha + i\beta$$

is a linear combination of real harmonic forms satisfying $*\alpha = \beta$.

The foliation $\mathcal{F}(\omega)$ has multipronged singularities at the zeros of ω. Away from these points, we can choose local coordinates such that $\omega = dz$ and the leaves of $\mathcal{F}(\omega)$ become horizontal lines in $\mathbb{C}$. In particular, the tangent space to $\mathcal{F}(\omega)$ is the kernel of β. Each leaf L of $\mathcal{F}(\omega)$ is naturally oriented by the condition $\alpha|L > 0$.

A *transverse measure* for $\mathcal{F}(\omega)$ is the specification of a Borel measure $\mu_\tau \geq 0$ on every smooth arc $\tau \subset X$ disjoint from $Z(\omega)$ and transverse to the leaves of the foliation. We require that μ_τ is compatible with restriction; that is, $\mu_\sigma = \mu_\tau|\sigma$ whenever $\sigma \subset \tau$. A transverse measure is *invariant* if it also compatible with the smooth maps between nearby transversal obtained by flowing along the leaves of $\mathcal{F}(\omega)$.

The standard transverse invariant measure for $\mathcal{F}(\omega)$ is defined by $\mu_\tau = \beta|\tau$ (where τ is oriented so the measure is positive.) There may be many others; for example, any closed leaf L of $\mathcal{F}(\omega)$ supports a transverse atomic measure μ_τ with mass one at each point of $\tau \cap L$.

Moduli of Riemann surfaces. Fix a compact, oriented topological surface Σ_g of genus $g \geq 2$ with mapping–class group Mod_g. A point (X, f) in the associated *Teichmüller space* $\mathcal{T}_g$ is specified by a Riemann surface of genus g together with an orientation–preserving *marking* homeomorphism $f : \Sigma_g \to X$. By forgetting the marking, we obtain a natural map

$$\mathcal{T}_g \to \mathcal{T}_g / \mathrm{Mod}_g \cong \mathcal{M}_g,$$

presenting $\mathcal{T}_g$ as the orbifold universal cover of the *moduli space* $\mathcal{M}_g$ of Riemann surfaces of genus g.

The cotangent space to $\mathcal{T}_g$ at X is naturally identified with the space $Q(X)$ of holomorphic quadratic differentials on X, and the Teichmüller metric corresponds to the norm $\|q\| = \int_X |q|$. To describe the tangent space, let $M(X)$ denote the space of measurable Beltrami differentials on X with $\|\mu\| = \sup_X |\mu| < \infty$. The natural pairing

$$\langle q, \mu \rangle = \int_X q\mu = \int_X q(z)\mu(z)|dz|^2 \qquad (2.2)$$

between $Q(X)$ and $M(X)$ then allows one to identify the tangent space $\mathrm{T}_X \mathcal{T}_g$ with the quotient space $M(X)/Q(X)^\perp$.

For later reference, we note that when $\|q\| = \|\mu\| = 1$, we have

$$\mathrm{Re}\langle q, \mu \rangle \leq 1, \textit{ and equality holds iff } \mu = \overline{q}/|q|. \qquad (2.3)$$

This observation is an infinitesimal form of uniqueness of the Teichmüller mapping.

Moduli of forms. Consider the holomorphic vector bundle over $\mathcal{T}_g$ whose fiber over X is $\Omega(X)$. Removing the zero section, we obtain the space $\Omega\mathcal{T}_g$

of marked holomorphic 1–forms. The associated sphere bundle, whose fibers are

$$\Omega_1(X) = \{\omega \in \Omega(X) \,:\, \|\omega\|_X = 1\},$$

will be denoted by $\Omega_1\mathcal{T}_g$. Taking the quotient by Mod_g yields the corresponding bundles $\Omega\mathcal{M}_g$ and $\Omega_1\mathcal{M}_g$ over moduli space.

Dynamics and geodesics. There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{T}_g$, characterized by the property that, for $L = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we have

$$L \cdot (X, \omega) = (Y, \eta)$$

if and only if there is a map $f : X \to Y$, compatible with markings, such that

$$f^*(\eta) = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \end{pmatrix}. \tag{2.4}$$

The orbits $(X_t, \omega_t) = a_t \cdot (X, \omega)$ of the diagonal group

$$A = \left\{ a_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \,:\, t \in \mathbb{R} \right\} \subset \mathrm{SL}_2(\mathbb{R})$$

project to Teichmüller geodesics in $\mathcal{T}_g$, parameterized by arclength, and satisfying

$$\frac{dX_t}{dt} = \left[-\frac{\overline{\omega}_t}{\omega_t} \right] \in M(X_t)/Q(X_t)^\perp \tag{2.5}$$

for all $t \in \mathbb{R}$. For $t > 0$, the natural affine map $f_t : (X, |\omega|) \to (X_t, |\omega_t|)$ is area–preserving and shrinks the leaves of $\mathcal{F}(\omega)$ by a factor of e^{-t}. In particular, if all the leaves of $\mathcal{F}(\omega)$ are closed, then $[X_t]$ converges to a stable Riemann surface in $\partial\mathcal{M}_g$ by pinching these closed curves.

Dynamics and cohomology. The vector bundle $H^1 \to \mathcal{T}_g$ with fibers $H^1(X, \mathbb{R})$ is both trivial and flat with respect to the connection provided by the marking isomorphisms $H^1(X, \mathbb{R}) \to H^1(\Sigma_g, \mathbb{R})$. The same is true when it is pulled back to $\Omega\mathcal{T}_g$. Over this space we have $H^1 = W \oplus W^\perp$, where the splitting

$$H^1(X, \mathbb{R}) = W(X, \omega) \oplus W(X, \omega)^\perp \tag{2.6}$$

on each fiber is obtained by taking the image of the direct sum

$$\Omega(X) = (\mathbb{C}\omega) \oplus (\mathbb{C}\omega)^\perp$$

under the map $\omega \mapsto \mathrm{Re}\,\omega$. Since $W(X, \omega)$ is $*$-invariant, the summands in (2.6) are also orthogonal with respect to the symplectic form. Thus the following result is immediate from (2.4):

Proposition 2.1 *The sub-bundles W and $W^\perp$ are flat over any* $\mathrm{SL}_2(\mathbb{R})$ *orbit in* $\Omega\mathcal{T}_g$.

Put differently, we have $W(L \cdot (X, \omega)) = W(X, \omega)$ when both are identified with subspaces of $H^1(\Sigma_g, \mathbb{R})$, and similarly for $W(X, \omega)^\perp$.

3 Currents and cones

In this section we describe the connection between measured foliations and closed, positive currents. We then show that the shape of the convex cone in cohomology determined by these currents is uniformly controlled over compact subsets of moduli space. This control can be expressed in terms of the Hodge norm as a reverse Cauchy–Schwarz inequality, which we state as follows.

Theorem 3.1 *Let $K \subset \mathcal{M}_g$ be a compact set. Then for any closed, positive current ξ carried by the horizontal foliation $\mathcal{F}(\omega)$ of a holomorphic 1–form $\omega \in \Omega(X)$ with $X \in K$, we have*

$$\|\omega\|_X \cdot \|\xi\|_X \le C(K) \left| \int_X \omega \wedge \xi \right| . \tag{3.1}$$

Here $C(K) > 0$ is a constant depending only on K. Geometrically, this results says that the length of the current ξ in the metric $|\omega|$ controls the Hodge norm of its harmonic representative.

Currents and foliations. Recall that a 1–dimensional current ξ on X is an element of the dual of the space of smooth 1–forms (see e.g. [dR], [GH, §3.1]). Since X is oriented, currents on X can be thought of as forms with distributional coefficients. The current ξ is *closed* if $d\xi = 0$; equivalently, if $\int_X \xi \wedge df = 0$ for every smooth function f on X. Any closed current determines a cohomology class $[\xi] \in H^1(X, \mathbb{R})$.

Given a nonzero holomorphic form $\omega = \alpha + i\beta$, let

$$P(\omega) = \{\text{currents } \xi \; : \; d\xi = 0, \xi \wedge \beta = 0 \quad \text{and} \quad \alpha \wedge \xi \ge 0\} . \tag{3.2}$$

The final positivity condition means $\int_X (f\alpha) \wedge \xi \ge 0$ for all smooth $f \ge 0$ on X; in particular, ξ is required to satisfy infinitely many linear inequalities. We refer to $P(\omega)$ as the space of *closed, positive currents* carried by $\mathcal{F}(\omega)$. It is a closed, convex cone in the natural topology on currents.

Proposition 3.2 *There is a natural bijection between the closed, positive currents carried by $\mathcal{F}(\omega)$ and its transverse invariant measures.*

Proof. A transverse invariant measure determines a current $\xi \in P(\omega)$ by integration along the (oriented) leaves of $\mathcal{F}(\omega)$ locally weighted by μ_τ. Conversely, if $\xi \in P(\omega)$, then the fact that $\xi \wedge \beta = 0$ implies ξ is locally a distributional multiple of β; the fact that ξ is closed implies it is locally the pullback of a distribution on a transversal τ; and positivity implies this distribution is a measure $\mu_\tau \geq 0$. ∎

The cohomology class of a measured foliation. Each transverse invariant measure determines a cohomology class, by the correspondence $\mu_\tau \mapsto \xi \mapsto [\xi] \in H^1(X, \mathbb{R})$. Recall that the foliation $\mathcal{F}(\omega)$ is *minimal* if each of its leaves is dense in X.

Proposition 3.3 *If $\mathcal{F}(\omega)$ is minimal, its transverse invariant measures are determined by their cohomology classes. Equivalently, the natural map $P(\omega) \to H^1(X, \mathbb{R})$ is injective.*

Proof. It suffices to show that the values of $\int_C \xi$ for $C \in H_1(X, \mathbb{Z})$ determine $\mu_\tau(\tau)$ for every transversal τ. By minimality μ_τ has no atoms. Let L be a dense leaf of $\mathcal{F}(\omega)$. Then $L \cap \tau$ is dense in τ, so we can find an increasing sequence of subarcs $\tau_1 \subset \tau_2 \subset \tau_3 \cdots \subset \tau$ such that $\bigcup \tau_n$ has full measure in τ and each τ_n has endpoints in L. By adding a piece of L to connect these endpoints, we obtain a closed loop C_n with $[C_n] \in H_1(X, \mathbb{Z})$. Using these loops we find:

$$\mu_\tau(\tau) = \lim \mu_\tau(\tau_n) = \lim \int_{C_n} \xi.$$

Since the integrals above only depend on $[\xi] \in H^1(X, \mathbb{R})$, the proof is complete. ∎

Corollary 3.4 (Katok) *If $\mathcal{F}(\omega)$ is minimal, it carries at most $g = g(X)$ mutually singular, ergodic, transverse invariant measures.*

Proof. The cohomology classes of these measures are linearly independent by the previous result; and since the corresponding currents are given by integration along the leaves of the same foliation $\mathcal{F}(\omega)$, they lie in a Lagrangian subspace of $H^1(X, \mathbb{R})$, which has dimension g. ∎

See e.g. [Ka, Theorem 1], [V2, Theorem 0.5] and [Fi, Theorem 1.29] for other perspectives on Corollary 3.4.

Proof of Theorem 3.1. We will first prove a cone inequality for a single form $\omega = \alpha + i\beta \in \Omega(X)$, $X \in K$, normalized so that $\|\omega\|_X = 1$. By definition (3.2) we have $|\int \omega \wedge \xi| = \int \alpha \wedge \xi$. Thus our goal is to prove an inequality of the form

$$\|\xi\|_X \leq C(K) \int_X \alpha \wedge \xi \tag{3.3}$$

for all $\xi \in P(\omega)$.

Choose a sequence of smooth, closed 1–forms $\delta_1, \ldots, \delta_{2g}$ that represent an orthonormal basis for $H^1(X, \mathbb{R})$ with respect to the inner product (2.1). We may assume that these forms all vanish on an open neighborhood U of $Z(\omega)$. Since the smooth area form $\alpha \wedge \beta \geq 0$ only vanishes at the zeros of ω, there is a constant $M > 0$ such that

$$|\beta \wedge *\delta_i| \leq M\alpha \wedge \beta \tag{3.4}$$

pointwise on X, for $i = 1, 2, \ldots, 2g$.

Now for any $\xi \in P(\omega)$, the current ξ is locally a limit of smooth currents of the form $f_n\beta$ with $f_n \geq 0$. Since (3.4) implies that

$$|(f_n\beta) \wedge *\delta_i| \leq M\alpha \wedge (f_n\beta)$$

as measures on X, in the limit we obtain

$$|\xi \wedge *\delta_i| \leq M\alpha \wedge \xi,$$

and hence

$$\langle \xi, \delta_i \rangle_X \leq M \int_X \alpha \wedge \xi$$

for all i. This implies that

$$\|\xi\|_X^2 = \sum_1^{2g} \langle \xi, \delta_i \rangle_X^2 \leq 2gM^2 \left(\int_X \alpha \wedge \xi \right)^2,$$

and taking the square–root of both sides yields the desired inequality (3.3).

We now allow the form (X, ω) to move in $\Omega_1\mathcal{M}_g$. Since the zero set $Z(\omega)$ and the Hodge norm vary continuously with the form, it is easy to extend the argument just given to obtain a uniform cone constant in a neighborhood of (X, ω). By properness of the projection map $\Omega_1\mathcal{M}_g \to \mathcal{M}_g$, we can then obtain a uniform cone constant $C(K)$ for all unit–norm forms (X, ω) with X in a given compact set $K \subset \mathcal{M}_g$. But once the cone inequality (3.1) holds for ω it also holds for all the positive real multiples of ω, so the proof is complete. ∎

4 The complementary period mapping

In this section we study the variation of the Hodge norm along a geodesic in $\mathcal{T}_g$. Its relationship to the period mapping $\tau : \mathcal{T}_g \to \mathfrak{H}_g$ will be presented at the end.

Norms. Let X_t be the geodesic in $\mathcal{T}_g$ generated by a holomorphic 1–form

$$\omega_0 = \alpha_0 + i\beta_0 \in \Omega_1(X_0).$$

Since all the Riemann surfaces in $\mathcal{T}_g$ are marked by Σ_g, we have a natural isomorphism

$$H^1(X_0, \mathbb{R}) \cong H^1(X_t, \mathbb{R}) \tag{4.1}$$

for all $t \in \mathbb{R}$. Using this identification, we can transport the Hodge norms on X_t to a varying family of norms $\|\cdot\|_{X_t}$ on the fixed vector space $H^1(X_0, \mathbb{R})$.

It is easy to see that

$$\|\alpha_0\|_{X_t} = e^t \quad \text{and} \quad \|\beta_0\|_{X_t} = e^{-t}$$

for all $t \in \mathbb{R}$. The next result says that the Hodge norm of any cohomology class orthogonal to these moves more slowly.

Theorem 4.1 *Fix a compact set $K \subset \mathcal{M}_g$. Then there is a constant $\lambda = \lambda(K) > 0$ such that for all $[\delta_0] \in H^1(X_0, \mathbb{R})$ with $\|\delta_0\|_{X_0} = 1$ and*

$$\int_{X_0} \delta_0 \wedge \alpha_0 = \int_{X_0} \delta_0 \wedge \beta_0 = 0, \tag{4.2}$$

and all $t > 0$, we have

$$e^{\lambda T - t} \quad \leq \quad \|\delta_0\|_{X_t} \quad \leq \quad e^{t - \lambda T}.$$

Here $T = |\{s \in [0, t] \,:\, X_s \in K\}|$.

Proof. Let $(X_t, \omega_t) = a_t \cdot (X_0, \omega_0)$, and write

$$\omega_t = \alpha_t + i\beta_t.$$

Then $dX_t/dt = \dot{X}_t = [-\overline{\omega}_t/\omega_t]$ by equation (2.5).

Let $\eta_t \in \Omega_1(X_t)$ be the unique unit norm 1–form such that

$$\frac{[\delta_0]}{\|\delta_0\|_{X_t}} = [\operatorname{Re} \eta_t]$$

under the identification (4.1). By assumption, we have $\langle \eta_0, \omega_0 \rangle_{X_0} = 0$, and thus

$$\langle \eta_t, \omega_t \rangle_{X_t} = 0$$

for all $t \in \mathbb{R}$ by Proposition 2.1.

Define a function κ on $\Omega\mathcal{M}_g$ by

$$\kappa(X, \omega) = \sup \left\{ |\langle \eta^2, \overline{\omega}/\omega \rangle| \; : \; \eta \in \Omega_1(X) \quad \text{and} \quad \langle \eta, \omega \rangle_X = 0 \right\}. \qquad (4.3)$$

(The brackets denote the natural pairing (2.3) between tangent and cotangent vectors to $\mathcal{T}_g$ at X.) Since $\eta = \omega$ is excluded by the orthogonality condition, we have $\kappa(X, \omega) < 1$ (see equation (2.3)). It can also be readily verified that κ is continuous on $\Omega_1\mathcal{M}_g$, and hence

$$\lambda = \lambda(K) = 1 - \sup\{\kappa(X, \omega) \; : \; X \in K\} > 0.$$

Let $N(t) = \log \|\delta_0\|_{X_t}$. Computing the variation of the Hodge norm (see the Appendix, Corollary A.2), we find that

$$N'(t) = -\operatorname{Re}\langle \eta_t, \dot{X}_t \rangle,$$

and hence

$$|N'(t)| \le |\langle \eta_t, \dot{X}_t \rangle| = |\langle \eta_t, \overline{\omega}_t/\omega_t \rangle| \le \kappa(X, \omega_t).$$

This shows that $|N'(t)| < 1$ for all t, and that $|N'(t)| \le 1 - \lambda$ whenever $X_t \in K$. Since $N(0) = 0$, this implies that $|N(t)| \le t - \lambda T$ for all t, and then exponentiation yields the theorem above. ■

Conceptual framework. The idea behind the proof above can be expressed as follows. First, the Hodge norm on $H^1(X, \mathbb{R})$ provides the same information as the period matrix $\tau_{ij}(X)$, which is recorded by the holomorphic *period map* to Siegel space,

$$\tau : \mathcal{T}_g \to \mathfrak{H}_g.$$

Second, the choice of a 1–form $\omega = \alpha + i\beta$ on X determines a natural splitting

$$H^1(X, \mathbb{R}) = W \oplus W^{\perp}, \tag{4.4}$$

where $W = *W$ is spanned by α and β. Third, the form ω generates a holomorphic, isometric *complex geodesic*

$$F : \mathbb{H} \to \mathcal{T}_g,$$

related to the real geodesic by $X_t = F(ie^{2t})$. The splitting (4.4) is constant over this geodesic (see Proposition 2.1) and accordingly the period map $\tau \circ F$ can be written as

$$\mathbb{H} \to \mathbb{H} \times \mathfrak{H}_{g-1} \subset \mathfrak{H}_g,$$

where the two factors of the product $\mathbb{H} \times \mathfrak{H}_{g-1}$ record the Hodge structures on W and $W^{\perp}$ respectively. Indeed, with a suitable choice of coordinates on the first factor, we can write $\tau \circ F(s) = (s, \sigma(s))$.

It is then straightforward to show, using Ahlfors variational formula, that the complementary period map

$$\sigma : \mathbb{H} \to \mathfrak{H}_{g-1}$$

is a contraction for the Kobayashi metric. In fact, we have

$$\|D\sigma(s)\| = \kappa(X_s, \omega_s) < 1$$

for all s, and the upper bound can be replaced by $1 - \lambda(K) < 1$ provided $X_s \in K \subset \mathcal{M}_g$. To complete the proof, one need only observe that the rate of change of $\sigma(s)$ controls the rate at which the Hodge norm varies for a class in $W^{\perp}$ (cf. [Mc3, Prop. 3.1]).

For more details and similar discussions, see e.g. [Ah], [Roy], [Mc1, Theorem 4.2], [Mc3, §3], the Appendix, and the works [Fo], [AF] and [FMZ] on ergodic averages and Lyapunov exponents.

5 Unique ergodicity with bounds

With the previous results in place, it is now easy to prove Theorems 1.1 and 1.2.

Narrowing the cone. We begin with Theorem 1.2. Let $K \subset \mathcal{M}_g$ be a compact set, let $\omega = \alpha + i\beta$ be a 1–form on $X \in K$ with $\|\omega\|_X = 1$, and let X_t be the Teichmüller ray generated by (X, ω). Let

$$P_1(\omega) = \{\xi \in P(\omega) \,:\, \langle \beta, \xi \rangle = 1\}.$$

Let T denote the amount of time that X_t spends in K for $t \geq 0$, and let $C(K), \lambda(K) > 0$ be the constants provided by Theorems 3.1 and 4.1.

Theorem 5.1 *We have* $\|\xi - \beta\|_X \leq C(K)e^{-\lambda(K)T}$ *for all* $\xi \in P_1(\omega)$*.*

Proof. Let $(X_t, \omega_t) = a_t \cdot (X, \omega)$, and write $\omega_t = \alpha_t + i\beta_t$. The Teichmüller mapping $f_t : X \to X_t$ provides a natural identification between $H^1(X_t, \mathbb{R})$ and $H^1(X, \mathbb{R})$, under which we have

$$[\alpha_t] = e^{-t}[\alpha_0] \quad \text{and} \quad [\beta_t] = e^t[\beta_0] \tag{5.1}$$

by equation (2.4). Since $\|\omega_t\|_{X_t} = \|\beta_t\|_{X_t} = 1$, this gives

$$\|\beta_0\|_{X_t} = e^{-t}.$$

Recall that $\langle \alpha, \xi \rangle = 0$ by the definition (3.2) of $P(\omega)$. Since $\langle \beta, \xi \rangle = 1$, we can write

$$\xi = \beta + \delta, \quad \text{where} \int_X \omega \wedge \delta = 0.$$

The Teichmüller mapping f_t transports ξ to a current $\xi_t \in P(\omega_t)$ which we can similarly write as

$$\xi_t = e^{-t}\beta_t + \delta_t, \quad \text{where} \int_{X_t} \omega_t \wedge \delta_t = 0.$$

By Theorem 3.1, we then have

$$\|\delta_t\|_{X_t} \leq C(K) \left| \int_{X_t} \omega_t \wedge \xi_t \right| = C(K)e^{-t} \tag{5.2}$$

whenever $X_t \in K$.

Note that the cohomology classes $[\xi_t]$ and $[\delta_t]$ do not depend on t. The first is constant because we use f_t to identify cohomology groups as t varies, and the second is constant because the span of $[\alpha_t]$ and $[\beta_t]$ does not depend on t (cf. Proposition 2.1).

Suppose $s = \sup\{t \geq 0 \ : \ X_t \in K\}$ is finite. Then we also have

$$\|\delta_s\|_{X_s} \geq \|\delta\|_X \cdot e^{-s+\lambda(K)T},$$

by Theorem 4.1. Setting $t = s$ in (5.2) and combining these inequalities gives

$$\|\delta\|_X \leq C(K)e^{-\lambda(K)T}.$$

Since $\delta = \xi - \beta$, the proof is complete. The case $s = \infty$ is similar. ■

Theorem 1.2 is then equivalent to:

Corollary 5.2 *The diameter of the intersection of* $[P(\omega)] \subset H^1(X, \mathbb{R})$ *with the unit sphere in the Hodge norm is bounded by* $2C(K)e^{-\lambda(K)T}$.

Proof. Consider any $\xi \in P(\omega)$ with $\|\xi\|_X = 1$. Then $r = \langle \xi, \beta \rangle > 0$ by Theorem 3.1, and $r \leq 1$ since $\|\beta\|_X = 1$. The preceding result then gives

$$\|\xi - \beta\|_X \leq \|r^{-1}\xi - \beta\|_X \leq C(K)e^{-\lambda(K)T},$$

where the first inequality comes from the fact that both ξ and β lie on the unit sphere. ∎

Unique ergodicity: Proof of Theorem 1.1. Suppose the geodesic ray X_t generated by (X, ω) is recurrent. Then X_t spends an infinite amount of time in some fixed compact set $K \subset \mathcal{M}_g$, and thus $[P_1(\omega)] \subset H^1(X, \mathbb{R})$ is a single point by the preceding result. By recurrence, $\mathcal{F}(\omega)$ has no cylinders or loops of saddle connections; if it did, they would pinch and force X_t to infinity. Hence $\mathcal{F}(\omega)$ has a dense leaf, which implies the map $P(\omega) \to H^1(X, \mathbb{R})$ is injective, by Proposition 3.3. Thus $P_1(\omega)$ itself is a single point, and hence $\mathcal{F}(\omega)$ is uniquely ergodic. ∎

6 Billiards and equidistribution

In this section we prove Theorem 1.3 on equidistribution of billiards.

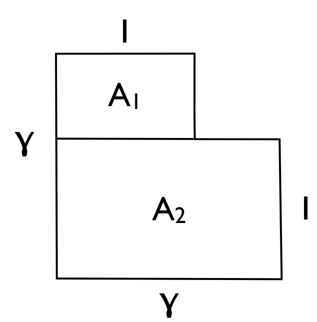

Figure 2. The golden table P.

The golden table. Let $\gamma = (1 + \sqrt{5})/2$ be the golden ratio, and let P denote the symmetric L–shaped polygon $P \subset \mathbb{C}$ shown in Figure 2, whose short and long sides have lengths 1 and γ respectively. By gluing parallel edges of P together by horizontal and vertical translations, we obtain a holomorphic 1–form $(X,\omega) = (P, dz)/\sim \in \Omega\mathcal{M}_g$, $g = 2$, whose stabilizer is the lattice

$$\Gamma = \mathrm{SL}(X,\omega) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(\mathbb{R}).$$

Its quotient $V = \mathbb{H}/\Gamma$ is the $(2,5,\infty)$ hyperbolic orbifold.

By a well–known result of Veech, the fact that Γ is a lattice implies every billiard trajectory in P is either periodic or uniformly distributed [V1]. (The same result holds for the regular n–gon, and P is closely related to the case $n = 5$.)

Continued fractions. Recall that the *cusps* of a Fuchsian group are the fixed points of its parabolic elements. The slopes s of periodic trajectories for P are essentially the same as the cusps of Γ. [1] Since the cusps form a single orbit $\Gamma \cdot \infty$, every periodic slope can be written as a finite *golden continued fraction*,

$$s = [a_1, a_2, a_3, \ldots, a_N] = a_1\gamma + \cfrac{1}{a_2\gamma + \cfrac{1}{a_3\gamma + \cdots \cfrac{1}{a_N\gamma}}}.$$

This expression can be computed recursively and made unique by requiring that $x - a_1\gamma \in (-\gamma/2, \gamma/2]$, and similarly for each subsequent $a_i \in \mathbb{Z}$. We refer to the number of a_i as the *length* $N = N(s)$.

Proof of Theorem 1.3. Consider a sequence of periodic slopes $s_n \to s$. As usual (cf. [MT]), a billiard trajectory in P at slope s_n corresponds to a closed leaf L_n of the foliation $\mathcal{F}(\omega_n)$ of X, $\omega_n = (1 + is_n)^{-1}\omega$, and to prove equidistribution of trajectories in P it suffices to prove equidistribution of L_n in X. We may assume that s itself is a periodic direction, otherwise equidistribution is immediate from unique ergodicity at slope s. In fact, since Γ acts transitively on periodic slopes, we may assume $s = 0$ and $\omega_n \to \omega$.

Let $\omega = \alpha + i\beta$. Each closed leaf $L_n \subset X$ determines a current of integration on X; dividing through by the length of L_n in the $|\omega|$–metric, we

[1] It can be shown that the cusps of Γ coincide with $\mathbb{Q}(\sqrt{5}) \cup \{\infty\}$; see [Le, Satz 2], [Mc1, Thm. A.1].

obtain a sequence of bounded currents $\xi_n \in P(\omega_n)$ such that $\langle \beta, \xi_n \rangle_X \to 1$. Pass to a subsequence such that $\xi_n \to \xi \in P(\omega)$.

Our goal is to show that $\xi = \beta$. To this end, note that X naturally decomposes into a pair of horizontal cylinders $A_1 \cup A_2$, corresponding to the two rectangles in Figure 2. Using the fact that the slope of L_n tends to zero, it is easy to see that L_n is equidistributed in each cylinder individually. This implies that $\xi | A_i$ is a multiple of $\beta | A_i$ for $i = 1, 2$, and thus

$$\xi = c_1(\beta | A_1) + c_2(\beta | A_2) \tag{6.1}$$

for some $c_1, c_2 \geq 0$.

Now we use the fact that $N(s_n) \to \infty$. Let $K \subset V \subset \mathcal{M}_2$ be the compact set obtained by deleting a small neighborhood of the cusp of the Teichmüller curve $V = \mathbb{H}/\Gamma$. The length of the continued fraction $N(s_n)$ essentially counts the number of times the Teichmüller ray generated by (X, ω_n) makes an excursion into the cusp; hence the amount of time T_n it spends in K is comparable to $N(s_n) \to \infty$. Applying Theorem 1.2, we find that

$$\|\xi_n - \beta_n\|_X \to 0,$$

and hence $[\xi] = [\beta]$ in $H^1(X, \mathbb{R})$. Using (6.1) we can then conclude that $\xi = \beta$ as currents, and hence L_n becomes uniformly distributed on X as $n \to \infty$. ∎

Sample slopes. The first two examples in Figure 1 depict periodic billiard trajectories in P for the sequence of slopes $s_n = 1/(n\gamma)$. In the first example the trajectories start near the right endpoint of the bottom edge of P, and they all lie in A_2; in the second example, they start near the left endpoint, and they converge to a limiting measure that is not uniform (it assigns too much mass to A_1). The third example is uniformly distributed; it corresponds to the sequence of slopes with continued fractions $s_n = [0, n, 1, 1, \ldots, 1]$ with $N(s_n) = n + 3$.

A Appendix: Variation of the Hodge norm

We will show that a classical result of Ahlfors gives:

Theorem A.1 *Fix a cohomology class $C \in H^1(\Sigma_g, \mathbb{R})$. Consider any $(X, \omega) \in \Omega\mathcal{T}_g$ such that $[\operatorname{Re}\omega] = C$. Then for any variation of $X \in \mathcal{T}_g$, the Hodge norm of C satisfies*

$$(\|C\|_X^2)^\bullet = -2 \operatorname{Re}\langle \omega^2, \dot{X} \rangle. \tag{A.1}$$

Here a variation in X is described by a smooth path $X(t)$ in $\mathcal{T}_g$ with $X(0) = X$. We use the shorthand $\dot{X} = X'(0)$, and adopt a similar convention for other quantities that depend on t. Note that the quadratic differential ω^2 represents a cotangent vector to $\mathcal{T}_g$ at X, so it pairs naturally with the tangent vector $\dot{X}$ as in equation (2.3).

Proof. Fix a standard symplectic basis $(a_1, \ldots, a_g), (b_1, \ldots, b_g)$ for $H^1(\Sigma_g, \mathbb{R})$. The associated *Siegel period matrix* for X is defined by

$$\tau_{ij} = \int_{b_j} \omega_i,$$

where $(\omega_1, \ldots, \omega_g)$ is the basis for $\Omega(X)$ characterized by $\int_{a_i} \omega_j = \delta_{ij}$. The matrix $\sigma_{ij} = \operatorname{Im} \tau_{ij}$ is symmetric and positive–definite, and the norm of a general form $\eta = \sum_1^g s_i \omega_i \in \Omega(X)$ is given by

$$\|\eta\|_X^2 = \int_X |\eta|^2 = s^t \sigma \overline{s}.$$

Since equation (A.1) is homogeneous, we can assume $\|\omega\|_X = \|C\|_X = 1$. We can then choose a symplectic basis such that $\omega_1 = \omega$. With this normalization, we have

$$\langle C, a_i \rangle = \operatorname{Re} \int_{a_i} \omega_1 = \delta_{i1}. \tag{A.2}$$

Now consider a variation $X(t)$ of X. Then ω_i, τ_{ij} and σ_{ij} vary as well. By Ahlfors' variational formula [Ah, eq. (7)], we have

$$\dot{\tau}_{ij} = -2i \, \langle \omega_i \omega_j, \dot{X} \rangle. \tag{A.3}$$

Let $\omega(t) = \sum s_i(t) \omega_i(t)$ be the unique form in $\Omega(X(t))$ satisfying $[\operatorname{Re} \omega(t)] = C$. Then

$$(\|C\|_X^2)^{\bullet} = (s^t \sigma \overline{s})^{\bullet}.$$

Since $\omega(0) = \omega_1$, we have $s_i(0) = \delta_{i1}$. By equation (A.2) we also have $\operatorname{Re} s_i = \delta_{i1}$, and hence $\operatorname{Re} \dot{s} = 0$. Using the fact that $\sigma^t = \sigma$, this gives

$$\dot{s}^t \sigma \overline{s} + s^t \sigma \dot{\overline{s}} = 2(\operatorname{Re}(\dot{s})^t \sigma \overline{s}) = 0,$$

and therefore

$$(s^t \sigma \overline{s})^{\bullet} = s \dot{\sigma} \overline{s} = \dot{\sigma}_{11}.$$

Formula (A.1) then follows directly from Ahlfors variational formula (A.3). ∎

Here is an equivalent formulation, used §4:

Corollary A.2 *For any nonzero $C \in H^1(\Sigma_g, \mathbb{R})$, we have*

$$(\log \|C\|_X)^{\bullet} = -\operatorname{Re}\langle \omega^2, \dot{X} \rangle,$$

where $[\operatorname{Re}\omega] = C/\|C\|_X$.

Notes and references. A variant of Theorem A.1, with a different proof, is given in [Fo, Lemma 2.1′]. A precursor to (A.3) appears in [Ra, (7)], where the factor $1/2\pi i$ should be replaced by 1. Ahlfors' formula (A.3) is sometimes stated without the factor $-2i$, which results from the identity $dz \wedge d\overline{z} = -2i|dz|^2$.

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