Prym varieties and Teichmüller curves

Curtis T. McMullen*

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Abstract

This paper gives a uniform construction of infinitely many primitive Teichmüller curves $V \subset \mathcal{M}_g$ for $g = 2$, 3 and 4.

Contents

1 Introduction

A *Teichmüller curve* $V \subset \mathcal{M}_g$ is a totally geodesic algebraic curve in the moduli space of Riemann surfaces of genus g.

We say V is *primitive* if it is generated by an Abelian differential of minimal genus. Every Teichmüller curve has a unique primitive representative in its commensurability class (see §2).

In this paper we will show:

Theorem 1.1 *There exist infinitely many primitive Teichmüller curves* $V \subset \mathcal{M}_g$ *for* $g = 2$, 3 *and* 4.

The case $g = 2$ was treated previously in [Mc1] using Jacobians with real multiplication. In this paper we use Prym varieties and pseudo-Anosov diffeomorphisms to present a uniform construction for genus $2 \leq g \leq 4$.

We also construct new closed, $\mathrm{SL}_2(\mathbb{R})$-invariant loci in $\Omega\mathcal{M}_g$ for $g \leq 5$, and new Teichmüller curves generated by strictly quadratic differentials.

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Prym varieties. Let $\Omega(X)$ denote the vector space of holomorphic 1-forms on $X \in \mathcal{M}_g$, and let $\rho : X \to X$ be a holomorphic involution. The variety

$$\mathrm{Prym}(X, \rho) = (\Omega(X)^-)^*/H_1(X, \mathbb{Z})^-$$

is the subtorus of the Jacobian of X determined by the negative eigenspace of ρ.

Let $\mathcal{O}_D \cong \mathbb{Z}[T]/(T^2 + bT + c)$, $D = b^2 - 4c$, be the real quadratic order of discriminant $D > 0$. An Abelian variety A admits *real multiplication by* $\mathcal{O}_D$ if $\dim_{\mathbb{C}} A = 2$ and $\mathcal{O}_D$ occurs as an indivisible, self-adjoint subring of $\mathrm{End}(A)$.

Let $W_D \subset \mathcal{M}_g$ denote the locus of all Riemann surfaces X such that

- there is a holomorphic involution $\rho : X \to X$,
- $\mathrm{Prym}(X, \rho)$ admits real multiplication by $\mathcal{O}_D$, and
- there is an eigenform $\omega \in \Omega(X)^-$ for the action of $\mathcal{O}_D$, with a unique zero on X (of multiplicity $2g - 2$).

The curve W_D is nonempty only if $g = 2, 3$ or 4, as can be seen by considering the genus of X/ρ (§3).

We can now state a more precise form of Theorem 1.1.

Theorem 1.2 *The locus $W_D \subset \mathcal{M}_g$ is a finite union of Teichmüller curves, all of which are primitive provided that D is not a square.*

Theorem 1.3 *For $g = 2$, 3 or 4, there are infinitely many primitive Teichmüller curves contained in $\bigcup W_D \subset \mathcal{M}_g$.*

We will also see that every irreducible component V of $W_D \subset \mathcal{M}_g$ is commensurable to another Teichmüller curve $V' \subset \mathcal{M}_g$ that is generated by a strictly quadratic differential (§6).

Curve systems. To prove Theorem 1.2, we show the locus of Prym eigenforms is closed and invariant under the Teichmüller geodesic flow (§3). For Theorem 1.3 we use pseudo-Anosov mappings to construct explicit examples of Prym eigenforms with varying discriminants (§5).

The examples in genus 2, 3 and 4 correspond to the L, S and X–shaped polygons in the first column of Figure 1. Each such polygon Q determines a holomorphic 1-form $(X, \omega) = (Q, dz)/\sim$ by identifying opposite sides. There is a unique involution $\rho : X \to X$ making ω into a Prym form, given by the

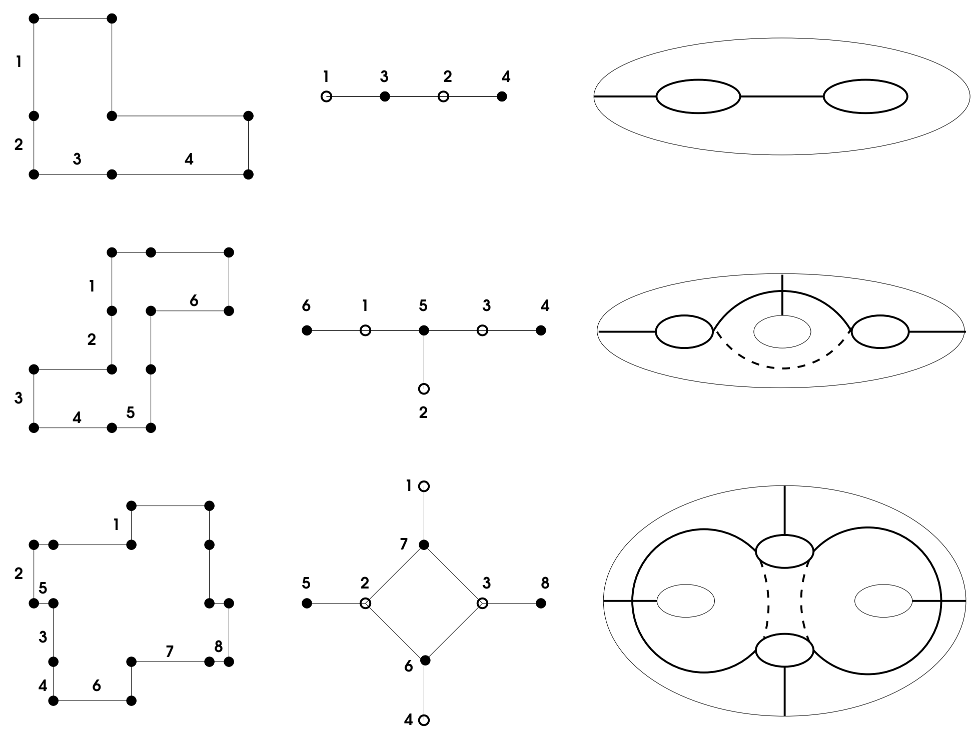

Figure 1. Generators for Teichmüller curves in genus 2, 3 and 4.

hyperelliptic involution when X has genus 2, and by a 180° rotation of Q when X has genus 3 or 4.

The vertical and horizontal foliations of ω decompose X into cylinders X_i, whose intersection matrix C_{ij} is recorded by the adjacent Coxeter graph. The last column gives a topological picture of the configuration of cylinders.

Given positive integer weights $m_i > 0$, symmetric under ρ, we can choose the heights $h_i > 0$ of X_i so they form an eigenvector satisfying

$$\mu h_i = \sum_j m_i C_{ij} h_j.$$

Then by [Th], a suitable product of Dehn twists gives an affine, pseudo-Anosov mapping

$$\phi : (X, \omega) \to (X, \omega).$$

The endomorphism $T = \phi_* + \phi_*^{-1}$ of $\mathrm{Prym}(X, \rho)$ makes ω into an eigenform for real multiplication by $\mathcal{O}_D$, where $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\mu^2)$, and thus $W_D \neq \emptyset$. By varying the weights (m_i) we obtain infinitely many different fields $\mathbb{Q}(\mu^2)$, and hence we obtain infinitely many primitive Teichmüller curves $V \subset \bigcup W_D$.

Classification. The primitive Teichmüller curves in genus 2 are classified in [Mc2] and [Mc3]; in particular, it is shown (using [Mo]) that there is only one such curve lying outside $\bigcup W_D \subset \mathcal{M}_2$. It would be interesting to have a similar analysis in genus 3 and 4.

Billiards. We close with a remark connecting these Teichmüller curves to billiards in *rational polygons* (those whose angles are rational multiples of π).

Via an unfolding construction, any rational polygon P gives rise to a holomorphic 1-form (X, ω), which in turn generates a complex geodesic $f : \mathbb{H} \to \mathcal{M}_g$. If f happens to cover a Teichmüller curve $V = f(\mathbb{H})$, then P is a *lattice polygon of genus g*. By [V1], the billiard flow in a lattice polygon has optimal dynamics: for example, every trajectory is either periodic or uniformly distributed.

The lattice polygons of genus 2 are classified in [Mc3]. At present only finitely many primitive lattice polygons are known in each genus $g \geq 3$. Letting $T(p, q)$ denote the triangle with internal angles $(\pi/p, \pi/q, \pi(1-1/p-1/q))$, the known examples include:

1. The triangles $T(3, 4)$ and $T(4, 8)$ in genus 3, and
2. The triangles $T(5, 10)$ and $T(3, 12)$ in genus 4.

In fact $T(n, 2n)$ is a lattice triangle for all $n \geq 3$ [Wa], [Vo2]; and $T(3,4)$ is the first of three sporadic examples associated to the Coxeter diagrams E_6, E_7 and E_8 [Vo1], [KS], [Lei, §7]. The final example $T(3,12)$ was recently discovered by W. P. Hooper [Ho].

Using Corollary 3.6, it is straightforward to check that each triangle above generates a Teichmüller curve V belonging to the infinite family $\bigcup W_D \subset \mathcal{M}_g$. Thus these four triangles furnish particular instances of Theorem 1.2.

2 Teichmüller curves

This section presents general results on holomorphic 1-forms, quadratic differentials and Teichmüller curves; for additional background, see [KZ], [Mc4, §3] and references therein.

Holomorphic 1-forms. Let $\mathcal{M}_g$ denote the moduli space of compact Riemann surfaces of genus g, and let $\Omega(X)$ denote the space of holomorphic 1-forms (Abelian differentials) on $X \in \mathcal{M}_g$. The nonzero 1-forms assemble to form a bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ (in the orbifold sense), which admits a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ satisfying

$$\int_C A \cdot \omega = A \left(\int_C \omega \right).$$

Strata. The action of $\mathrm{SL}_2(\mathbb{R})$ leaves invariant the *strata* $\Omega\mathcal{M}_g(p_i) \subset \Omega\mathcal{M}_g$, consisting of forms with zeros of multiplicities $(p_1, \ldots, p_n)$, $\sum p_i = 2g - 2$.

Affine automorphisms. The stabilizer of a given form is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, denoted $\mathrm{SL}(X, \omega)$.

A homeomorphism $f : (X, \omega) \to (Y, \eta)$ is *affine* if it has the form

$$f(x + iy) = (ax + by) + i(cx + dy)$$

in local charts where $z = x + iy$ and $\omega = \eta = dz$. The group $\mathrm{SL}(X, \omega)$ can also be described as the image of the group of orientation-preserving affine automorphisms under the derivative map

$$D : \mathrm{Aff}^+(X, \omega) \to \mathrm{SL}_2(\mathbb{R})$$

given by $Df = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We say $p \in X$ is a *periodic point* if its stabilizer has finite index in $\mathrm{Aff}^+(X, \omega)$.

Primitive and commensurable forms. A *covering map* $f : (X, \omega) \to (Y, \eta)$ is a surjective holomorphic map $f : X \to Y$ such that $f^*(\eta) = \omega$. The map f is allowed to have branch points, which necessarily lie in the zero set of ω.

A holomorphic 1-form is *primitive* if it does not cover a form of smaller genus.

The 1-forms (X_1, ω_1) and (X_2, ω_2) are *commensurable* if there is a compact Riemann surface Y and a pair of surjective holomorphic maps $f_i : Y \to X_i$ such that

(i) $f_1^*(\omega_1) = f_2^*(\omega_2)$, and
(ii) the map f_i is branched over periodic points of (X_i, ω_i).

In this case the groups $\mathrm{SL}(X, \omega_i)$, $i = 1, 2$ are also commensurable (see e.g. [GJ]).

Let $g(X)$ denote the genus of X. By [Mo, Thm. 2.6], we have:

Theorem 2.1 *Every holomorphic 1-form admits a covering map*

$$f : (X, \omega) \to (X_0, \omega_0)$$

to a primitive form, which is unique provided $g(X_0) > 1$.

Proof. Consider the directed system of all closed subgroups $B \subset \mathrm{Jac}(X)$ such that $\omega | B = 0$, and let X_B denote the normalization of the image of X in $\mathrm{Jac}(X)/B$. Clearly (X, ω) covers a unique form (X_B, ω_B), and every quotient (Y, η) of (X, ω) eventually occurs, by taking $B = \mathrm{Ker}(\mathrm{Jac}(X) \to \mathrm{Jac}(Y))$.

If $g(X_B) > 1$ for all B, then (X_B, ω_B) must stabilize as B increases, and its stable value is the unique primitive form (X_0, ω_0) covered by (X, ω). ■

The primitive form (X_0, ω_0) can also be made unique when it has genus one, by adding the requirement that f_* is surjective on π_1.

Corollary 2.2 *The map f induces an inclusion* $\mathrm{SL}(X, \omega) \hookrightarrow \mathrm{SL}(X_0, \omega_0)$.

Proof. By uniqueness, we have $(Y_0, \eta_0) = A \cdot (X_0, \omega_0)$ whenever $(Y, \eta) = A \cdot (X, \omega)$, which implies $\mathrm{SL}(X, \omega) \hookrightarrow \mathrm{SL}(X_0, \omega_0)$. ■

Corollary 2.3 *If* $\mathrm{SL}(X, \omega)$ *is a lattice, then f is branched over periodic points of* (X_0, ω_0).

Proof. In this case the maps that lift to X form a subgroup G of finite index in $\mathrm{Aff}^+(X_0, \omega_0)$, and G permutes the branch points of f. ■

Trace fields. The *trace field* of $\mathrm{SL}(X,\omega)$ is given by

$$K = \mathbb{Q}(\operatorname{tr} A \ : \ A \in \mathrm{SL}(X,\omega)).$$

Theorem 2.4 *If $A \in \mathrm{SL}(X,\omega)$ is a hyperbolic element, then:*

1. *The trace field of $\mathrm{SL}(X,\omega)$ is $K = \mathbb{Q}(\operatorname{tr} A)$, and*
2. *The periods of ω span a 2-dimensional vector space over K.*

See [KS, Appendix], [Mc4, §9].

Quadratic differentials. Let $Q(X)$ denote the space of holomorphic quadratic differentials $q(z)\,dz^2$ on $X \in \mathcal{M}_{g,n}$, with at most simple poles at the marked points of X. The nonzero quadratic differentials form a bundle $Q\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$, admitting a natural action of $\mathrm{PSL}_2(\mathbb{R})$. The stabilizer of a given differential is denoted $\mathrm{PSL}(X,q)$. The definitions of the affine group, periodic points and commensurable forms generalize naturally.

Teichmüller curves. The projection to moduli space of an orbit

$$\mathrm{PSL}_2(\mathbb{R}) \cdot (X,q) \subset Q\mathcal{M}_g$$

yields a holomorphic *complex geodesic* $f : \mathbb{H} \to \mathcal{M}_g$. If $\mathrm{PSL}(X,q)$ is a lattice, then f factors through the quotient $V = \mathbb{H}/\,\mathrm{PSL}(X,q)$, and (X,q) generates a *Teichmüller curve*

$$f : V \to \mathcal{M}_g.$$

Squares of 1-forms. The *squaring map* $s : \Omega\mathcal{M}_g \to Q\mathcal{M}_g$, given by $s(X,\omega) = (X,\omega^2)$, is equivariant with respect to the actions of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$. In particular, we have

$$\mathrm{PSL}(X,\omega^2) = \mathrm{SL}_2(X,\omega)/(\pm I),$$

so if one group is a lattice, so is the other.

Double covers. A differential (X,q) is *strictly quadratic* if it is not the square of a 1-form. In this case the Riemann surface of $\sqrt{q}$ determines a canonical branched double cover $\pi : (Y,\omega^2) \to (X,q)$ on which q becomes a square. The branch points of π correspond to the simple poles and zeros of odd order of q.

Theorem 2.5 *The groups $\mathrm{PSL}(Y,\omega^2)$ and $\mathrm{PSL}(X,q)$ are commensurable.*

Proof. It suffices to prove the corresponding statement for affine groups.

The branch points of π are poles and zeros of q, hence they are periodic points; thus a subgroup of finite index in $\mathrm{Aff}^+(X,q)$ lifts to $\mathrm{Aff}^+(Y,\omega)$.

For the converse, let ρ be the involution generating the deck group of Y over X. Then ρ belongs to the finite set $F = \{\sigma \in \mathrm{Aff}^+(Y,\omega) \, : \, D\sigma = -I\}$. Since $-I$ is in the center of $\mathrm{SL}_2(\mathbb{R})$, we have $\phi F \phi^{-1} = F$ for all affine automorphisms ϕ, and thus the centralizer G of ρ has finite index in $\mathrm{Aff}^+(Y,\omega)$. Since it commutes with ρ, G descends to $\mathrm{Aff}^+(X,q)$. ■

Using the preceding result to reduce to the case of 1-forms, we obtain:

Corollary 2.6 *If* (X_i, q_i), $i = 1, 2$ *are commensurable quadratic differentials, then the groups* $\mathrm{PSL}(X_i, q_i)$ *are also commensurable.*

Corollary 2.7 *Every quadratic differential is commensurable to the square of a primitive holomorphic 1-form* (X,ω)*, which is unique up to sign unless* $g(X) = 1$.

Primitive curves. A Teichmüller curve $f : V \to \mathcal{M}_g$ is *primitive* if it is generated by the square of a primitive 1-form, and two Teichmüller curves are *commensurable* if they are generated by commensurable quadratic differentials. The preceding results imply:

Theorem 2.8 *Every Teichmüller curve is commensurable to a unique primitive Teichmüller curve* $f : V \to \mathcal{M}_g$.

(Note that all forms of genus one generate the same Teichmüller curve, namely $f : \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \xrightarrow{\sim} \mathcal{M}_1$.)

3 Prym varieties

In this section we introduce Prym varieties and their eigenforms.

Prym varieties. Let X be a compact Riemann surface equipped with an automorphism $\rho : X \to X$ of order two. The action of ρ determines a splitting

$$\Omega(X) = \Omega(X)^- \oplus \Omega(X)^+$$

of the Abelian differentials into even and odd forms, as well as sublattices $H_1(X,\mathbb{Z})^{\pm} \subset H_1(X,\mathbb{Z})$ consisting of even and odd cycles. The *Prym variety*

$$P = \mathrm{Prym}(X,\rho) = (\Omega(X)^-)^*/H_1(X,\mathbb{Z})^-$$

is the sub-Abelian variety of $\mathrm{Jac}(X) = \Omega(X)^*/H_1(X, \mathbb{Z})$ determined by the forms which are odd with respect to ρ.

We refer to $\Omega(X)^- \cong \Omega(P)$ as the space of *Prym forms* on (X, ρ). Note that $\Omega(X)^+$ is canonically isomorphic to $\Omega(Y)$, $Y = X/\rho$, and thus $\dim P = g(X) - g(Y)$.

The variety P is canonically polarized by the intersection pairing on $H_1(X, \mathbb{Z})^-$.

Examples.

1. Let ρ be the canonical involution on a hyperelliptic Riemann surface. Then $\mathrm{Prym}(X, \rho) = \mathrm{Jac}(X)$, since every form on X is odd.

2. Let ρ be the deck transformation of an unramified double covering $\pi : X \to Y$. Then $g(Y) = (g(X) + 1)/2$, and therefore $\dim \mathrm{Prym}(X, \rho) = (g(X) - 1)/2$.

3. Let (Y, q) be a strictly quadratic differential. Then the Riemann surface of $\sqrt{q}$ provides a canonical double covering $\pi : X \to Y$, with deck transformation ρ, such that $\pi^*(q) = \omega^2$ for some Prym form ω.

By Riemann-Hurwitz, we have:

Theorem 3.1 *If* $\dim \mathrm{Prym}(X, \rho) = h$, *then* $h \le g(X) \le 2h + 1$.

Examples (1) and (2) above provide the extreme values of h.

Real multiplication. Let $D > 0$ be an integer congruent to 0 or 1 mod 4, and let

$$\mathcal{O}_D \cong \mathbb{Z}[T]/(T^2 + bT + c),$$

$D = b^2 - 4c$, be the real quadratic order of discriminant D.

Let $\mathrm{End}(P)$ denote the endomorphism ring of a polarized Abelian variety $P \cong \mathbb{C}^g/L$. We can regard elements of $\mathrm{End}(P)$ as complex-linear maps $T : \mathbb{C}^g \to \mathbb{C}^g$ such that $T(L) = L$. An endomorphism is *self-adjoint* if it satisfies

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

with respect to the symplectic pairing on $x, y \in L$.

The variety P admits *real multiplication by* $\mathcal{O}_D$ if $\dim_{\mathbb{C}} P = 2$, and there is a proper subring $R \cong \mathcal{O}_D \subset \mathrm{End}(L)$ generated by a self-adjoint endomorphism T. (Here *proper* means if $U \in \mathrm{End}(L)$, and $nU \neq 0$ belongs to R, then $U \in R$.)

Prym eigenforms. Now suppose $P = \mathrm{Prym}(X, \rho)$ is a Prym variety with real multiplication by $\mathcal{O}_D$. Then $\mathcal{O}_D$ also acts on $\Omega(P) \cong \Omega(X)^-$, and the space of odd forms splits into a pair of one-dimensional eigenspaces. We say $\omega \in \Omega(X)^-$ is a *Prym eigenform* if $0 \neq \mathcal{O}_D \cdot \omega \subset \mathbb{C}\omega$.

Let $\Omega E_D \subset \Omega\mathcal{M}_g$ denote the space of all Prym eigenforms for real multiplication by $\mathcal{O}_D$. (Note that (X, ω) may be an eigenform in more than one way; in general, neither ρ nor the action of $\mathcal{O}_D$ is uniquely determined by ω).

By Theorem 3.1, $\Omega E_D \neq \emptyset$ only if $2 \leq g \leq 5$. Eigenforms of these genera are supplied by Theorem 5.1 and Corollary 6.3 below.

Theorem 3.2 *The locus ΩE_D of Prym eigenforms for real multiplication by $\mathcal{O}_D$ is a closed,* $\mathrm{SL}_2(\mathbb{R})$*-invariant subset of $\Omega\mathcal{M}_g$.*

Proof. Let $(X_n, \omega_n) \in \Omega E_D$ converge to $(X, \omega) \in \Omega\mathcal{M}_g$. Each ω_n is odd with respect to an involution ρ_n of X_n, and each is an eigenform for a self-adjoint endomorphism T_n of $\mathrm{Prym}(X_n, \rho_n)$ generating $\mathcal{O}_D$. Writing $D = b^2 - 4c$, we can assume $T_n^2 + bT_n + c = 0$.

Passing to a subsequence, we can assume $\rho_n \to \rho \in \mathrm{Aut}(X)$ and thus

$$P_n = \mathrm{Prym}(X_n, \rho_n) \to P = \mathrm{Prym}(X, \rho)$$

as an Abelian variety. Since the eigenvalues of T_n are independent of n, the norm of its derivative with respect to the Kähler metric on P_n is constant, and hence $T_n \to T \in \mathrm{End}(P)$ after passing to a subsequence. Consequently (X, ω) is a Prym eigenform for T, and hence ΩE_D is closed in $\Omega\mathcal{M}_g$.

Next we show $\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega) \subset \Omega E_D$. Note that the space of odd forms on (X, ρ) splits into a pair of complex lines,

$$\Omega(X)^- = (\mathbb{C}\omega) \oplus (\mathbb{C}\omega)^\perp,$$

mapping under $\eta \mapsto \mathrm{Re}(\eta)$ to the two-dimensional eigenspaces

$$H^1(X, \mathbb{R})^- = S_1 \oplus S_2$$

of T.

Let $(X', \omega') = A \cdot (X, \omega)$ for some $A \in \mathrm{SL}_2(\mathbb{R})$. Then there is a canonical real-affine mapping $\phi : X \to X'$ with derivative $D\phi = A$ [Mc1, §3]. Since $D\rho = -I$ is in the center of $\mathrm{SL}_2(\mathbb{R})$, ω' is odd with respect to the automorphism $\rho' = \phi \circ \rho \circ \phi^{-1}$ of X'. The isomorphism

$$\phi_* : H^1(X, \mathbb{R}) \to H^1(X', \mathbb{R})$$

sends ρ to ρ', and sends T to an automorphism T' of $H^1(X', \mathbb{R})^-$ with eigenspaces $S_i' = \phi_*(S_i)$.

Since ϕ is affine, the forms $(\operatorname{Re}\omega', \operatorname{Im}\omega')$ span S_1', and thus $S_1' \oplus S_2'$ corresponds to splitting $\mathbb{C}\omega' \oplus (\mathbb{C}\omega')^\perp$ under the isomorphism $H^1(X', \mathbb{R})^- \cong \Omega(X')^-$. Thus T' determines a holomorphic endomorphism of $\operatorname{Prym}(X', \rho')$, and hence ω' is an eigenform for real multiplication by $\mathcal{O}_D$. ■

Weierstrass curves. One can obtain finer $\mathrm{SL}_2(\mathbb{R})$-invariant sets by intersecting the eigenform locus with the strata $\Omega\mathcal{M}_g(p_i)$. The smallest such intersection is the *Weierstrass locus*

$$\Omega W_D = \Omega E_D \cap \Omega\mathcal{M}_g(2g-2),$$

consisting of eigenforms with a unique zero. (When $g = 2$, this zero is located at a Weierstrass point of X.) The projection of ΩW_D to moduli space is the *Weierstrass curve* W_D.

Lemma 3.3 *The locus $W_D \subset \mathcal{M}_g$ is nonempty only if $g = 2, 3$ or 4.*

Proof. We have $\Omega E_D \neq \emptyset$ only if $2 \leq g \leq 5$; and if $g = 5$ then $X \mapsto X/\rho$ is an unramified double cover, and therefore ω has an even number of zeros so it cannot belong to ΩW_D. ■

Theorem 3.4 *The Weierstrass curve $W_D \subset \mathcal{M}_g$ is a finite union of Teichmüller curves. Each such curve is primitive, provided D is not a square.*

Proof. Since $\Omega\mathcal{M}_g(2g-2)$ is closed in $\Omega\mathcal{M}_g$, so is ΩW_D. Thus W_D is a closed algebraic subset of $\mathcal{M}_g$.

To see $\dim_{\mathbb{C}} W_D = 1$, let $(X_0, \omega_0) \in \Omega W_D$ be a Prym eigenform for the involution $\rho \in \operatorname{Aut}(X_0)$ and the endomorphism T of $H^1(X_0, \mathbb{C})$ generating $\mathcal{O}_D$. Let $S \subset H^1(X_0, \mathbb{C})^-$ be the 2-dimensional eigenspace of T containing $[\omega]$, and recall that the period map gives a local holomorphic homeomorphism between $\Omega\mathcal{M}_g(2g-2)$ and $H^1(X_0, \mathbb{C})$ [V2], [MS, Lemma 1.1]. This map sends the nearby Prym eigenforms for (T, ρ) into S; thus $\dim \Omega W_D = 2$ and $\dim W_D = 1$.

Since ΩW_D is $\mathrm{SL}_2(\mathbb{R})$-invariant, its projection to $\mathcal{M}_g$ is a union of complex Teichmüller geodesics, and hence each irreducible component V of W_D is a Teichmüller curve.

Now suppose $(X, \omega) \in \Omega W_D$ has the form $\pi^*(Y, \eta)$, where $\pi : X \to Y$ has degree $d \geq 2$. Provided D is not a square we can assume $g(Y) \geq 2$. Since ω

has a unique zero p, so does η; thus the map π is totally ramified at p and unbranched elsewhere. This implies $\chi(X) = d\chi(Y) - d + 1$; therefore d is odd and $\chi(X) \leq 3(-2) - 2 = -8$, contradicting the fact that $g(X) \leq 4$. Thus (X, ω) must be primitive, so the Teichmüller curve it generates is primitive as well. ■

The case $D = d^2$. We remark that if (X, ω) belongs to ΩW_{d^2}, then $\mathrm{Prym}(X, \rho)$ is isogenous to a product of elliptic curves, and ω is the pullback of a form of genus one. In particular, the Teichmüller curves in W_{d^2} are never primitive.

Traces and eigenforms. We conclude with a useful criterion for recognizing eigenforms.

Theorem 3.5 *Let $\omega \neq 0$ be a Prym form with $\dim \mathrm{Prym}(X, \rho) = 2$, and suppose there is a hyperbolic element $A \in \mathrm{SL}(X, \omega)$. Then we have*

$$(X, \omega) \in \Omega E_D,$$

where $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\mathrm{tr}\, A)$.

Proof. Let $\phi : (X, \omega) \to (X, \omega)$ be an affine automorphism with derivative $D\phi = A$. Since $D\rho = -I$ is in the center of $\mathrm{SL}_2(\mathbb{R})$, the map $\phi^n \rho \phi^{-n}$ is a holomorphic involution on X for every n. The set of such involutions is finite, so after replacing ϕ with ϕ^n if necessary we can assume $\phi\rho = \rho\phi$. Then $H^1(X, \mathbb{R})^-$ splits into a pair of 2-dimensional eigenspaces $S \oplus S^\perp$ under the action of $T = \phi_* + \phi_*^{-1}$, where $S = \langle \mathrm{Re}\, \omega, \mathrm{Im}\, \omega \rangle$. Since S maps to a complex line under the isomorphism $H^1(X, \mathbb{R})^- \cong \Omega(X)^-$, so does $S^\perp$, and thus T gives a self-adjoint endomorphism of $P = \mathrm{Prym}(X, \rho)$. This endomorphism satisfies $T^* \omega = (\mathrm{tr}\, A)\omega$.

Now note that $\mathrm{tr}\, A$ is an eigenvalue of multiplicity two for the action of T on $H^1(X, \mathbb{Z})^- \cong \mathbb{Z}^4$. Thus

$$K = \mathbb{Q}(T) \subset \mathrm{End}(P) \otimes \mathbb{Q}$$

is either a real quadratic field (isomorphic to $\mathbb{Q}(\mathrm{tr}\, A)$), or a copy of $\mathbb{Q} \oplus \mathbb{Q}$ (if $\mathrm{tr}\, A \in \mathbb{Q}$). Since ω is an eigenform for the order $K \cap \mathrm{End}(P)$, it belongs to ΩE_D for some D with $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\mathrm{tr}\, A)$. ■

Corollary 3.6 *Suppose $\omega \neq 0$ is a Prym form with a unique zero, and $\dim \mathrm{Prym}(X, \rho) = 2$. Then the following are equivalent.*

1. *There is a hyperbolic element $A \in \mathrm{SL}(X, \omega)$.*

2. *The group $\mathrm{SL}(X, \omega)$ is a lattice.*

3. *The form (X, ω) generates a Teichmüller curve $V \subset W_D$, and $\mathbb{Q}(\sqrt{D})$ is the trace field of $\mathrm{SL}(X, \omega)$.*

Proof. Clearly (3) $\implies$ (2) $\implies$ (1), and (1) implies that $(X, \omega) \in \Omega W_D$ by the preceding result. ∎

Remark. The results of this section were established for genus 2 in [Mc1] (see also [Mc4]); the proofs above follow similar lines. Additional background on Prym varieties can be found in [Mum] and [BL, Ch. 12].

4 Curve systems

In this section we discuss Thurston's construction of quadratic differentials such that $\mathrm{PSL}(X, q)$ contains a pair of independent parabolic elements.

Multicurves. Let Z_g be a closed, oriented topological surface of genus g. A *multicurve* $A \subset Z_g$ is a union of disjoint, essential, simple closed curves, no two of which bound an annulus.

A pair of multicurves A, B *bind* the surface Z_g if they meet only in transverse double points, and every component of $Z - (A \cup B)$ is a polygonal region with at least 4 sides (running alternately along A and B).

Indexing the components of A and B so that $A = \bigcup_1^a \gamma_i$ and $B = \bigcup_{a+1}^{a+b} \gamma_i$, we can form the symmetric matrix

$$C_{ij} = i(\gamma_i, \gamma_j) = \begin{cases} |\gamma_i \cap \gamma_j|, & \text{if } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient to record this incidence matrix by a *Coxeter graph* $\Gamma(A, B)$, with $a + b$ vertices and C_{ij} edges from vertex i to vertex j (compare [Lei]).

Quadratic differentials. Next we assign a positive weight m_i to each curve γ_i (or equivalently to each vertex of the Coxeter graph $\Gamma(A, B)$). Let

$$\mu = \sigma(m_i C_{ij}) > 0$$

be the spectral radius of the matrix $(m_i C_{ij})$. By the Perron-Frobenius theorem there is a positive eigenvector (h_i), unique up to scale, satisfying

$$\mu h_i = \sum_1^{a+b} m_i C_{ij} h_j.$$

Take one rectangle

$$R_p = [0, h_i] \times [0, h_j] \subset \mathbb{C}$$

for each intersection point $p \in \gamma_i \cap \gamma_j$, $i < j$, and glue (R_p, dz^2) to (R_q, dz^2) whenever p and q are joined by an edge of $A \cup B$. The result is a holomorphic quadratic differential $(X, q) \in Q\mathcal{M}_g$, with the following features:

- Up to a positive factor, (X, q) is uniquely determined by the data (A, B, m_i), and therefore inherits its symmetries.
- The horizontal and vertical foliations of q decompose X into cylinders X_i of height h_i homotopic to γ_i.
- We have $\operatorname{mod}(X_i) = m_i/\mu$. Indeed, the modulus of X_i is given by h_i/c_i, where
$$c_i = \sum i(\gamma_i, \gamma_j) h_j = \mu h_i / m_i.$$
- The zeros of q of multiplicity $k > 0$ correspond to the components of $Z_g - (A \cup B)$ with $2k + 4$ sides.
- We have $q = \omega^2$ for some $\omega \in \Omega(X)$ iff the curves (γ_i) can be oriented so their algebraic intersection numbers satisfy $\gamma_i \cdot \gamma_j = i(\gamma_i, \gamma_j)$ for all $i < j$.

Dehn twists. Now suppose the weights (m_i) are integers. Let τ_i denote a right Dehn twist around γ_i, and consider the products

$$\begin{aligned} \tau_A &= \tau_1^{m_1} \circ \cdots \circ \tau_a^{m_a}, \\ \tau_B &= \tau_{a+1}^{m_{a+1}} \circ \cdots \circ \tau_{a+b}^{m_{a+b}}. \end{aligned} \tag{4.1}$$

Then by [Th] we have:

Theorem 4.1 *The multitwists τ_A, τ_B are represented by affine automorphisms of (X, q) satisfying*

$$D\tau_A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad D\tau_B = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}.$$

Proof. The linear part of an affine right multitwist τ^m on a horizontal cylinder of modulus M satisfies $D\tau^m = \left(\begin{smallmatrix} 1 & m/M \\ 0 & 1 \end{smallmatrix}\right)$. Since $m_i/\operatorname{mod}(X_i) = \mu$ for all i, the linear parts of the factors of τ_A all agree; hence they assemble to form an affine automorphism of (X, q). The same applies to τ_B. ∎

Taking products of powers of τ_A and τ_B, it is easy to see:

Corollary 4.2 *The group* $\mathrm{PSL}(X, q)$ *contains infinitely many hyperbolic elements.*

Corollary 4.3 *The trace field of* $\mathrm{PSL}(X, q)$ *is* $\mathbb{Q}(\mu^2)$*,*

Proof. Choose n large enough that $\operatorname{tr} D(\tau_A^n \tau_B) = 2 - n\mu^2 < -2$ and apply Theorem 2.4. ∎

Note that $\deg(\mu^2/\mathbb{Q}) \leq \max(a, b)$, since the intersection matrix has the form $C = \left(\begin{smallmatrix} 0 & I \\ I^t & 0 \end{smallmatrix}\right)$.

Remark. The Coxeter graph $\Gamma(A, B)$ determines (A, B) up to finitely many choices. Indeed, a connected graph Γ plus a finite amount of ordering data determines a pattern for banding together cylinders to obtain a *ribbon graph* (in the sense of [Pen] or [Ko]). Each ribbon graph determines a surface with boundary; by gluing in disks, we obtain the finitely many curve systems satisfying $\Gamma(A, B) = \Gamma$.

5 Examples in genus 2, 3 and 4

In this section we use specific configurations of simple closed curves to show:

Theorem 5.1 *For* $g = 2, 3$ *or* 4*, there are infinitely many primitive Teichmüller curves contained in* $\bigcup W_D \subset \mathcal{M}_g$*.*

Prym systems. Let $A \cup B \subset Z_g$ be a binding pair of multicurves as in §4. We say (A, B) is a *Prym system* if:

1. There is a unique component of $Z_g - (A \cup B)$ with more than 4 sides;
2. The loops $(\gamma_i)_1^{a+b}$ forming A and B can be oriented so that $i(\gamma_i, \gamma_j) = \gamma_i \cdot \gamma_j$ for $i < j$;
3. There is an orientation-preserving symmetry $\rho : Z_g \to Z_g$ of order two, such that $\rho(A) = A$ and $\rho(B) = B$; and

4. The genus of Z_g/ρ is $g-2$.

Condition (1) implies ρ is unique up to isotopy.

Let $C_{ij} = i(\gamma_i, \gamma_j)$ as usual, and let $\lambda(A, B, m_i) = \sigma(m_i C_{ij})^2$.

Theorem 5.2 *For any system of positive integral weights (m_i), symmetric under ρ, the data (A, B, m_i) determines a 1-form*

$$(X, \omega) \in \Omega W_D$$

such that $\mathbb{Q}(\lambda(A, B, m_i)) = \mathbb{Q}(\sqrt{D})$.

Proof. Let (X, q) be the quadratic differential determined by (A, B, m_i) as in §4. It is unique up to a positive multiple. By condition (2) in the definition of a Prym system, there is a 1-form such that $(X, q) = (X, \omega^2)$. The form ω has a unique zero by (1). By (3) it satisfies $\rho^*(\omega) = -\omega$, so by (4) it is a Prym form with $\dim \operatorname{Prym}(X, \rho) = 2$.

According to Corollary 4.3, $\mathrm{SL}(X, \omega)$ has trace field $K = \mathbb{Q}(\lambda)$, where $\lambda = \sigma(m_i C_{ij})^2$. Moreover K is generated by the trace of a hyperbolic element, by Theorem 2.4 and Corollary 4.2. Thus $(X, \omega) \in \Omega E_D$ is a Prym eigenform, and $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\lambda)$ by Corollary 3.6. ■

Theorem 5.3 *Every Teichmüller curve $V \subset W_D$ can be specified by a suitable weighted Prym system (A, B, m_i).*

Proof. Let $s \neq t$ be slopes of saddle connections for a Prym eigenform (X, ω) generating V. Then the geodesics of slopes s and t decompose $(X, |\omega|)$ into cylinders (cf. [V1]), with core curves $A = \bigcup_1^a \gamma_i$ and $B = \bigcup_{a+1}^{a+b} \gamma_i$ that fill X. Since ω has a unique zero, there is a unique component of $X - (A \cup B)$ with more than 4 sides; and since ω^2 is ρ-invariant, so are A and B.

Let $C_{ij} = i(\gamma_i, \gamma_j)$. Parallel cylinders have rational ratios of moduli, and thus there exist integers $m_i > 0$ such that the multitwists τ_A, τ_B defined by (4.1) are affine automorphisms of (X, ω). The corresponding heights $h_i > 0$ are therefore an eigenvector for $(m_i C_{ij})$, and the weighted Prym system (A, B, m_i) determines V. ■

Prym systems of types L, S and X. By Theorem 3.4 the 1-form (X, ω) in Theorem 5.2 generates a Teichmüller curve

$$V(A, B, m_i) \subset W_D \subset \mathcal{M}_g,$$

which is primitive if D is not a square.

To prove Theorem 5.1, we give examples of Prym systems (A, B) for genus 2, 3 and 4. These are shown in Figure 1.

The first column shows a typical 1-form $(X, \omega) = (Q, dz)/\sim$ determined by the Prym system (A, B, ρ). Here the sides of $Q \subset \mathbb{C}$ are identified by pure translations, $z \mapsto z + x$ or $z \mapsto z + iy$. The vertices of Q are identified to a single point p, namely the unique zero of ω. The sides of Q labeled i form the heights of cylinders X_i homotopic to γ_i, where $A \cup B = \bigcup \gamma_i$.

There is a unique involution $\rho \in \mathrm{Aut}(X)$ fixing p. For the Prym systems of genus 3 and 4, ρ is realized by a 180° rotation about the center of Q; for genus 2, it is the hyperelliptic involution. In all cases ρ sends ω to $-\omega$.

The second column shows the Coxeter diagram $\Gamma(A, B)$, with vertices numbered by the corresponding cylinders $X_i \subset X$. The white and black vertices represent loops in A and B respectively. The involution ρ gives the unique order-two symmetry of $\Gamma(A, B)$ preserving the colors of vertices.

The final column shows the topological configuration of curves $A \cup B$ on a surface of genus g.

Based on the shape of the polygon Q, we call these the *Prym systems of types L, S and X.*

Weights. For brevity, let L_n, S_n and X_n denote the Prym systems (A, B, m_i) of types L, S and X with weights $m_i = n > 0$ on the endpoints of the Coxeter diagram, and $m_i = 1$ on the remaining vertices. (Concretely, this means $(m_i) = (n, 1, 1, n)$ for L_n, $(1, n, 1, n, 1, n)$ for S_n, and $(n, 1, 1, n, n, 1, 1, n)$ for X_n.)

Theorem 5.4 *For $n > 0$, we have*

$$\begin{aligned} \mathbb{Q}(\lambda(L_n)) &= \mathbb{Q}(\sqrt{1+4n}), \\ \mathbb{Q}(\lambda(S_n)) &= \mathbb{Q}(\sqrt{1+2n}) \quad \textit{and} \\ \mathbb{Q}(\lambda(X_n)) &= \mathbb{Q}(\sqrt{1+n}). \end{aligned}$$

Proof. A straightforward computation shows

$$\lambda(L_n) = \sigma(m_i C_{ij})^2 = \sigma \begin{pmatrix} 0 & 0 & n & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & n & 0 & 0 \end{pmatrix}^2 = \frac{1 + 2n + \sqrt{1+4n}}{2};$$

similarly, $\lambda(S_n) = 1 + n + \sqrt{1+2n}$ and $\lambda(X_n) = 2 + n + 2\sqrt{1+n}$. ■

Proof of Theorem 5.1. The Prym systems of type L constructed above yield a sequence of Teichmüller curves $V(L_n) \subset W_{D_n} \subset \mathcal{M}_2$ satisfying

$$\mathbb{Q}(\sqrt{D_n}) = \mathbb{Q}(\lambda(L_n)) = \mathbb{Q}(\sqrt{4n+1}).$$

Since infinitely many different real quadratic fields appear as $n \to \infty$, infinitely many different primitive Teichmüller curves occur in the sequence $V(L_n)$. The same reasoning applies in genus 3 and 4. ■

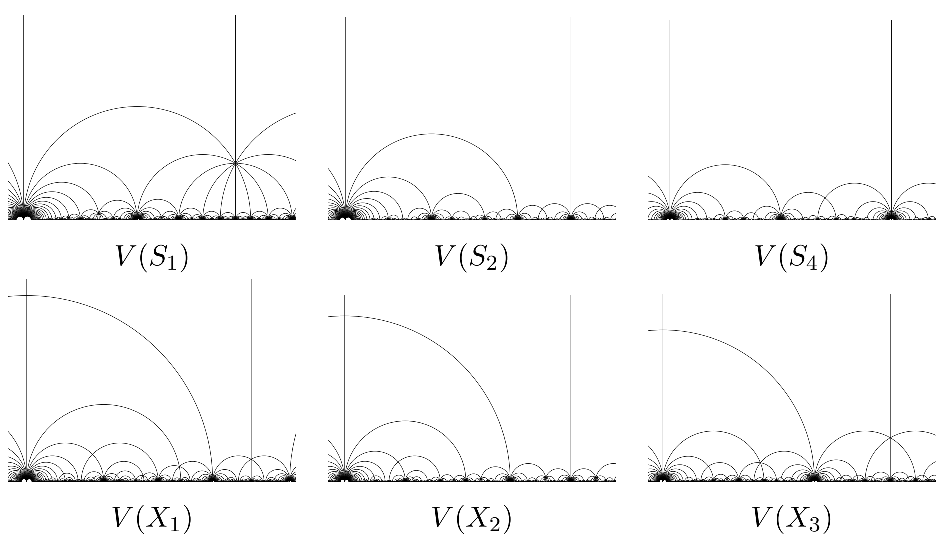

Figure 2. A sampler of Teichmüller curves in genus 3 and 4.

A sampler. Reflection groups of finite index in $\mathrm{GL}(X,\omega)$ were determined for several curves of the form $V(L_n)$ in [Mc1, §9]. The same algorithm can be applied to the curves $V(S_n)$ and $V(X_n)$, yielding the tilings of the hyperbolic plane shown in Figure 2.

Note that the curve $V(S_1)$, coming from the Coxeter graph E_6 with $(m_i) = (1,1,1,1,1,1)$, is also generated by billiards in the triangle $T(3,4)$ mentioned in the Introduction [Lei, §7].

6 Quadratic differentials

In this section we construct related Teichmüller curves in genus 2, 3 and 4, coming not from 1-forms, but from holomorphic quadratic differentials.

Theorem 6.1 *Every irreducible component $V \subset W_D \subset \mathcal{M}_g$ is commensurable to a Teichmüller curve $V' \subset \mathcal{M}_g$ generated by a strictly quadratic differential.*

Twists. Let $(X,\omega) \in \Omega\mathcal{M}_g$ be a Prym form for (X,ρ). Then ω^2 descends to a strictly quadratic differential q on the quotient space $Y = X/\rho$. The covering $\pi : X \to Y$ determines a cohomology class $\kappa \in H^1(Y - B, \mathbb{Z}/2)$, where B is the branch locus of π. (Note that q may have simple poles along B.) Given $0 \neq \xi \in H^1(Y, \mathbb{Z}/2)$, let $\pi_\xi : X_\xi \to Y$ be the branched covering determined by $\xi + \kappa \in H^1(Y - B, \mathbb{Z}/2)$, and let $q_\xi = \pi_\xi^*(q)$. Note that ξ vanishes on peripheral loops around points of B, so π and π_ξ have the same local branching behavior.

Definition. The differential (X_ξ, q_ξ) is the *twist* of (X,ω) by ξ.

Theorem 6.2 *The twists of (X,ω) are strictly quadratic differentials such that*

1. *(X_ξ, q_ξ) and (X,ω^2) are commensurable,*
2. *$g(X_\xi) = g(X)$, and*
3. *q_ξ has zeros of the same multiplicities as ω^2.*

Proof. The forms (X,ω^2) and (Y,q) are commensurable by Theorem 2.5, and the forms (Y,q) and (X_ξ, q_ξ) are commensurable because π_ξ is branched only over the zeros and poles of q.

Since π and π_ξ have the same local branching behavior, X and X_ξ have the same genus. Similarly q_ξ and ω^2 have zeros of the same multiplicity;

in particular, q_ξ is locally the square of a 1-form. However the 1-form $\sqrt{q_\xi}$ cannot be defined globally, since $\xi \neq 0$; thus q_ξ is a strictly quadratic differential. ∎

Proof of Theorem 6.1. Suppose V is generated by $(X, \omega) \in \Omega W_D$ and $g = 3$ or 4. Then the twists (X_ξ, ω_ξ) of (X, ω) provide $4^{g-2} - 1 > 0$ strictly quadratic differentials, also of genus g and commensurable to (X, ω). These twists generate the required Teichmüller curves $V' \subset \mathcal{M}_g$.

For $g = 2$ we use a different construction. Let $\pi : X \to \mathbb{P}^1 = X/\rho$ be the quotient of X by the hyperelliptic involution. Then we can write $\omega^2 = \pi^*(q_0)$, where q_0 is a quadratic differential on $\mathbb{P}^1$ with 5 poles and 1 simple zero. Normalize so that two poles are at $z = 0$ and ∞, let $s(z) = z^2$ and let $q_1 = s^*(q_0)$. Then q_1 has 6 poles and 2 simple zeros. Passing to the double cover of $\mathbb{P}^1$ branched over the poles of q_1, we obtain a strictly quadratic differential (Y, q) of genus two with 4 simple zeros. By Theorem 2.5, (Y, q) is commensurable to (X, ω^2), and thus it too generates a Teichmüller curve $V' \subset \mathcal{M}_g$. ∎

A similar construction in genus two has recently appeared in [Vas].

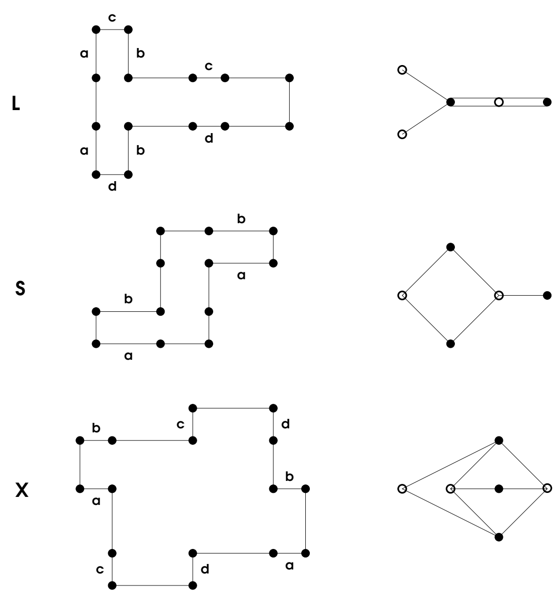

Figure 3. Quadratic differentials in genus 2, 3 and 4.

Corollary 6.3 *The locus $\bigcup_{D \neq d^2} \Omega E_D$ has infinitely many irreducible components for genus $g = 5$.*

Proof. Let (X_ξ, q_ξ) be a twist of a form (X, ω) of genus 3 in $\Omega E_{D'}$. Then by passing to the Riemann surface of $\sqrt{q_\xi}$, we obtain a commensurable form (Y, ω) of genus 5 on an unramified double $Y \to X$. By construction ω is a Prym form for the deck transformation ρ of Y/X, and $\mathrm{SL}(Y, \omega)$ is a lattice because it is commensurable to $\mathrm{PSL}(X_\xi, q_\xi)$. Thus by Theorem 3.5, (Y, ω) belongs to ΩE_D, where $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{D'})$.

By the proof of Theorem 5.1, we can vary the choice of (X, ω) to obtain infinitely many distinct fields $\mathbb{Q}(\sqrt{D'})$, and hence infinitely many different, non-square values of D. ■

Gluing diagrams. The left column of Figure 3 shows gluing diagrams $(X, q) = (Q, dz^2)/\sim$ for strictly quadratic differentials commensurable to the Prym forms of types L, S and X. Edges labeled with the same letter are glued together by maps of the form $z \mapsto -z + c$; the remaining edges are glued by horizontal and vertical translations, as in Figure 1. The corresponding incidence graphs are shown at the right.

We remark that the graph $\Gamma \cong Ah_5$ for the twist of type S is one of the simplest minimal hyperbolic Coxeter diagrams (see [Hum, §6.9]). Our investigation of Prym forms began with a systematic study of such Coxeter diagrams, leading to this example.

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Mathematics Department
Harvard University
1 Oxford St
Cambridge, MA 02138-2901