Braid groups and Hodge theory

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Abstract

This paper gives an account of the unitary representations of the braid group that arise via the Hodge theory of cyclic branched coverings of $\mathbb{P}^1$, highlighting their connections with ergodic theory, complex reflection groups, moduli spaces of 1-forms and open problems in surface topology.

Contents

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1 Introduction

Every configuration of distinct points $(b_1, \ldots, b_n)$ in $\mathbb{C}$ determines a compact Riemann surface X by the equation

$$y^d = (x - b_1)(x - b_2) \cdots (x - b_n). \tag{1.1}$$

This paper gives an account of the unitary representations of the braid group, and the geometric structures on moduli space, that arise via this branched-covering construction. It also develops new connections with arithmetic groups, Teichmüller curves and ergodic theory, and highlights open problems in surface topology and complex reflection groups.

The general approach presented here pivots on the classification of certain arithmetic subgroups of $\mathrm{U}(r, s)$ which envelop the image of the braid group (§7). The case $\mathrm{U}(0, s)$ yields metrics of positive curvature on moduli space, finite representations of the braid group, algebraic definite integrals and rigid factors of the Jacobian of X. The case $\mathrm{U}(1, s)$ leads to the complex hyperbolic metrics on moduli space considered by Picard, Deligne–Mostow and Thurston. Other unitary groups $\mathrm{U}(r, s)$ provide moduli space with natural indefinite metrics. In particular, the hyperelliptic case (§12) yields an action of the braid group which, together with results of Ratner and Kapovich, illuminates the topology and ergodic theory of the period map and its connection with the foliation of $\mathcal{M}_{0,n}^*$ by Teichmüller geodesics.

This section presents a overview of the discussion.

Action of the braid group. We begin with topological considerations. Let $\zeta_d = e^{2\pi i/d}$.

The braid group B_n is the fundamental group of the space $\mathbb{C}^{\langle n \rangle}$ of finite sets $B \subset \mathbb{C}$ with $|B| = n$. Each point configuration B determines, via equation (1.1), a compact Riemann surface X, whose cohomology group $H^1(X)$ forms the fiber of a flat, complex vector bundle over $\mathbb{C}^{\langle n \rangle}$. The natural monodromy representation

$$\rho : B_n \cong \pi_1(\mathbb{C}^{\langle n \rangle}, B) \to \mathrm{Aut}(H^1(X))$$

preserves the interesection form $(\sqrt{-1}/2) \int \alpha \wedge \overline{\beta}$, and commutes with the action of the deck group $\mathbb{Z}/d$ of $X/\widehat{\mathbb{C}}$; thus it also preserves the eigenspace decomposition

$$H^1(X) = \bigoplus_{q^d=1} H^1(X)_q = \mathrm{Ker}(T^* - qI),$$

where $T(x, y) = (x, \zeta_d y)$.

As we will see in §3, the signature (r, s) of the Hermitian vector space $H^1(X)_q$ can be determined by drawing a line through 1 and q in $\mathbb{C}$; then r is the number of nth roots of unity strictly below the line, and s is the number strictly above. The signature changes from definite to hyperbolic to higher rank, and then back again, as q moves around the circle (Figure 1).

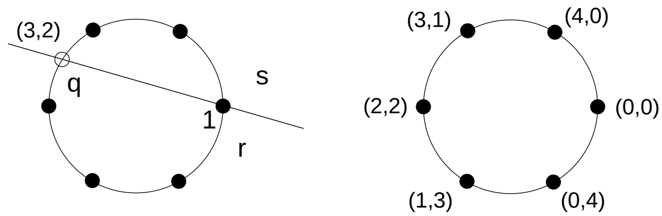

Figure 1. The signature of $H^1(X)_q$ for $n = 6$.

Restricting attention to a single eigenspace, we obtain an irreducible unitary representation

$$\rho_q : B_n \to \mathrm{U}(H^1(X)_q) \cong \mathrm{U}(r, s).$$

In fact, B_n acts by a complex reflection group of type $A_{n-1}(q)$ (see §5); there is a spanning set $(e_i)_1^{n-1}$ for $H^1(X)_q$ adapted to the standard generators τ_i of the braid group, such that

$$\langle e_i, e_j \rangle = \sqrt{-1} \begin{pmatrix} \overline{q} - q & 1 - \overline{q} & 0 & \dots & 0 \\ q - 1 & \overline{q} - q & 1 - \overline{q} & \dots & 0 \\ & & \dots & & \\ 0 & \dots & 0 & q - 1 & \overline{q} - q \end{pmatrix},$$

and

$$\widetilde{\tau}_i^*(x) = x - (q+1)\frac{\langle x, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

(for $q \neq \pm 1$). In particular, the generating twists τ_i of B_n act with finite order on $H^1(X)_q$.

Although purely topological, these results are conveniently established using particular algebraic models such as $y^d = x^n - 1$ for X. We also use consistency of ρ_q under restriction to $B_i \times B_{n-i}$; geometrically, this amounts to letting X degenerate to a stable curve where a given Dehn twist becomes a holomorphic automorphism. (For a more algebraic approach to homological representations of B_n, see e.g. [KT].)

The period map. Now fix $q = \zeta_d^{-k}$ with $1/n < k/d < 1$, let $\mathcal{T}_{0,n}^*$ denote the Teichmüller space of n points in the plane, and let $\mathcal{H}^{r,s}$ denote the space of positive lines in $\mathbb{P}H^1(X)_q \cong \mathbb{P}\mathbb{C}^{r,s}$. The *period map*

$$f_q : \mathcal{T}_{0,n}^* \to \mathcal{H}^{r,s}$$

sends a point $(\mathbb{C}, B)$ in Teichmüller space to the positive line spanned by the holomorphic form $[dx/y^k] \in H^{1,0}(X) \subset H^1(X)$. Thus it records a part of the Hodge structure on X.

Using ideas familiar from polygonal billiards, in §6 we show that f_q is a local homeomorphism when $q^n \neq 1$. It is also equivariant with respect to the action of B_n via the mapping-class group on $\mathcal{T}_{0,n}^*$ and via the representation ρ_q on $\mathcal{H}^{r,s}$. Thus the period map gives the *moduli space* $\mathcal{M}_{0,n}^*$ the structure of a (G, X)-manifold, where $G = \mathrm{U}(r, s)$ and $X = \mathcal{H}^{r,s}$. A similar construction applies to $\mathcal{M}_{0,n}$ (the moduli space of n points in the sphere instead of in the plane), when $q = \zeta_n^{-2}$. (For more on period mappings in general, see e.g. [CMP].)

Arithmetic groups. It is a challenging problem to describe the image of the braid group under ρ_q. In particular, it is unknown when $A_{n-1}(q) \cong \rho_q(B_n)$ is a lattice in $\mathrm{U}(r, s)$.

To study the image of ρ_q in more detail, we begin by observing that the action of B_n on $H^1(X)_q$ preserves the module $\Lambda_{n,q} = H^1(X, \mathbb{Z}[q])_q$. Thus $\rho_q(B_n)$ is contained in the countable subgroup

$$\mathrm{U}(\Lambda_{n,q}) \subset \mathrm{U}(H^1(X)) \cong \mathrm{U}(r, s)$$

of unitary automorphisms of this module.

In §7 we determine all pairs (n, q) such that $\mathrm{U}(\Lambda_{n,q})$ is discrete (see Table 9). When discreteness holds, $\rho_q(B_n)$ is also discrete, and potentially arithmetic. There are infinitely many (n, q) such that $\mathrm{U}(\Lambda_{n,q})$ is discrete, but for $n > 12$ this only occurs when $\mathbb{Z}[q]$ is itself discrete (i.e. when $d = 2, 3, 4$ or 6).

Special cases. In §8—§13 we develop the interaction between f_q, $\rho_q(B_n)$ and $\mathrm{U}(\Lambda_{q,n})$ in more detail, in three special cases.

1. The finite case. Let q be a primitive dth root of unity. In §8 we show that $\rho_q(B_n)$ is finite iff $\mathrm{U}(\Lambda_{n,q})$ is finite. Referring to Table 9, we find there are exactly 9 pairs (n, d) such that this finiteness holds. In each case $H^1(X)_q$ has signature $(0, s)$ or $(s, 0)$, and the Hodge structure on the corresponding part of $H^1(X)$ is rigid; equivalently, we have an isogeny

$$\mathrm{Jac}(X) \sim J(X) \times A$$

where A is independent of X. For example, when X is defined by

$$y^{10} = (x - b_1)(x - b_2)(x - b_3)$$

its Jacobian has a rigid factor isomorphic to $(\mathbb{C}^2/\mathbb{Z}[\zeta_5])^2$. In the other 7 cases, the rigid factor is a product of elliptic curves.

Certain instances of this rigidity can be seen in terms of conformal mappings. For example, when all the (b_i) are real, the case $(n, d) = (3, 4)$ is related to the fact that any (45,45,45,135)-degree quadrilateral can be mapped conformally onto the complement of a symmetric slit in a right-isosceles triangle (Figure 2). These polygons, when doubled, represent the sphere with the metrics $|dx/y^3|$ and $|(x - a)dx/y^3|$ respectively. Since the second polygon develops, under reflection through its sides, onto a periodic pattern in the plane, the form $(x - a)dx/y^3$ is pulled back from the square torus, giving $\mathrm{Jac}(X)$ a factor of $\mathbb{C}/\mathbb{Z}[\sqrt{-1}]$.

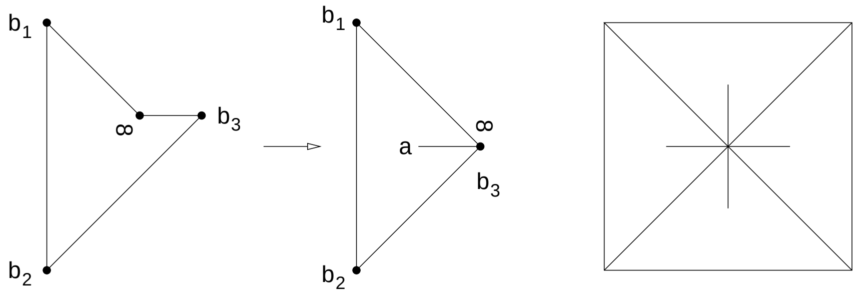

Figure 2. Quadrilaterals and squares.

By similar considerations, in §9 we determine 17 values of (n, μ) such that the definite integral

$$I(b_1, b_2, \ldots, b_n) = \int_{b_1}^{b_2} \frac{dx}{((x - b_1)(x - b_2) \cdots (x - b_n))^\mu}$$

is an algebraic function of its parameters. The integral is evaluated explicitly for $(n, \mu) = (4, 3/4)$, as an example.

2. The complex hyperbolic case. In §10 we turn the case where $H^1(X)_q$ has signature $(1, s)$. In this case the period map

$$f_q : \mathcal{T}^*_{0,n} \to \mathcal{H}^{1,s} \cong \mathbb{CH}^s$$

is a holomorphic map to the complex hyperbolic ball. Using the Schwarz lemma, we find $\rho_q(B_n) \subset \mathrm{U}(1, s)$ is a lattice whenever it is discrete. Examining Table 9, we obtain 24 cases where $\rho_q(B_n)$ is an arithmetic lattice. In particular we find that for $n = 4, 5, 6, 8$ and 12, the period map

$$f_q : \mathcal{M}_{0,n} \to \mathbb{CH}^{n-3}/\rho_q(B_n)$$

at $q = \zeta_n^{-2}$ presents moduli space as the complement of a divisor in a finite-volume, arithmetic, complex-hyperbolic orbifold. These results are special cases of the work of Deligne–Mostow and Thurston, which we review in §10. We also develop parallels between non-arithmetic lattices in $\mathrm{U}(1, 2)$ and the known Teichmüller curves on Hilbert modular surfaces.

Next, consider the following purely topological problem. Let $\mathrm{Sp}(X)^T \cong \mathrm{Sp}_{2g}(\mathbb{Z})^T$ denote the centralizer of T^* in the automorphism group of $H^1(X, \mathbb{Z})$, and let

$$\mathrm{Sp}(X)^T_d = \mathrm{Sp}(X)^T | \operatorname{Ker} \Phi_d(T^*)$$

where $\Phi_d(x)$ is the dth cyclotomic polynomial. We say a group homomorphism is *almost onto* if the image has finite index in the target.

Problem: Is the natural map $B_n \to \mathrm{Sp}(X)^T_d$ almost onto?

In §11 we connect this problem to arithmeticity of lattices. We show that for $n = 3$, the answer is yes iff the $(2, 3, p)$ triangle group is arithmetic, where p is the order of $d-2$ in $\mathbb{Z}/2d$. (There are 16 such cases.) More generally, the answer is yes whenever (d, n) appear in Table 9 with signature $(0, s)$ or $(1, s)$. On the other hand, using a non-arithmetic lattice in $\mathrm{U}(1, 2)$ constructed by Deligne and Mostow, we show the answer is *no* when $(n, d) = (4, 18)$.

3. The hyperelliptic case. In §12 we discuss the hyperelliptic case, where $d = 2$ and $q = -1$. In this case $H^1(X)_q = H^1(X)$, and $\rho_q(B_n)$ has finite index in $\mathrm{U}(\Lambda_{n,q}) \cong \mathrm{Sp}_{2g}(\mathbb{Z})$ by [A'C].

For convenience, assume $n = 2g + 1$ is odd and $g \geq 2$. Then there is a natural *Weierstrass foliation* of Teichmüller space by complex geodesics, each of which is sent into a straight line under the period map

$$f_q : \mathcal{T}^*_{0,n} \to \mathcal{H}^{g,g}.$$

In fact there is an equivariant action of $\mathrm{SL}_2(\mathbb{R})$ on suitable circle bundles over the domain and range of f_q. Using this action we find:

1. The period map f_q is an infinite-to-one local homeomorphism, but not a covering map to its image;
2. It transports the discrete action of B_n on $\mathcal{T}^*_{0,n}$ to an ergodic action on $\mathcal{H}^{g,g}$; and

3. The complement of the image $f_q(\mathcal{T}^*_{0,n})$ is contained in a countable union of lines.

The last assertion is deduced using Ratner's theorem as in [Kap].

The Weierstrass foliation descends to a foliation $\mathcal{W}$ of $\mathcal{M}^*_{0,n}$. Although almost every leaf of $\mathcal{W}$ is dense, Veech showed the special leaf V passing through $[(b_i = \zeta_n^i)]$ is a properly embedded algebraic curve. In §13 we discuss this *Teichmüller curve* V from the perspective of braid groups and flat metrics on the sphere. We first show that $\pi_1(V)$ is the $(2, n, \infty)$ triangle group generated by the images of the braids

$$\alpha = \tau_1\tau_3 \ldots \tau_{2g-1} \quad \text{and} \quad \beta = \tau_2\tau_4 \ldots \tau_{2g}.$$

(Note that $\alpha\beta$ is a lift of the Coxeter element for the A_{n-1} diagram.) We then describe the abstract Euclidean polygons $Q = (\widehat{\mathbb{C}}, |\omega|)$ associated to the two orbifold points of V: one is obtained from a regular n-gon by folding its vertices to a single point, while the other is the double of an immersed, right-angled disk whose edge lengths form an eigenvector for the A_{n-1} adjacency matrix. These descriptions allow one to explicitly evaluate the period map at the orbifold points of V.

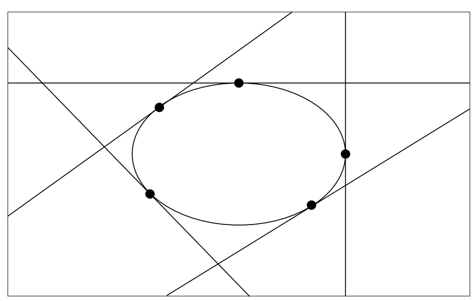

Figure 3. Line configuration associated to 5 points on a conic.

Remark: Cyclic multiple planes. One can also associate to each point configuration $(b_1, \ldots, b_n) \in \mathbb{C}^{\langle n \rangle}$ the cyclic covering X of $\mathbb{P}^2$ defined by the equation

$$z^d = L_1(x, y) \cdots L_n(x, y) Q(x, y),$$

where $L_i(x, y) = y - 2b_i x + b_i^2$ and $Q(x, y) = y - x^2$. This complex surface is branched over the conic Q, its tangent lines L_i at $P_i = (b_i, b_i^2)$, and (possibly) the line at infinity (Figure 3). After desingularizing X one obtains, for each eigenspace, a unitary action of B_n on $H^2(\widetilde{X})_q$. It would be interesting to study the associated period maps as Hodge-theoretic counterparts to the Lawrence–Krammer–Bigelow representations ξ_q of B_n [Law], [At], [Kr], [Bg2].

Notes and references. The *hypergeometric functions* of the form $F = \int_{b_1}^{b_2} \prod_1^n (x - b_i)^{-\mu_i}\, dx$ have been studied since the time of Euler; the present discussion is closely connected to works of Schwarz and Picard [Sch], [Pic]. For more on the classical theory of hypergeometric functions, see e.g. [Kl], [SG] and [Yo]. Deligne and Mostow developed a modern perspective on hypergeometric functions, and used their monodromy to exhibit non-arithmetic lattices in $\mathrm{U}(1, s)$ for certain $s \geq 2$ [Mos1], [DM1], [DM2]; see also [Sau] and the surveys [Mos2] and [Par]. Thurston recast the work of Deligne and Mostow in terms of shapes of convex polyhedra, by observing that the integrand of F determines a flat metric on the sphere with cone-type singularities. Convexity of the polyhedron corresponds to signature $(1, s)$ for the metric on moduli space [Th2]. For other perspectives, see [V3] and [Tr].

In this paper we have focused on the case where all μ_i assume a common rational value k/d. This case leads directly to algebraic curves and their Jacobians, and yields representations of the full braid group. At the same time it allows for non-convex polyhedra (as in Figure 2) and non-hyperbolic signatures (r, s), because $|dx/y^k|$ can be negatively curved at $x = \infty$.

Analogous Hodge-theory constructions have been used to study, instead of finite sets in $\mathbb{P}^1$, hypersurfaces of low degree in $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^n$, $n \leq 4$; see e.g. [DK], [Ko], [ACT1], [ACT2], [Al2], [Lo2]. The action of the mapping-class group of a *closed* surface on the homology of its finite abelian covers is studied in [Lo1].

We note that ρ_q is essentially a specialization of the Burau representation (§5); as a representation of the Hecke algebra $H_n(q)$, it corresponds to the partition $n = 1 + (n - 1)$ [J], [KT]. The Lawrence–Krammer–Bigelow representations ξ_q correspond to the next partition, $n = 2 + (n - 2)$. Our investigation began with the observation that ρ_q coincides with the quantum representations discussed in [AMU] when $n = 3$. It would be interesting to understand more fully the connection between classical and quantum representations.

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Notation. We use $\lfloor x \rfloor$ to denote the largest integer $\leq x$, and $\lceil x \rceil$ for the smallest integer $\geq x$. We let $\mathrm{U}(V)$ denote the unitary group of a Hermitian vector space V over $\mathbb{C}$. The unitary group of the standard Hermitian form of signature (r, s) on $\mathbb{C}^{r+s}$ is denoted by $\mathrm{U}(r, s)$, and the corresponding projective group by $\mathrm{PU}(r,s) \cong \mathrm{U}(r,s)/\,\mathrm{U}(1)$. The automorphism group of a real symplectic vector space is denoted by $\mathrm{Sp}(V) \cong \mathrm{Sp}_{2g}(\mathbb{R})$.

2 Topological preliminaries

A basic reference for the algebraic structure of the braid groups and mapping-class groups is [Bi]; see also [BZ], [KT] and [FM].

Moduli spaces and point configurations. Given $n \geq 2$, let $\mathbb{C}^{\langle \mathrm{n} \rangle}$ denote the space of unordered subsets $B \subset \mathbb{C}$ with $|B| = n$, and similarly for $\widehat{\mathbb{C}}^{\langle \mathrm{n} \rangle}$.

Each of these spaces is naturally a connected complex manifold; for example, $\mathbb{C}^{\langle \mathrm{n} \rangle}$ can be identified with the space of monic complex polynomials of degree n with nonvanishing discriminant.

The quotient of $\mathbb{C}^{\langle \mathrm{n} \rangle}$ by the group of affine automorphisms $z \mapsto az + b$ yields the moduli space of n-tuples of points in the plane, $\mathcal{M}^*_{0,n} = \mathbb{C}^{\langle \mathrm{n} \rangle} / \operatorname{Aut} \mathbb{C}$. Similarly $\mathcal{M}_{0,n} = \widehat{\mathbb{C}}^{\langle \mathrm{n} \rangle} / \operatorname{Aut} \widehat{\mathbb{C}}$ is the moduli space of n-tuples of points on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Since some point configurations have extra symmetries, these moduli space are orbifolds. The natural maps

$$\mathbb{C}^{\langle \mathrm{n} \rangle} \to \mathcal{M}^*_{0,n} \quad \text{and} \quad \mathcal{M}^*_{0,n} \to \mathcal{M}_{0,n} \tag{2.1}$$

are both fibrations, with fibers $\operatorname{Aut}(\mathbb{C})/\operatorname{Aut}(\mathbb{C}, B)$ and $(\widehat{\mathbb{C}} - B)/\operatorname{Aut}(\widehat{\mathbb{C}}, B)$ respectively.

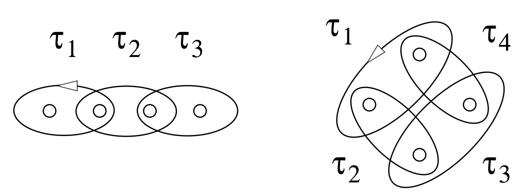

Figure 4. Right twists generating the braid group.

The braid group. The *braid group* B_n is the fundamental group of $\mathbb{C}^{\langle \mathrm{n} \rangle}$. The motion of B around any loop in $\mathbb{C}^{\langle \mathrm{n} \rangle}$ can be extended to a motion of $\mathbb{C}$ with *compact support* (fixing a neighborhood of ∞); this gives an isomorphism

$$B_n \cong \pi_1(\mathbb{C}^{\langle \mathrm{n} \rangle}, B) \cong \mathrm{Mod}_c(\mathbb{C}, B)$$

between the braid group and the compactly-supported mapping-class group of the pair $(\mathbb{C}, B)$.

Standard generators $\tau_1, \ldots, \tau_{n-1}$ for the braid group are given, for the configuration $B = \{1, 2, \ldots, n\} \subset \mathbb{C}$, by right Dehn half-twists around loops enclosing $[i, i+1]$ as in Figure 4. This means τ_i rotates the disk enclosing $[i, i+1]$ counter-clockwise by 180°. These $n-1$ generators satisfy the *braid relation*

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \tag{2.2}$$

for $i = 1, \ldots, n-2$, as well as the commutation relation $\tau_i \tau_j = \tau_j \tau_i$ for $|i - j| > 1$; and these generators and relations give a presentation for B_n.

Alternatively, for the more symmetric configuration

$$B = \{1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\}, \quad \zeta_n = \exp(2\pi i/n)$$

shown at the right in Figure 4, we can take τ_i to be the right half-twist interchanging ζ_n^i and ζ_n^{i+1}. Then a rotation of B by $2\pi/n$ is given by

$$\sigma = \tau_1 \tau_2 \cdots \tau_{n-1},$$

and we have

$$\tau_{i+1} = \sigma \tau_i \sigma^{-1}.$$

In particular, we obtain an additional twist $\tau_n = \sigma \tau_{n-1} \sigma^{-1}$. For $n \geq 3$, the center of B_n is generated by σ^n.

Mapping-class groups. Like the braid group, the mapping-class groups of n-tuples of points in the plane and on the sphere can be defined by

$$\mathrm{Mod}_{0,n}^* \cong \pi_1(\mathcal{M}_{0,n}^*) \cong \mathrm{Mod}(\mathbb{C}, B)$$

and

$$\mathrm{Mod}_{0,n} \cong \pi_1(\mathcal{M}_{0,n}) \cong \mathrm{Mod}(\widehat{\mathbb{C}}, B).$$

The groups at the right consist of the orientation-preserving diffeomorphisms $f : (\mathbb{C}, B) \to (\mathbb{C}, B)$ (resp. $f : (\widehat{\mathbb{C}}, B) \to (\widehat{\mathbb{C}}, B)$), up to isotopy. From (2.1) we have natural surjective maps $B_n \to \mathrm{Mod}_{0,n}^* \to \mathrm{Mod}_{0,n}$. The kernel of the first is the center of B_n; the kernel of the second is the smallest normal subgroup containing $(\tau_2 \cdots \tau_{n-1})^{n-1}$.

The braid relation (2.2) can also be written as

$$(\tau_i \tau_{i+1})^3 = (\tau_i \tau_{i+1} \tau_i)^2;$$

in particular, B_3 is a central extension of the $(2, 3, \infty)$ triangle group

$$\langle a, b \,:\, a^2 = b^3 = 1 \rangle \cong \mathrm{PSL}_2(\mathbb{Z}) \cong \mathrm{Mod}_{0,3}^*.$$

Cyclic covers. Fix an integer $d > 0$ and a point configuration $B = \{b_1, \ldots, b_n\} \subset \mathbb{C}$. Let $p(z) = \prod_1^n (z - b_i)$. There is a unique homomorphism $w : \pi_1(\mathbb{C} - B) \to \mathbb{Z}/d$ sending a loop to the sum of its winding numbers around the points of B; concretely, it is given by

$$w(\alpha) = \frac{1}{2\pi i} \int_\alpha \frac{p'(z)\, dz}{p(z)} \mod d.$$

Let $\pi : X \to \widehat{\mathbb{C}}$ denote the corresponding covering of $\mathbb{C} - B$, completed to a branched covering of $\widehat{\mathbb{C}}$. Let $T : X \to X$ be the generator of the deck group corresponding to $1 \in \mathbb{Z}/d$, and let $\widetilde{\infty}$ and $\widetilde{B}$ denote the preimages of ∞ and B under π. It is easy to see that $|\widetilde{B}| = |B|$ and

$$e = |\widetilde{\infty}| = \gcd(d, n);$$

in particular, X is unramified over infinity iff d divides n. By the Riemann-Hurwitz formula, the genus g of X is given by

$$g = \frac{(n-1)(d-1) + 1 - e}{2} \cdot$$

Eigenspaces. Recall that $H^1(X, \mathbb{C}) = H^1(X)$ carries a natural intersection form of signature (g, g), defined by $(\sqrt{-1}/2) \int \alpha \wedge \overline{\beta}$. Any orientation-preserving diffeomorphism of X preserves this form, so the map

$$f \mapsto (f^{-1})^* | H^1(X) \tag{2.3}$$

gives a natural unitary representation of the mapping-class group

$$\operatorname{Mod}(X) \to \operatorname{U}(H^1(X)) \cong \operatorname{U}(g, g).$$

Since $[T] \in \operatorname{Mod}(X)$ has order d, we also have an eigenspace decomposition

$$H^1(X) = \bigoplus_{q^d = 1} H^1(X)_q$$

where $H^1(X)_q = \operatorname{Ker}(T^* - qI)$. Let (r, s) denote the signature of the intersection form on $H^1(X)_q$. The mapping-classes $\operatorname{Mod}(X)^T$ commuting with T also preserve these eigenspaces, so we have a natural representation

$$\operatorname{Mod}(X)^T \to \operatorname{U}(H^1(X)_q) \cong \operatorname{U}(r, s)$$

for each q.

Clearly the invariant cohomology satisfies $H^1(X)_1 \cong H^1(\widehat{\mathbb{C}}) \cong 0$. For the sequel, *we assume* $q^d = 1$ *but* $q \neq 1$.

Lifting braids. To bring the braid group into the picture, observe that any compactly supported diffeomorphism $\phi : (\mathbb{C}, B) \to (\mathbb{C}, B)$ has a *unique* lift $\widetilde{\phi} : X \to X$ which is the identity on a neighborhood of $\widetilde{\infty}$. This lifting gives a natural map

$$B_n \cong \mathrm{Mod}_c(\mathbb{C}, B) \to \mathrm{Mod}(X)^T. \tag{2.4}$$

Passing to cohomology, we obtain a natural representation

$$\rho_q : B_n \to \mathrm{U}(H^1(X)_q) \cong \mathrm{U}(r, s).$$

Our goal is to understand this representation.

Lifting mapping classes. Note that any $\psi \in \mathrm{Mod}(\mathbb{C}, B)$ has a lift $\widetilde{\psi}$ to X which is well-defined up to a power of T; we pass to the braid group only to make the lift unique. Since $T|H^1(X)_q = qI$, the representation ρ_q descends to a map $\mathrm{Mod}(\mathbb{C}, B) \to \mathrm{PU}(r, s)$. On the other hand, if X is unramified over infinity (i.e. if $d|n$), then $\widetilde{\psi}$ is uniquely determined by the condition that it fixes $\widetilde{\infty}$ pointwise, so ρ_q factors through $\mathrm{Mod}(\mathbb{C}, B)$. Similarly, if X is unramified over ∞ then any $\psi \in \mathrm{Mod}(\widehat{\mathbb{C}}, B)$ has a lift to X which is well-defined up to the action of T. Thus we have a commutative diagram:

$$\begin{array}{ccc} B_n & \xrightarrow{\rho_q} & \mathrm{U}(r,s) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Mod}^*_{0,n} & \longrightarrow & \mathrm{PU}(r,s) \\ \downarrow & \nearrow & \\ \mathrm{Mod}_{0,n} & & \end{array} \tag{2.5}$$

whose diagonal arrows exist when $d|n$.

Invariant subsurfaces. To describe ρ_q, we first choose standard generators $\tau_1, \ldots, \tau_n$ for $B_n \cong \mathrm{Mod}_c(\mathbb{C}, B)$. Each τ_i is a half Dehn twist supported in an open disk $D_i \subset \mathbb{C}$ with $D_i \cap B = \{b_i, b_{i+1}\}$ (or $\{b_n, b_1\}$ when $i = n$). Let

$$X_i = \pi^{-1}(D_i) \subset X.$$

This is an open, connected, incompressible subsurface of X of genus $\lfloor (d-1)/2 \rfloor$, with one or two boundary components (depending on the parity of d; see Figure 5).

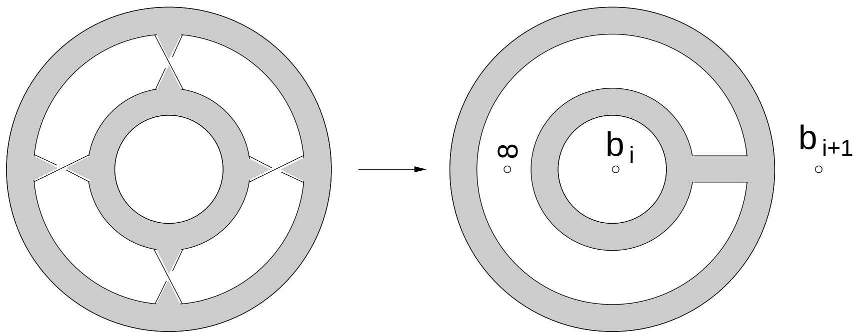

Figure 5. The surface X_i is built from the $\mathbb{Z}/d$-covering space of a pair of pants in $\widehat{\mathbb{C}}$ shown above, by capping off the inner and outer circles at the left with disks.

Let Y_i be the closed subsurface $X - X_i$. Since τ_i is supported on D_i and $\widetilde{\tau}_i$ fixes $\widetilde{\infty} \subset Y_i$, the lift $\widetilde{\tau}_i$ is supported on X_i. In particular, both X_i and Y_i are invariant under T and $\widetilde{\tau}_i$.

Action on cohomology. For $n \geq 3$, Y_i is connected. In this case, by Mayer-Vietoris, we have an equivariant exact sequence

$$0 \to H^1_c(X_i) \to H^1(X) \to H^1(Y_i) \to 0,$$

where $H^1_c(X_i)$ denotes cohomology with compact supports. We denote the corresponding exact sequence of eigenspaces of T by

$$0 \to H^1_c(X_i)_q \to H^1(X)_q \to H^1(Y_i)_q \to 0. \tag{2.6}$$

The same sequences are exact for $n = 2$, except when d is even. If d is even then Y_i is disconnected (it is a pair of disjoint disks), and we have instead the exact sequence

$$H^1(\partial X_i) \to H^1_c(X_i) \to H^1(X) \to H^1(Y_i) = 0. \tag{2.7}$$

In any case, since $\widetilde{\tau}_i$ is supported in X_i, we have

$$\widetilde{\tau}_i^* | H^1(Y_i) = I. \tag{2.8}$$

In the next section we will show that $\widetilde{\tau}_i^* | H^1_c(X_i) = -qI$.

Remark: stabilization. Note that if d divides d', then the corresponding branched covering X' factors through X, and we obtain an equivariant isomorphism

$$\iota : H^1(X)_q \cong H^1(X')_q.$$

Thus the representation ρ_q of B_n depends only on q, not on d. The isomorphism ι is not, however, unitary; it multiplies the intersection form by d'/d. To obtain a unitary isomorphism, one must rescale ι by $\sqrt{d/d'}$.

3 From Hodge theory to topology

In this section we analyze the action of the braid group further, by using holomorphic 1-forms to represent the cohomology of X.

Hodge structure. A given point configuration $B \subset \mathbb{C}$ determines a complex structure on the genus g cyclic covering space $\pi : X \to \widehat{\mathbb{C}}$ branched over $B \cup \{\infty\}$. The complex structure in turn determines the g-dimensional space of holomorphic 1-forms $\Omega(X)$ on X, and the Hodge decomposition

$$H^1(X) = H^{1,0}(X) \oplus H^{0,1}(X) \cong \Omega(X) \oplus \overline{\Omega}(X).$$

The intersection form

$$\langle \alpha, \beta \rangle = \frac{\sqrt{-1}}{2} \int_X \alpha \wedge \overline{\beta} \tag{3.1}$$

is positive-definite on $H^{1,0}(X)$ and negative-definite on $H^{0,1}(X)$.

Symmetric curves. Now assume B coincides with the nth roots of unity. Then X is isomorphic to the smooth algebraic curve given by

$$y^d = x^n - 1,$$

with the projection to $\widehat{\mathbb{C}}$ given by $\pi(x, y) = x$ and with the generating deck transformation of $X/\widehat{\mathbb{C}}$ given by

$$T(x, y) = (x, \zeta_d y).$$

We then have an additional symmetry $R : X \to X$ given by

$$R(x, y) = (\zeta_n x, y).$$

Eigenforms. Since the abelian group $\langle R, T \rangle$ preserves the intersection form and the Hodge structure, we have a decomposition

$$H^1(X) = \bigoplus H^1(X)_{jk}$$

of the cohomology of X into orthogonal eigenspaces

$$H^1(X)_{jk} = \operatorname{Ker}(R^* - \zeta_n^j I) \cap \operatorname{Ker}(T^* - \zeta_d^{-k} I),$$

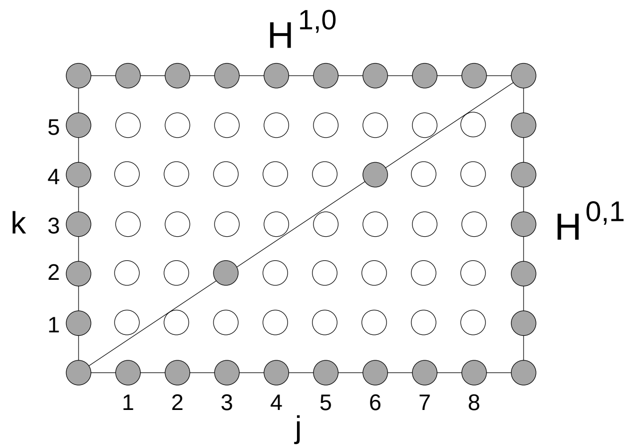

Figure 6. Occupied eigenspaces $H^1(X)_{jk}$ are shown in white. Those above the diagonal span $H^{1,0}(X)$, and those below span $H^{0,1}(X)$.

and a further refinement

$$H^1(X)_{jk} = H^{1,0}_{jk}(X) \oplus H^{0,1}_{jk}(X).$$

We can assume $0 \le j/n, k/d < 1$.

To populate these eigenspaces, we consider the meromorphic forms defined by

$$\omega_{jk} = \frac{x^{j-1}\,dx}{y^k}.$$

The divisor $(\omega_{jk}) = (dx/x) + j(x) - k(y)$ is easily calculated using the fact that the fibers of $X/\widehat{\mathbb{C}}$ have multiplicity d/e over ∞, d over B, and 1 elsewhere, where $e = \gcd(d,n) = |\widetilde{\infty}|$. Letting $\widetilde{B}$, $\widetilde{\infty}$ and $\widetilde{0}$ denote the preimages of B, 0 and ∞ on X, considered as divisors where each point has weight one, we obtain

$$\begin{aligned} (x) &= -(d/e)\widetilde{\infty} + \widetilde{0}, \\ (y) &= -(n/e)\widetilde{\infty} + \widetilde{B}, \quad \text{and} \\ (dx/x) &= -\widetilde{\infty} - \widetilde{0} + (d-1)\widetilde{B}, \end{aligned}$$

which yields

$$(\omega_{jk}) \;=\; (kn/e - jd/e - 1)\widetilde{\infty} + (j-1)\widetilde{0} + (d-1-k)\widetilde{B}. \tag{3.2}$$

Thus ω_{jk} is holomorphic for all j,k such that $0 < j/n < k/d < 1$, and hence in this range we have $\dim H^{1,0}(X)_{jk} \geq 1$. Similarly, by considering $\overline{\omega}_{jk}$, we find $\dim H^{0,1}(X)_{jk} \geq 1$ for $0 < k/d < j/n < 1$. But the total number of pairs (j,k) satisfying one condition or the other is exactly $(n-1)(d-1)+(1-e) = 2g = \dim H^1(X)$. Thus ω_{jk} and $\overline{\omega}_{jk}$ account for the full cohomology of X. In summary:

Theorem 3.1 *We have*

$$\dim H^1(X)_{jk} = \dim H^{1,0}(X)_{jk} = 1 \quad \text{if } 0 < j/n < k/d < 1, \quad \text{and}$$
$$\dim H^1(X)_{jk} = \dim H^{0,1}(X)_{jk} = 1 \quad \text{if } 0 < k/d < j/n < 1.$$

The remaining eigenspaces are zero.

The case of a degree $d = 6$ covering branched over $n = 9$ points (and infinity) is shown in Figure 6.

Corollary 3.2 *When $q = \zeta_d^{-k}$, the signature of the intersection form on $H^1(X,\mathbb{C})_q$ is given by $(r,s) = (\lceil n(k/d) - 1\rceil, \lceil n(1-k/d) - 1\rceil)$.*

Indeed, the signature (r,s) just counts the number of terms of type $(1,0)$ and $(0,1)$ in the expression $H^1(X)_q = \bigoplus_j H^1(X)_{jk}$, or equivalently the number of terms above and below (but not on) the diagonal in the kth row of Figure 6.

Corollary 3.3 *The eigenspace $H^1(X)_q$ has dimension $n-2$ if $q^n = 1$, and dimension $n-1$ otherwise.*

Genus zero quotients. We note that the Riemann surfaces $X/\langle R\rangle$, $X/\langle T\rangle$ and $X/\langle RT\rangle$ all have genus zero. This provides another explanation for the vanishing of eigenspaces — any R, T or RT-invariant cohomology class descends to the sphere, and $H^1(\widehat{\mathbb{C}}) = 0$.

Remark: Lefschetz formulas. The eigenspace decomposition of $H^1(X)$ shown in Figure 6 can be determined, alternatively, by applying the Lefschetz formulas

$$\operatorname{Tr}(g|H^1(X)) = 2 - \sum_{g(z)=z} 1 \quad \text{and} \quad \operatorname{Tr}(g|H^{0,1}) = 1 - \sum_{g(z)=z} (1-g'(z))^{-1}$$

to compute the characters of $G = \langle R,T\rangle$ acting on $H^1(X)$ and $H^{0,1}(X)$. The Chevalley-Weil formula similarly gives the characters of $G|H^{1,0}(X)$ for general Galois coverings [ChW]; see also [Hu].

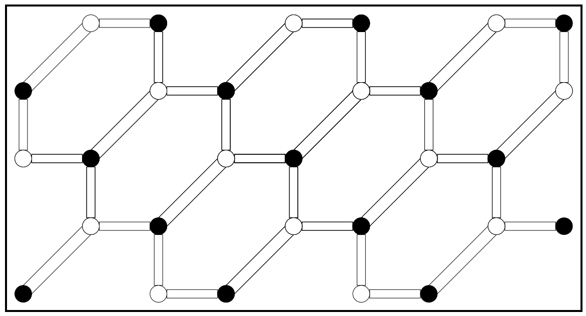

Figure 7. A model for the branched covering X.

Remark: the hexagon model. Here is an explicit model for $H^1(X,\mathbb{Z})$ as a symplectic lattice equipped with the action of $G = \mathbb{Z}/n \times \mathbb{Z}/d = \langle R, T\rangle$. The proof will also provide a global topological model for G acting on X.

Let $S = \{(1,0),(0,1),(-1,-1)\} \subset G$. Define an alternating form on the standard basis of the group ring $\mathbb{Z}[G] \cong \mathbb{Z}^{nd}$ by

$$[g,h] = \begin{cases} 1 & \text{if } g-h \in S, \\ -1 & \text{if } h-g \in S, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

and extend it by linearity to $\mathbb{Z}[G]$. Let $R \subset \mathbb{Z}[G]$ denote the radical of this form (the set of elements x such that $[x,y]=0$ for all y).

Theorem 3.4 *The space $H_1(X,\mathbb{Z})$ is isomorphic to $\mathbb{Z}[G]/R$ as a symplectic $\mathbb{Z}[G]$-module.*

Sketch of the proof. Let A and B be a pair of 2-disks, colored white and black respectively. Construct a surface Y_0 from $(A \sqcup B) \times G$ by joining $A \times \{g\}$ to $B \times \{g+s\}$ with a band whenever $s \in S$ (see Figure 7). The group G acts by deck transformations on Y_0, with quotient a surface Y_0/G of genus 1 with 1 boundary component.

Now put a half-twist in each band of Y_0 (as in Figure 5), to obtain a new surface Y_1. Then Y_1/G has genus 0 and 3 boundary components. By filling in the boundary components of Y_1 and Y_1/G with disks, we obtain closed surfaces Y and $Z \cong S^2$ and a degree nd branched covering

$$\pi : Y \to S^2$$

with deck group G.

Let $C \subset Y$ be the core curve of one of the hexagons appearing in Figure 7. It is then straightforward to check that the intersection number $(gC) \cdot (hC)$ in $H_1(Y, \mathbb{Z})$ is given by (3.3). By comparing the rank of this form with the first Betti number of Y, we find the inclusion $\mathbb{Z}[G]/R \to H^1(Y, \mathbb{Z})$ is an isomorphism.

Now one can also regard Y/G as an orbifold with signature $(n, d, nd/e)$, where $e = \gcd(d, n)$. The generators of G correspond loops around the points of orders n and d. Similarly, the map $p : X \to \widehat{\mathbb{C}}$ given by $p(x, y) = x^n$ presents $X/\langle R, T\rangle$ as a covering of the same orbifold, with singular points at 0, 1, ∞. The proof is completed by identifying $p : X \to \widehat{\mathbb{C}}$ with $\pi : Y \to S^2$. ■

4 Action of the braid group

In this section we determine the representation $\rho_q : B_n \to \mathrm{U}(H^1(X)_q)$.

Theorem 4.1 *There is a spanning set $(e_i)_1^n$ for $H^1(X)_q$ such that the intersection form is given by*

$$\begin{aligned} \langle e_i, e_i\rangle &= 2\,\mathrm{Im}\, q, \\ \langle e_i, e_{i+1}\rangle &= \sqrt{-1}(1-\overline{q}), \quad \textit{and} \\ \langle e_i, e_j\rangle &= 0 \quad \textit{if } |i-j| > 1; \end{aligned}$$

and the action of the braid group is given by

$$\widetilde{\tau}_i^*(x) = x - \frac{\sqrt{-1}}{2}\langle x, e_i\rangle e_i \tag{4.1}$$

if $q = -1$, and otherwise by

$$\widetilde{\tau}_i^*(x) = x - (q+1)\frac{\langle x, e_i\rangle}{\langle e_i, e_i\rangle} e_i. \tag{4.2}$$

(Here the indices $i, j \in \mathbb{Z}/n$ and $|i-j| > 1$ means $i - j \neq -1, 0$ or $1 \bmod n$.)

Derivatives at infinity. Recall that $e = |\widetilde{\infty}| = \gcd(d, n)$. Near infinity we have $y^d \sim x^n$, and thus $z = y^{d/e}/x^{n/e}$ maps $\widetilde{\infty}$ to the eth roots of unity. It follows that the pointwise stabilizer of $\widetilde{\infty}$ in $\langle R, T\rangle \cong \mathbb{Z}/n \times \mathbb{Z}/d$ is the cyclic group generated by RT. Similarly, a local coordinate near a point of $\widetilde{\infty}$ is provided by

$$t = 1/(x^a y^b),$$

where $ad + bn = e$. Computing in this coordinate, we find

$$(RT)'(\widetilde{\infty}) = \zeta_n^{-a}\zeta_d^{-b} = \zeta_{nd}^{-ad-bn} = \zeta_{nd}^{-e}. \tag{4.3}$$

Action of rotations. Now recall that $\sigma = \tau_1 \cdots \tau_{n-1} \in \mathrm{Mod}_c(\mathbb{C}, B)$ represents a right $1/n$ Dehn twist around a loop enclosing B. The actions of R and T on $H^1(X)$ are given transparently by the eigenspace description just presented. To bring the braid group into play, we show:

Proposition 4.2 *We have* $[\widetilde{\sigma}] = [RT]$ *in* $\mathrm{Mod}(X)$.

Proof. The mapping-class σ is represented by a homeomorphism which agrees with the rigid rotation $R_0(x) = \zeta_n x$ except on a small neighborhood of ∞. Thus $\widetilde{\sigma}$ agrees with some lift RT^i of R_0 outside a small neighborhood of $\widetilde{\infty}$. Now near infinity, σ twists by angle $-2\pi/n$ back to the identity. Since d/e sheets of X come together at each point of $\widetilde{\infty}$, upon lifting this becomes of twist by angle $(-2\pi e)/nd$. It follows that

$$(RT^i)'(\widetilde{\infty}) = \zeta_{nd}^{-e} = (RT)'(\widetilde{\infty})$$

by (4.3), and thus $i = 1$. ■

Corollary 4.3 *The kernel of the lifting map* $B_n \to \mathrm{Mod}(X)^T$ *is the cyclic central subgroup generated by* $\sigma^{nd/e}$.

Proof. The lifted map $\widetilde{\phi} \in \mathrm{Mod}(X)^T$ determines $[\phi] \in \mathrm{Mod}(\mathbb{C}, B)$, so the kernel of lifting must be contained in the kernel of the map $B_n \to \mathrm{Mod}(\mathbb{C}, B)$ which is generated by σ^n; now apply the preceding Proposition. ■

Action of twists. We can now determine the action of twists on $H^1(X)$. As in §2 we assume τ_i is supported in a disk with $D_i \cap B = \{b_i, b_{i+1}\}$, and let $X_i = \pi^{-1}(D_i)$.

Theorem 4.4 *For each* i *and* $q \neq 1$, *we can choose* $e_i \neq 0$ *such that*

$$H^1_c(X_i)_q = \mathbb{C}e_i, \quad \tau_i^*(e_i) = -qe_i, \quad \textit{and} \quad \langle e_i, e_i \rangle = -\operatorname{Im} q.$$

Proof. Since the statement concerns only the covering $X_i \to D_i$, which is branched over 2 points, it suffices to treat the case where $n = 2$, $i = 1$ and X is defined by $y^d = x^2 - 1$. In this case $R^* = -I$ on $H^1(X)$; indeed, $y : X \to \widehat{\mathbb{C}}$

presents X as a hyperelliptic Riemann surface, with $R(x,y) = (-x,y)$ its hyperelliptic involution. Moreover $\tau_1 = \sigma$, so we have

$$\widetilde{\tau}_1^* = \widetilde{\sigma}^* = (RT)^* = -T^* \quad \text{on } H^1(X) \tag{4.4}$$

by Proposition 4.2.

Now recall from (2.7) that for each q we have an exact sequence

$$H^1(\partial X_i)_q \to H^1_c(X_i)_q \to H^1(X)_q \to 0.$$

If $q \neq -1$, then the first term is zero and we get an equivariant isomorphism $H^1_c(X_i)_q \cong H^1(X)_q$. By Theorem 3.1, the latter space has dimension one and satisfies $H^1(X)_q = H^{1,0}(X)_q$ when $\operatorname{Im} q = \operatorname{Im} \zeta_d^{-k} > 0$ (i.e. $1/2 < k/d < 1$), and $H^1(X)_q = H^{0,1}(X)_q$ when $\operatorname{Im} q < 0$. Thus $H^1(X_i)_q$ admits a basis vector with $\langle e_i, e_i\rangle = \operatorname{Im} q$; and we have

$$\widetilde{\tau}_1^*(e_i) = -T^*(e_i) = -q e_i$$

by (4.4).

On the other hand, when $q = -1$ we have $H^1(X)_q = 0$, and by equation (2.6) we have an equivariant isomorphism

$$H^1(\partial X_i)_{-1} \cong H^1_c(X_i)_{-1}.$$

In this case d is even and ∂X_i has two components, which are preserved by $\widetilde{\tau}_1$ but interchanged by T; thus $H^1(\partial X_i)_{-1}$ is a one-dimensional Lagrangian subspace of $H^1_c(X_i)$, spanned by a vector e_i fixed by $\widetilde{\tau}_i^*$ and satisfying $\langle e_i, e_i\rangle = \operatorname{Im} q = 0$. ■

Corollary 4.5 *The lifted twist satisfies* $\widetilde{\tau}_i^* = -T^*$ *on* $H^1_c(X_i) \subset H^1(X)$.

Example of genus 1. Consider the case $(n,d,q) = (4,2,-1)$. Then the elliptic curve $y^2 = x^4 - 1$ can be identified with the Riemann surface $X = \mathbb{C}/\mathbb{Z}[\sqrt{-1}]$ in such a way that $R(z) = \sqrt{-1}z$, X_1 is a vertical annulus, and the lift of τ_1 is the full right twist $\widetilde{\tau}_1(z) = z + \operatorname{Im} z$. The other lifted twists are given (up to isotopy) by $\widetilde{\tau}_{i+1} = R^i \widetilde{\tau}_1 R^{-i}$, $i = 2,3,4$. Noting that

$$\widetilde{\tau}_1^*(dz) = dz + d \operatorname{Im} z = dz - \frac{\sqrt{-1}}{2}(dz - d\overline{z}),$$

we find the action of the braid group on

$$H^1(X)_q = H^1(X) = \mathbb{C}[dz] \oplus \mathbb{C}[d\overline{z}]$$

is given by

$$\widetilde{\tau}_i^*(x) = x - \frac{\sqrt{-1}}{2}\langle x, e_i\rangle e_i,$$

where $e_1 = dz - d\overline{z}$ and $e_{i+1} = (R^{-i})^* e_1$. Since $\mathbb{C}e_i$ is the unique eigenspace for $\widetilde{\tau}_i^*|H^1(X)$, it must coincide with $H^1_c(X_i)$. With these normalizations, we have $\langle e_i, e_i\rangle = 0$ and

$$\langle e_1, e_2\rangle = \langle e_i, e_{i+1}\rangle = \sqrt{-1}\,\langle dz - d\overline{z}, dz + d\overline{z}\rangle = 2\sqrt{-1}$$

for all i.

Proof of Theorem 4.1. We can assume X is defined by $y^d = x^n - 1$ and the disks D_i are chosen symmetrically with respect to rotation, as in Figure 4; then $R(X_i) = X_{i+1}$. Using Theorem 4.4, we can choose $e_i \in H^1(X)_q$ such that

$$H^1_c(X_i)_q = \mathbb{C}e_i, \quad \langle e_i, e_i\rangle = 2\operatorname{Im} q \quad \text{and} \quad R^*(e_{i+1}) = e_i.$$

Then $\langle e_i, e_j\rangle = 0$ if $|i-j| > 1$, since X_i and X_j are disjoint up to isotopy.

If $q = -1$, then $\langle e_i, e_i\rangle = 0$ as well, and e_i can be normalized so that $\langle e_i, e_{i+1}\rangle = 2\sqrt{-1}$ and $\widetilde{\tau}_i^*$ acts by (4.1), by comparison with the case $y^2 = x^4 - 1$ above. If $q = 1$ then $H^1(X)_q = 0$ and the theorem also holds.

Now suppose $q \neq -1$ (and $q \neq 1$, as usual). Since the value of $A = \langle e_i, e_{i+1}\rangle = \langle e_1, e_2\rangle$ is independent of n, to compute it we may assume $n = d$. Then by Theorem 3.1, 1 and q^{-1} do not occur as eigenvalues of $R|H^1(X)_q$, and thus $\sum_1^n e_i = \sum_1^n q^{-i} e_i = 0$. Pairing these vectors with e_1, we find

$$A + 2\operatorname{Im} q + \overline{A} = qA + 2\operatorname{Im} q + \overline{q}\overline{A} = 0,$$

which gives $A = \sqrt{-1}(1 - \overline{q})$.

Once the inner product is known, we can also compute that

$$\left\langle \sum_{i=1}^n \zeta_d^{-ji} e_i, e_1 \right\rangle = \zeta_d^j A + 2\operatorname{Im} q + \zeta_d^{-j}\overline{A} \neq 0$$

whenever $\zeta_d^j \neq 1$ or q^{-1}; thus the eigenspaces of $R|H^1(X)_q$ that *do* occur are all contained in the span of $(e_i)_1^n$. It follows that $(e_i)_1^n$ spans $H^1(X)_q$.

The intersection form on $H^1_c(X_i)$ is nondegenerate, so we have an orthogonal splitting

$$H^1(X)_q = H^1_c(X_i)_q \oplus H^1_c(X_i)_q^\perp \cong H^1_c(X_i)_q \oplus H^1(Y_i)_q$$

preserved by $\widetilde{\tau}_i^*$. Since $\widetilde{\tau}_i^*$ acts by $-q$ on the first factor by the identity on the second (cf. (2.8)), its action on $H^1(X)_q$ is given by the complex reflection formula (4.2). ∎

Remark: The cases $q = \pm 1$. When $q = 1$ we have $H^1(X)_q = 0$, and the formulas in Theorem 4.1 give $\langle e_i, e_j \rangle = 0$. On the other hand, if we rescale by $(\operatorname{Im} q)^{-1}$ as $q \to 1$ in S^1, then the inner product converges to the Coxeter matrix

$$\langle e_i, e_j \rangle' = \begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \cdots & & -1 \\ -1 & & & & -1 & 2 \end{pmatrix}$$

for the affine diagram $\widetilde{A}_n$, and the action of B_n on $H^1(X)_q$ converges to the action of S_n as a reflection group.

As $q \to -1$ in S^1, no rescaling is necessary: we have $(q+1)/\langle e_i, e_i \rangle \to \sqrt{-1}/2$, and thus the transvection (4.1) arises as a limit of the complex reflections (4.2). In the case $q = -1$ we can also express the action of B_n by

$$\tau_i^*(x) = x + [x, e_i] e_i,$$

where $[e_i, e_{i+1}] = 1$ and $[e_i, e_j] = 0$ for $|i - j| > 1$. This alternating form is a positive multiple of the usual intersection pairing on $H^1(X, \mathbb{R})$.

5 Complex reflection groups

In this section we put the action of the braid group on $H^1(X)_q$ in context by relating it to Artin systems, complex reflection groups and the Burau representation.

Artin systems. Let Γ be a finite graph with vertex set S. The associated *Artin group* $A(\Gamma)$ is the group freely generated by S, modulo the following relations:

1. If s and t are joined by an edge of Γ, then $sts = tst$ (we say s and t *braid*);

2. Otherwise, s and t commute.

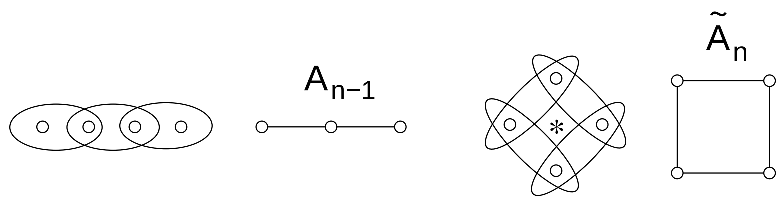

Figure 8. The braid groups of n points in $\mathbb{C}$ and $\mathbb{C}^*$ correspond to the Coxeter diagrams A_{n-1} and $\widetilde{A}_n$.

(For weighted graphs one allows longer relations of the form $stst \cdots s = tsts \cdots t$, but we will only need the simply-laced systems above.)

Complex reflections. Let V be a finite-dimensional complex vector space, and suppose $q \in S^1$, $q \neq -1$. A linear map $T : V \to V$ is a *q-reflection* if it acts by $v \mapsto -qv$ on a 1-dimensional subspace of V, and by the identity on a complementary subspace.

Every Artin group admits a finite-dimensional representation

$$\widetilde{\rho}_q : A(\Gamma) \to \mathrm{GL}(\mathbb{C}^S)$$

such that each $s \in S$ acts by a q-reflection through e_s. One such representation is given by the action

$$s(x) = x - (q+1)\frac{\langle x, e_s \rangle}{\langle e_s, e_s \rangle} e_s \tag{5.1}$$

with respect to the (possibly degenerate) real-valued inner product satisfying

$$\langle e_s, e_t \rangle = \begin{cases} 2 & \text{if } s = t, \\ 0 & \text{if } s \text{ and } t \text{ commute, and} \\ -\sec(\theta/2) & \text{if } s \text{ and } t \text{ braid.} \end{cases}$$

Here $\theta = \arg q \in (-\pi, \pi)$. The value $-\sec(\theta/2)$ is determined (up to sign) by the condition that the braid relation $sts = tst$ holds in the subspace $\mathbb{C}^{\{s,t\}}$.

Phase shift. More generally, we can modify this representation of $A(\Gamma)$ by specifying a *phase shift* $\phi(s,t) \in S^1$ whenever s and t braid (cf. [Mos1]), to obtain a new inner product

$$\langle e_s, e_t \rangle = \phi(s,t) \sec(\theta/2).$$

We assume $\phi(s,t) = \overline{\phi(t,s)}$, so this inner product is Hermitian. Then the action defined by (5.1) for the new inner product still respects the braiding and commutation relations.

The shifted representation of $A(\Gamma)$ depends only on q and the cohomology class of ϕ in $H^1(D, S^1)$ (represented by the cochain sending the edge $[s,t]$ to $\phi(s,t) \in S^1$.) In particular, if Γ is a tree, then the q-reflection representation is unique up to conjugacy, and we refer to its image as a *complex reflection group* of type $\Gamma(q)$.

Reduction. The quotient of $\mathbb{C}^S$ by the subspace $N = \{x \, : \, \langle x, y\rangle = 0 \; \forall y\}$ yields the *reduced* complex reflection representation

$$\rho_q : A(\Gamma) \to \mathrm{U}(\mathbb{C}^S/N).$$

Proposition 5.1 *If Γ is connected, then ρ_q is irreducible for all $q \neq -1$.*

Proof. Let $V \supset N$ be an invariant subspace of $\mathbb{C}^S$ with $V \neq N$. Then $\langle v, e_s\rangle \neq 0$ for some $v \in V$ and $s \in S$. Therefore $v - sv = (q+1)\langle v, e_s\rangle e_s \in V$ and hence $e_s \in V$. Now if s and t braid, then $\langle e_s, e_t\rangle \neq 0$ and thus $e_t \in V$ as well. Connectivity of Γ then implies $V = \mathbb{C}^S$. ■

Coxeter groups and Shephard groups. The *Shephard group* $A(\Gamma, d)$ is the quotient of $A(\Gamma)$ by the relations $s^d = 1$ for all $s \in S$ (cf. [KM]). Note that ρ_q factors through $A(\Gamma, d)$ whenever $(-q)^d = 1$. The quotient $A(\Gamma, 2)$ is the usual *Coxeter group* associated to Γ, and ρ_1 is its usual reflection representation.

Braid groups. Recall that the Coxeter diagrams A_n and $\widetilde{A}_n$ are simply an interval and a circle, each with n vertices. The braid group B_n, with its standard generators, can be identified with the Artin group of type A_{n-1}. Other spherical and affine Coxeter diagrams arise from braid groups for other orbifolds [Al1]. In particular, the Artin group of the type $\widetilde{A}_n$ can be identified with the braid group $\widetilde{B}_n$ of n points in $\mathbb{C}^*$ moving with total winding number (about $z = 0$) equal to zero (see Figure 8).

Theorem 5.2 *The braid group acts on $H^1(X)_q$ as a reduced complex reflection group of type $A_{n-1}(q)$ with generators $s_i = \widetilde{\tau}_i^*$.*

Proof. The inner product on $H^1(X)_q$ given in Theorem 4.1 satisfies

$$\frac{|\langle e_i, e_{i+1}\rangle|}{|\langle e_i, e_i\rangle|} = \frac{|1-q|}{2|\operatorname{Im} q|} = \frac{1}{|1+q|} = \frac{1}{2}\sec\frac{\theta}{2},$$

so it is proportional to the inner product for $A_{n-1}(q)$ up to phase factor; but this phase factor determines an equivalent representation, since A_{n-1} is a tree. ∎

Corollary 5.3 *The action of B_n on $H^1(X)_q$ is irreducible.*

The lifted representation. If we declare that the vectors $(e_i)_1^n$ are linearly independent, the formulas in Theorem 4.1 give a lifting of $\widetilde{\tau}_i^*$ to a matrix $T_i \in \mathrm{GL}_n(\mathbb{C})$ for $i = 1, \ldots, n$. It is given explicitly by

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -q & q \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-3}$$

and $T_i = P^{i-2} T_2 P^{2-i}$, where P is a cyclic permutation of the coordinates $(e_1, \ldots, e_n)$. The cyclically ordered braid relations are still satisfied, so we obtain a lifted representation

$$\widetilde{\rho}_q : \widetilde{B}_n \to \mathrm{GL}_n(\mathbb{C});$$

and the same inner product considerations show:

Theorem 5.4 *The lifted representation $\widetilde{\rho}_q$ gives a complex reflection group of type $\widetilde{A}_n(q)$, with phase shift $\phi(s_i, s_{i+1}) = \sqrt{-1}(1 - \overline{q})/|1 - q|$.*

Note however that $\widetilde{\rho}_q$ does *not* in general factor through B_n, because the relation $(\tau_1 \cdots \tau_{n-1})\tau_n = \tau_1(\tau_1 \cdots \tau_{n-1})$ does not lift to a relation between the matrices T_i.

The Burau representation. The classical *Burau representation* $\widetilde{\beta} : B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ is given by

$$\widetilde{\beta}(\tau_i) = I_{i-1} \oplus \begin{pmatrix} 1 - t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}.$$

These matrices preserve the vector $(1, 1, 1, \ldots, 1)$; taking the quotient yields the *reduced* Burau representation β.

Theorem 5.5 *For $q^n \neq 1$, the representation $\rho_q : B_n \to \mathrm{U}(H^1(X)_q)$ is dual to the reduced Burau representation specialized to $t = q$.*

(When $q^n = 1$, the reduced Burau representation has an extra trivial summand of dimension one.)

Proof. Let $t = q$. Then $\widetilde{\beta}(\tau_1)$ is a q-reflection through $v_1 = (1, -q, 0, \ldots, 0)$, and similarly for $\widetilde{\beta}(\tau_i)$. When $q^n \neq 1$ the vectors $(v_1, \ldots, v_{n-1})$ are linearly independent modulo $(1, 1, 1, \ldots, 1)$, so the reduced Burau representation, like ρ_q, gives a complex reflection group of type $A_{n-1}(q)$. Note, however, that $\rho_q(\tau_i) = (\widetilde{\tau}_i^*)^{-1}$, because the braid group is acting on cohomology (see equation (2.3)). Thus $\rho_q(\tau_i)$ is actually a $\overline{q}$-reflection, and hence the representations ρ_q and β are dual. ∎

One can also see this duality by noting that the reduced Burau representation gives the action of $\mathrm{Mod}_c(\mathbb{C}, B)$ on the first homology of the associated infinite cyclic covering of $\mathbb{C} - B$.

Examples: $\boldsymbol{n = 3}$. For $n = 3$ and $p > 6$, the image of $A_{n-1}(-\zeta_p)$ in $\mathrm{PU}(1,1)$ is simply the $(2,3,p)$-triangle group acting on the hyperbolic plane. It has the presentation

$$\Gamma(2,3,p) = \{a, b \,:\, a^2 = b^3 = (a^{-1}b)^p = e\},$$

where a is the image of $\tau_1\tau_2\tau_1$ and b is the image of $\tau_1\tau_2$. (Note that $a^2 = b^3$ is the image of the central element $(\tau_1\tau_2)^3 \in B_3$, so it is trivial by irreducibility.) This group arises as $\rho_q(B_3)$ when $q = \zeta_d^{-k}$ and $k/d = 1/2 \pm 1/p$. For $p = 3, 4, 5$, we obtain the Platonic symmetry groups A_4, S_4 and A_5 in $\mathrm{PU}(2)$; while for $p = 6$, the reduction of $A_{n-1}(-\zeta_p)$ lies in $\mathrm{PU}(1)$, so it is trivial.

Question 5.6 *For what values of n and q is the $A_n(q)$ reflection group a lattice in* $\mathrm{U}(r,s)$*?*

This question is open even for $A_n(\zeta_3)$ with $n \geq 7$. For related questions, see §11.

Notes. A Hermitian form preserved by the Burau representation is given in [Sq]; see also [Pe] and [KT, Ch.3]. It is known that the Burau representation is unfaithful for $n \geq 5$ [Bg1].

The reflection groups $A_n(q)$ are particular representations of the Hecke algebras $H_n(q)$. For more on complex reflection groups, Coxeter diagrams and braids, see e.g. [Sh], [Cox], [Mos1], [Mos2], [Al1], [J] and [BMR].

6 The period map

In this section we introduce the period map

$$f_q : \mathcal{T}^*_{0,n} \to \mathcal{H}^{r,s}$$

from the Teichmüller space of n points in the plane to the space $\mathcal{H}^{r,s}$ of positive lines in $\mathbb{C}^{r,s}$. This map records, for each configuration $B \subset \mathbb{C}$ and for $q = \zeta_d^{-k}$, the line in $H^1(X)_q \cong \mathbb{C}^{r,s}$ spanned by the holomorphic 1-form $[dx/y^k]$. It is defined so long as $1/n < k/d < 1$.

We then show:

Theorem 6.1 *The period map is a holomorphic local diffeomorphism when $q^n \neq 1$. Otherwise it is a submersion, with 1-dimensional fibers.*

Corollary 6.2 *Provided $q^n \neq 1$, the period map gives $\mathcal{M}^*_{0,n}$ the structure of a (G, X)-manifold, where $G = \mathrm{U}(r, s)$ and $X = \mathcal{H}^{r,s}$.*

Note: When $q = -1$, the structure group can be reduced from $\mathrm{U}(g, g)$ to $\mathrm{Sp}_{2g}(\mathbb{R})$ (see §12).

Corollary 6.3 *The moduli space $\mathcal{M}^*_{0,n}$ is endowed with a natural Kähler metric of constant negative curvature for each $k/d \in (1/n, 2/n)$, and constant positive curvature for each $k/d \in (1-1/n, 1)$.*

These metrics are modeled on the symmetric spaces $\mathbb{CP}^{n-2}$ and $\mathbb{CH}^{n-2}$ respectively. (For more on the complex hyperbolic case, see §10 and [Th2].)

Theorem 6.4 *When $q = \zeta_n^{-2}$, the period map descends to a holomorphic local diffeomorphism $f_q : \mathcal{T}_{0,n} \to \mathbb{CH}^{n-3}$.*

Corollary 6.5 *The period map for $q = \zeta_n^{-2}$ gives $\mathcal{M}_{0,n}$ the structure of a complex hyperbolic orbifold.*

Positive lines. Given (r, s) with $r > 0$, we let

$$\mathcal{H}^{r,s} \cong \mathrm{U}(r, s)/\,\mathrm{U}(r-1, s) \times \mathrm{U}(1)$$

denote the space of positive lines in the Hermitian vector space $\mathbb{C}^{r,s}$. Thus $\mathcal{H}^{r,0} \cong \mathbb{CP}^{r-1}$ and $\mathcal{H}^{1,s} \cong \mathbb{CH}^s$. In general, $\mathcal{H}^{r,s}$ carries an invariant Hermitian metric of signature $(r-1, s)$. (It should not be confused with the bounded symmetric domain $\mathrm{U}(r,s)/(\mathrm{U}(r)\times\mathrm{U}(s))$, whose natural metric is always Riemannian [Sat, Appendix, §3].)

Holomorphic 1-forms. Let $\Omega(X)$ denote the g-dimensional vector space of holomorphic 1-forms ω on $X \in \mathcal{M}_g$. Assume $\omega \neq 0$, and let $Z \subset X$ denote its zero set.

The *absolute periods* of ω are given by $\int_C \omega$, where C ranges all closed loops on X. The *relative periods* include, more generally, integrals along paths joining the zeros of ω. The relative periods are recorded by the cohomology class

$$[\omega] \in H^1(X,Z) = \mathrm{Hom}(H_1(X,Z),\mathbb{C}),$$

and the absolute periods by its image in $H^1(X)$.

Moduli spaces, Teichmüller spaces and periods. We will regard the moduli spaces

$$\mathcal{M}_{0,n}^* = \mathcal{T}_{0,n}^*/\operatorname{Mod}_{0,n}^* \quad \text{and} \quad \mathcal{M}_g = \mathcal{T}_g/\operatorname{Mod}_g$$

as quotients of the associated Teichmüller spaces of marked Riemann surfaces. Any two different markings of the same surface are related by the action of the mapping-class group.

The set of pairs (X,ω) with $\omega \neq 0$ forms the moduli space of 1-forms

$$\Omega\mathcal{M}_g \to \mathcal{M}_g.$$

This moduli space is a union of strata $\Omega\mathcal{M}_g(p_1,\ldots,p_s)$ labeled by partitions of $2g-2$; a form (X,ω) lies in a given stratum iff the zero set of ω consists of s distinct points with multiplicities $(p_1,\ldots,p_s)$.

The bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ pulls back to a bundle $\Omega\mathcal{T}_g \to \mathcal{T}_g$ which is stratified in the same way. The cohomology groups $H^1(X)$ form a trivial bundle over $\mathcal{T}_g$; thus if we fix a basepoint $(X_0,\omega_0) \in \Omega\mathcal{T}_g$, we have a natural *absolute period map*

$$\alpha : \Omega\mathcal{T}_g \to H^1(X_0)$$

given by $\alpha(X,\omega) = [\omega] \in H^1(X) \cong H^1(X_0)$.

Similarly, the cohomology groups $H^1(X,Z)$ form a locally trivial bundle over any stratum, so we have a locally defined *relative period map*

$$\pi : \Omega\mathcal{T}_g(p_i) \to H^1(X_0,Z_0).$$

By [V2] and [MS, Lemma 1.1], these relative period maps are holomorphic local homeomorphisms.

Point configurations. Now fix an integer $n \geq 2$, a degree $d > 1$ and a level k with $1/n < k/d < 1$. Set $q = \zeta_d^{-k}$ and $e = \gcd(d, n)$. As usual, we associate to a point configuration $B = (b_1, \ldots, b_n) \in \mathbb{C}^{\langle n \rangle}$ the branched covering space

$$X : y^d = (x - b_1) \cdots (x - b_n)$$

with deck transformation T. The level k picks out a holomorphic eigenform $\omega = dx/y^k \in H^{1,0}(X)_q$, and $F(B) = (X, \omega)$ gives a holomorphic map

$$F : \mathbb{C}^{\langle n \rangle} \to \Omega\mathcal{M}_g(p_1, \ldots, p_s).$$

Here the partition (p_i) depends only on (n, k, d); it can be read off from the formula

$$(\omega) = (kn/e - d/e - 1)\widetilde{\infty} + (d - 1 - k)\widetilde{B},$$

which is a special case of equation (3.2).

Note that X and the divisor (ω) depend only the pair $(\mathbb{C}, B)$ up to isomorphism, and (ω) determines ω up to scale. Thus F descends to a map $\mathcal{M}_{0,n}^* \to \mathbb{P}\Omega\mathcal{M}_g$. If $d|n$ then X is unramified over infinity, so it only depends on the pair $(\widehat{\mathbb{C}}, B)$; and if $k/d = 2/n$ as well, then the same is true of the divisor $(\omega) = (n-3)\widetilde{B}$. Summing up, we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^{\langle n \rangle} & \xrightarrow{F} & \Omega\mathcal{M}_g \\ \downarrow & & \downarrow \\ \mathcal{M}_{0,n}^* & \longrightarrow & \mathbb{P}\Omega\mathcal{M}_g \\ \downarrow & \nearrow & \downarrow \\ \mathcal{M}_{0,n} & \dashrightarrow & \mathcal{M}_g \end{array} \tag{6.1}$$

whose bottom arrow exists when $d|n$, and whose diagonal arrow exists when we also have $k/d = 2/n$.

The period map. Lifting to the universal cover of the domain, we obtain a map

$$\widetilde{F} : \widetilde{\mathbb{C}^{\langle n \rangle}} \to \Omega\mathcal{T}_g$$

whose image we denote by $(\Omega\mathcal{T}_g)_q$. By construction, the absolute periods of any form in the image of $\widetilde{F}$ determine a positive vector

$$[\omega] \in H^1(X)_q \cong H^1(X_0)_q \cong \mathbb{C}^{r,s},$$

because the trivialization of $H^1(X)$ over $(\Omega\mathcal{T}_g)_q$ respects the action of T. The line $[\mathbb{C}\omega]$ depends only the location of B in Teichmüller space, so $\alpha \circ \widetilde{F}$ descends to a function

$$f_q : \mathcal{T}^*_{0,n} \to \mathcal{H}^{r,s}$$

which we call the *period map.*

Since $\mathcal{H}^{r,s}$ is constructed from the flat bundle $H^1(X)_q$ over $\mathbb{C}^{\langle n \rangle}$ with holonomy $\rho_q : B_n \to \mathrm{U}(r,s)$, the map f_q is equivariant with respect to the actions coming from $B_n \to \mathrm{Mod}^*_{0,n}$ and $B_n \to \mathrm{PU}(r,s)$ (see equation (2.5)).

Lemma 6.6 *The map $\widetilde{F}$ covers a bijection*

$$\mathcal{T}^*_{0,n} \to (\mathbb{P}\Omega\mathcal{T}_g)_q,$$

except when $d|n$ and $k/d = 2/n$, in which case it factors through a bijection on $\mathcal{T}_{0,n}$.

Proof. For each $(X, \omega) \in (\Omega\mathcal{T}_g)_q$ there is a $T \in \mathrm{Aut}(X)$ which is in the same isotopy class as the deck transformation T_0 at the basepoint X_0, and which satisfies $T^*(\omega) = q\omega$. The data (X, ω, T) generally allows us to reconstruct the original configuration $(\widehat{\mathbb{C}}, B, \infty) \in \mathcal{T}^*_{0,n}$, by setting

$$\widetilde{B} \cup \widetilde{\infty} = Z(\omega) \cup \mathrm{Fix}(T^e) \tag{6.2}$$

and passing to the quotient $X/T \cong \widehat{\mathbb{C}}$. This reconstruction succeeds unless $d|n$ and $k/d = 2/n$. In this special case, X is unramified over ∞ and $(\omega) = (d-3)\widetilde{B}$, so (6.2) does not hold. But then (X, ω) depends only on the location of $[B]$ in $\mathcal{T}_{0,n}$, and the same procedure recovers $(\widehat{\mathbb{C}}, B)$. ∎

Lemma 6.7 *The relative period map from $(\Omega\mathcal{T}_g)_q$ to $H^1(X_0, Z_0)_q$ is a local homeomorphism.*

Proof. Recall that the relative period map gives a local homeomorphism $\Omega\mathcal{T}_g \supset U \to H^1(X_0, Z_0)$. The points in $H^1(X_0, Z_0)_q$ correspond to eigenforms (X, ω, T) with the same combinatorics as (X_0, ω_0, T_0), so they come from branched coverings of the sphere. ∎

Corollary 6.8 *The image of $\mathcal{M}_{0,n}^*$ in $\mathbb{P}\Omega\mathcal{M}_g$ is locally a union of linear submanifolds with respect to period coordinates.*

Lemma 6.9 *The map $H^1(X_0, Z_0)_q \to H^1(X_0)_q$ is surjective.*

Proof. The map $H^1(X_0, Z_0) \to H^1(X_0)$ is surjective, and T has finite order so its action is semisimple. ■

Proof of Theorems 6.1 and 6.4. The three lemmas above show f_q is a composition of submersions. By Corollary 3.3 we have $\dim \mathcal{H}^{r,s} = n-2$ if $q^n \neq 1$ and $n-3$ otherwise. Since $\dim \mathcal{T}_{0,n}^* = n-2$, f_q is a local homeomorphism in the first case and f_q has one-dimensional fibers in the second. When $d|n$ and $k/d = n/2$ the map f_q factors through $\mathcal{T}_{0,n}$ by Lemma 6.6, and $(r,s) = (1, n-3)$ by Corollary 3.2 so its target is $\mathbb{CH}^{n-3}$. ■

Proof of Corollaries 6.5 and 6.3. By Corollary 3.2 we have $(r,s) = (n-3, 0)$ or $(1, n-2)$ in these two cases, so f_q gives an equivariant map to $\mathbb{CP}^{n-2}$ or $\mathbb{CH}^{n-2}$. Pulling back the $U(r,s)$-invariant metric on the target gives the desired $\mathrm{Mod}_{0,n}^*$-invariant metric on the domain. ■

Example: conformal mapping. For a concrete instance of the period map, suppose $n = 3$ and $q = \zeta_d^{-k}$ with $1/3 < k/d \leq 1$. We can identify the moduli space $\widetilde{\mathcal{M}}_{0,3}^*$ of *ordered* triples of points in $\mathbb{C}$ with $\mathbb{C} - \{0, 1\}$. Then any $t \neq 0, 1$ determines a Riemann surface of the form

$$y^d = x(x-1)(x-t),$$

and the (multivalued) period map $f_q : \widetilde{\mathcal{M}}_{0,3}^* \to \mathbb{P}^1$ is given by

$$f_q(t) = [\textstyle\int_0^1 dx/y^k : \int_1^\infty dx/y^k].$$

This *Schwarz triangle function* is univalent on the upper halfplane, and $T = f_q(\mathbb{H})$ is an equilateral circular triangle with internal angles $2\pi|k/d - 1/2|$; compare [SG, §14, §16.6]. The triangle T is Euclidean for $k/d = 2/3$, spherical for $k/d > 2/3$, and hyperbolic for $k/d < 2/3$.

Filtrations. We remark that differentials $p(x)\,dx/y^k$ with $\deg p < i$ determine a filtration

$$0 \subset F_1 \subset F_2 \subset \cdots F_r \subset H^1(X)_q,$$

whose variation as a function of B could also be studied. We have focused our attention on the first term so that the period map becomes a local homeomorphism, providing a geometric structure on moduli space.

Notes. An elegant characterization of those classes $C \in H^1(X)$ which lie in $H^{1,0}(X)$ for *some* complex structure on X is given in [Kap] (see also §12 below). Another approach to Theorem 6.1 appears in [DM1, Prop. 3.9].

7 Arithmetic groups

We now turn to arithmetic constraints on the image of the braid group.

Consider the $\mathbb{Z}[q]$-module

$$\Lambda_{n,q} = H^1(X, \mathbb{Z}[q])_q \subset H^1(X)_q.$$

The unitary automorphisms preserving this module form a countable subgroup

$$\mathrm{U}(\Lambda_{n,q}) \subset \mathrm{U}(H^1(X)_q) \cong \mathrm{U}(r,s).$$

Since our representation of the braid group preserves $H^1(X, \mathbb{Z}[q])$, it also preserves $\Lambda_{n,q}$, and hence it factors as:

$$\rho_q : B_n \to \mathrm{U}(\Lambda_{n,q}) \subset \mathrm{U}(r,s).$$

In this section we will show:

Theorem 7.1 *The values of* $n \geq 3$ *and* $q \neq 1$ *such that* $\mathrm{U}(\Lambda_{n,q})$ *is a discrete subgroup of* $\mathrm{U}(H^1(X)_q) \cong \mathrm{U}(r,s)$ *are those given in table Table 9 and their complex conjugates. In these cases, either:*

1. $\mathrm{U}(\Lambda_{n,q})$ *is an arithmetic lattice in* $\mathrm{U}(r,s)$*, or*
2. *We have* $q = -1$*, and* $\mathrm{U}(\Lambda_{n,q}) \cong \mathrm{Sp}_{2g}(\mathbb{Z})$ *is a lattice in* $\mathrm{Sp}_{2g}(\mathbb{R}) \subset \mathrm{U}(g,g)$.

Corollary 7.2 *The image of the braid group* $\rho_q(B_n)$ *is discrete in the cases appearing in Table 9.*

Table of arithmetic groups. In Table 9, p gives the order of $\rho_q(\tau_i)$ in $\mathrm{U}(\Lambda_{n,q})$. Note that the values $k/d = 1/6, 1/4, 1/3$ and $1/2$ occur for every $n \geq 3$. These are exactly the cases where $\mathbb{Z}[q]$ is a discrete subring of $\mathbb{C}$, and they account for *all* cases of discreteness when $n > 12$.

For the entries with $n = 3$, $\mathrm{U}(\Lambda_{n,q})$ is an arithmetic triangle group of signature $(2, 3, p)$ (or its finite image in $\mathrm{U}(1)$, when $p = 6$). The complete list of arithmetic triangle groups can be found in [Tak]. Note that the $(2, 3, 5)$

n	k/d	(r,s)	p
3	1/6	(0,2)	3
3	1/4	(0,2)	4
3	1/10	(0,2)	5
3	3/10	(0,2)	5
3	1/3	(0,1)	6
3	5/14	(1,1)	7
3	3/8	(1,1)	8
3	7/18	(1,1)	9
3	2/5	(1,1)	10
3	9/22	(1,1)	11
3	5/12	(1,1)	12
3	3/7	(1,1)	14
3	7/16	(1,1)	16
3	4/9	(1,1)	18
3	11/24	(1,1)	24
3	7/15	(1,1)	30
3	1/2	(1,1)	∞
4	1/6	(0,3)	3
4	1/4	(0,2)	4
4	3/10	(1,2)	5
4	1/3	(1,2)	6
4	5/14	(1,2)	7
4	3/8	(1,2)	8
4	2/5	(1,2)	10
4	5/12	(1,2)	12
4	4/9	(1,2)	18
4	1/2	(1,1)	∞

n	k/d	(r,s)	p
5	1/6	(0,4)	3
5	1/4	(1,3)	4
5	3/10	(1,3)	5
5	1/3	(1,3)	6
5	3/8	(1,3)	8
5	2/5	(1,2)	10
5	5/12	(2,2)	12
5	1/2	(2,2)	∞
6	1/6	(0,4)	3
6	1/4	(1,4)	4
6	3/10	(1,4)	5
6	1/3	(1,3)	6
6	3/8	(2,3)	8
6	5/12	(2,3)	12
6	1/2	(2,2)	∞
7	1/6	(1,5)	3
7	1/4	(1,5)	4
7	3/10	(2,4)	5
7	1/3	(2,4)	6
7	3/8	(2,4)	8
7	5/12	(2,4)	12
7	1/2	(3,3)	∞
8	1/6	(1,6)	3
8	1/4	(1,5)	4
8	3/10	(2,5)	5
8	1/3	(2,5)	6
8	3/8	(2,4)	8
8	5/12	(3,4)	12
8	1/2	(3,3)	∞

n	k/d	(r,s)	p
9	1/6	(1,7)	3
9	1/4	(2,6)	4
9	3/10	(2,6)	5
9	1/3	(2,5)	6
9	5/12	(3,5)	12
9	1/2	(4,4)	∞
10	1/6	(1,8)	3
10	1/4	(2,7)	4
10	3/10	(2,6)	5
10	1/3	(3,6)	6
10	5/12	(4,5)	12
10	1/2	(4,4)	∞
11	1/6	(1,9)	3
11	1/4	(2,8)	4
11	1/3	(3,7)	6
11	5/12	(4,6)	12
11	1/2	(5,5)	∞
12	1/6	(1,9)	3
12	1/4	(2,8)	4
12	1/3	(3,7)	6
12	5/12	(4,6)	12
12	1/2	(5,5)	∞

n	k/d	(r,s)	p
$n > 12$	1/6	$(\lceil n/6-1\rceil, \lceil 5n/6-1\rceil)$	3
	1/4	$(\lceil n/4-1\rceil, \lceil 3n/4-1\rceil)$	4
	1/3	$(\lceil n/3-1\rceil, \lceil 2n/3-1\rceil)$	6
	1/2	$(\lceil n/2-1\rceil, \lceil n/2-1\rceil)$	∞

Table 9. For $q = \zeta_d^{-k}$ above, the braid group maps into an arithmetic lattice $\mathrm{U}(\Lambda_{n,q}) \subset \mathrm{U}(r,s)$.

triangle group occurs twice in our table, once for each complex place of $\mathbb{Q}(\zeta_5)$. In all other cases (including those with $n > 3$), we have $1/p + k/d = 1/2$.

Arithmetic perspectives. To establish Theorem 7.1, we will relate the group $\mathrm{U}(\Lambda_{n,q})$ to the full $\mathbb{Q}$-algebraic group $G \cong \mathrm{Sp}_{2g}(\mathbb{R})^T$ of automorphisms of $H^1(X)$ commuting with T and respecting the symplectic structure.

1. We first note that the real points of the centralizer factor as follows:

$$\begin{aligned} G(\mathbb{R}) &\cong \mathrm{Sp}_{2g}(\mathbb{R})^T \\ &\cong \mathrm{Sp}(H^1(X,\mathbb{R})_{-1}) \times \prod_{\mathrm{Im}\, q>0} \mathrm{U}(H_1(X)_q). \end{aligned}$$

   There are $\lfloor (d-1)/2 \rfloor$ terms in the second factor, one for each d-th root of unity q in the upper halfplane. This factorization of $G(\mathbb{R})$ is obtained from the splitting of $H^1(X,\mathbb{R})$ into eigenspaces $V_{q+\overline{q}}$ of $T + T^{-1}|H^1(X,\mathbb{R})$. We have also used the fact that for $q \neq \pm 1$, T determines a complex structure on $V_{q+\overline{q}}$ which yields an isomorphism

$$\mathrm{Sp}(V_{q+\overline{q}})^T \cong \mathrm{U}(H_1(X)_q).$$

2. A fundamental result of Borel and Harish-Chandra then implies that

   $G(\mathbb{Z}) \cong \mathrm{Sp}_{2g}(\mathbb{Z})^T$ *is a lattice in* $G(\mathbb{R}) \cong \mathrm{Sp}_{2g}(\mathbb{R})^T$.

   Indeed, the real characters of the symplectic and unitary factors of $G(\mathbb{R})$ satisfy $X_{\mathbb{R}}(\mathrm{Sp}_{2h}(\mathbb{R})) = X_{\mathbb{R}}(\mathrm{U}(r,s)) = 0$, and hence $X_{\mathbb{Q}}(G) \subset X_{\mathbb{R}}(G) = 0$ as well. By [BH, Thm. 9.4], this implies $\mathrm{vol}(G(\mathbb{R})/G(\mathbb{Z})) < \infty$.

3. Let $\Phi_e(x)$ denote the cyclotomic polynomial for the primitive e-th roots of unity, and let $G_e = G|\operatorname{Ker}(\Phi_e(T^*))$ We then have a similar factorization of $\mathbb{Q}$-algebraic groups, namely $G = \prod_{e|d} G_e$. Note that G_1 is trivial, G_2 is the symplectic group of $\operatorname{Ker}(T^* + I)$, and for $e \geq 3$ we have

$$G_e(\mathbb{R}) \cong \prod_{\Phi_e(q)=0, \mathrm{Im}\, q>0} \mathrm{U}(H_1(X)_q).$$

   Let $G_e(\mathbb{Z}) \subset G_e(\mathbb{R})$ denote the stabilizer of the lattice $L_e = \operatorname{Ker}(\Phi_e(T^*)) \cap H^1(X,\mathbb{Z})$. Then $G(\mathbb{Z})$ is commensurable to $\prod_{e|d} G_e(\mathbb{Z})$, and as above we find:

   $G_e(\mathbb{Z})$ *is a lattice in* $G_e(\mathbb{R})$.

4. Now suppose q is a primitive d-th root of unity. Then:

   The projection of $G_d(\mathbb{Z})$ to $\mathrm{U}(H_1(X)_q)$ is commensurable to $\mathrm{U}(\Lambda_{n,q})$.

   This comes from the fact that the $\mathbb{Z}[T]$-module L_e projects orthogonally to a $\mathbb{Z}[q]$-module commensurable to $\Lambda_{n,q}$ in $H^1(X)_q$. Clear $G(\mathbb{Z})$ preserves $\Lambda_{n,q}$, so we also find:

   The group $G(\mathbb{Z})$ projects to a subgroup of finite index in $\mathrm{U}(\Lambda_{n,q})$.

5. The lattice property of $G_d(\mathbb{Z})$ then implies:

   $\mathrm{U}(\Lambda_{n,q})$ is a discrete, arithmetic subgroup of $\mathrm{U}(H_1(X)_q)$ iff the other factors of $G_d(\mathbb{R})$ are compact.

6. Suppose $q = \zeta_d^k$ with $0 < k < d/2$ and $\gcd(k, d) = 1$. Using Corollary 3.2 to compute the signature (r', s') of $H^1(X)_{q'}$ for the other primitive d-th roots of unity, we deduce:

   $\mathrm{U}(\Lambda_{n,q}) \subset \mathrm{U}(H_1(X)_q)$ is a lattice $\iff$
   Every $k' \neq k$ in the range $0 < k' < d/2$ with $\gcd(k', d) = 1$ satisfies $k' \leq d/n$.

Table 9 is now readily verified by applying the criterion above, completing the proof of Theorem 7.1.

Notes and references. More generally, one can show that if $T \in \mathrm{Sp}_{2g}(\mathbb{Q})$ is semisimple, then $\mathrm{Sp}_{2g}(\mathbb{Z})^T$ is a lattice in $\mathrm{Sp}_{2g}(\mathbb{R})^T$ iff the irreducible factors of $\det(xI-T)$ are reciprocal polynomials (their roots are invariant under $z \mapsto 1/z$). (In particular, $\mathrm{Sp}_{2g}(\mathbb{Z})^T$ is a lattice in $\mathrm{Sp}_{2g}(\mathbb{R})^T$ whenever $\det(xI-T)$ is irreducible.)

The pseudo-Anosov mapping in genus 3 studied in [AY] and [Ar] gives an interesting example where the centralizer $\mathrm{Sp}_6(\mathbb{Z})^T$ is not a lattice in $\mathrm{Sp}_6(\mathbb{R})^T$. In this example, the reciprocal sextic polynomial $\det(xI - T)$ factors as a product $(x^3 + x^2 + x - 1)(x^3 - x^2 - x - 1)$ of two non-reciprocal cubics, and $\mathrm{Sp}_6(\mathbb{Z})^T$ is commensurable to $T^{\mathbb{Z}}$ in $\mathrm{Sp}_6(\mathbb{R})^T \cong \mathbb{R}^* \times \mathbb{C}^*$.

For background on arithmetic and algebraic groups, see e.g. [BH], [Bor2, §23], [Bor1], [Mg, Ch. I]. More on centralizers and conjugacy classes can be found in [SS], [Re] and [Gr].

8 Factors of the Jacobian

In this section we elaborate some of the algebraic geometry underlying the cases where $\rho_q(B_n)$ is finite. In particular, we describe a basis of elliptic differentials for the representative examples $(n, d) = (3, 4)$ and $(3, 6)$.

We begin by showing:

Theorem 8.1 *The representation of the braid group $\rho_q(B_n)$ at a primitive dth root of unity is finite iff*

> $n = 3$ *and* $d = 3, 4, 6$ *or* 10*; or*
> $n = 4$ *and* $d = 4$ *or* 6*; or*
> $n = 5, 6$ *and* $d = 6$.

Proof. In the 8 cases given we find, consulting Table 9, that $H^1(X)_q$ is definite for all primitive dth roots of unity q; hence $U(\Lambda_{n,q})$, and its subgroup $\rho_q(B_n)$, are both finite.

To see the list is complete for $n = 3$, note that if $-q$ is a primitive pth root of unity and $p > 6$, then $\rho_q(B_n)$ maps onto the $(2, 3, p)$ triangle group (see §5) and hence it is infinite. In particular, for $q = -1$ we have $\rho_q(B_n) \cong \mathrm{SL}_2(\mathbb{Z})$.

To see the list is complete for $n > 3$, let

$$g = \tau_1 \tau_2^{-1} \tau_3 \tau_4^{-1} \cdots \tau_n^{\pm 1}$$

and let $A = \rho_q(g)$. Observe that $\rho_q(B_n)$ contains a copy of $\rho_q(B_{n-1})$, so the list of d for which $\rho_q(B_n)$ is finite only gets shorter as n increases; and if $\rho_q(B_n)$ is finite, then the spectral radius $r(A)$ must be 1.

For $n = 4$ we need only test $d = 3, 4, 6$ and 10. We have seen that $d = 4$ and 6 give finite groups, and we can rule out $d = 3$ and 10 by checking that $r(A) > 1$ for $q = \zeta_3$ and $q = \zeta_{10}^3$. Similarly, for $n = 5$ we can rule out $d = 4$ by checking that $r(A) > 1$ at $q = \zeta_4$; and for $n = 7$ we can rule out $d = 6$ by checking that $r(A) > 1$ at $q = \zeta_6$. (Here we have used the fact that the representations ρ_q at different primitive dth roots of unity are all Galois conjugate.) ■

Corollary 8.2 *The image of the braid group $\rho_q(B_n)$ is finite iff $\mathrm{U}(\Lambda_{n,q})$ is finite.*

Jacobians. When $\rho_q(B_n)$ is finite, the Hodge structure on the corresponding part of the $H^1(X)$ is rigid; equivalently, we have an isogeny

$$\mathrm{Jac}(X) \to J(X) \times A$$

where $\dim A > 0$ and A is independent of X. We refer to A as a *rigid factor* of the Jacobian of X.

Theorem 8.3 *The Jacobian of the curve X defined by*

$$y^d = (x - b_1) \cdots (x - b_n)$$

(for n distinct points) has a rigid factor in the 8 cases shown in Table 10.

n	d	$g(X)$	Factor of $\mathrm{Jac}(X)$
3	3	1	$\mathbb{C}/\mathbb{Z}[\zeta_3]$
3	4	3	$(\mathbb{C}/\mathbb{Z}[\zeta_4])^2$
3	6	4	$(\mathbb{C}/\mathbb{Z}[\zeta_3])^2$
3	10	9	$(\mathbb{C}^2/\mathbb{Z}[\zeta_5])^2$

n	d	$g(X)$	Factor of $\mathrm{Jac}(X)$
4	4	3	$(\mathbb{C}/\mathbb{Z}[\zeta_4])^2$
4	6	7	$(\mathbb{C}/\mathbb{Z}[\zeta_3])^3$
5	6	10	$(\mathbb{C}/\mathbb{Z}[\zeta_3])^4$
6	6	10	$(\mathbb{C}/\mathbb{Z}[\zeta_3])^4$

Table 10. Rigid factors of the Jacobian.

In Table 10, $g(X)$ indicates the genus of X, and the rigid factor listed comes from $\bigoplus H^1(X)_q$ over the primitive dth roots of unity.

The factor $\mathbb{C}^2/\mathbb{Z}[\zeta_5]$, which occurs for $(n, d) = (3, 10)$, denotes quotient of $\mathbb{C}^2$ by the image of $\mathbb{Z}[\zeta_5]$ under $x \mapsto (x, x')$, where $x \mapsto x'$ is the Galois automorphism satisfying $\zeta_5' = \zeta_5^2$. The other factors listed are elliptic curves with automorphisms of orders 4 or 6.

Note that the cases $(n, d) = (3, 4)$ and $(4, 4)$ yield the same set of curves X, as do the cases $(5, 6)$ and $(6, 6)$. Note also that curves of type $(n, d) = (3, 6)$ cover curves of type $(3, 3)$, so they also have a second rigid factor $A' \cong \mathbb{C}/\mathbb{Z}[\zeta_3]$.

Proof of Theorem 8.3. Let $S = \bigoplus H^1(X)_q$, where the sum is taken over the primitive dth roots of unity. This subspace is defined over $\mathbb{Q}$, so it meets $H^1(X, \mathbb{Z})$ in a lattice $S_{\mathbb{Z}}$ of the same rank as $\dim_{\mathbb{C}} S$.

For the 8 cases listed, the induced Hodge structure on S is given by

$$S = S^{1,0} \oplus S^{0,1} = \left(\bigoplus_{\mathrm{Im}\, q > 0} H^1(X)_q \right) \oplus \left(\bigoplus_{\mathrm{Im}\, q < 0} H^1(X)_q \right),$$

where again the sum is over the primitive dth roots of unity (see Corollary 3.2). That is, the Hodge structure is refined by the eigenspace decomposition of $T|S$, which is independent of the complex structure on X; and hence the complex torus $A = S^{1,0}/S_{\mathbb{Z}}$ is a rigid factor of $\mathrm{Jac}(X)$. The dimension of A is determined by Corollary 3.3, and the CM-type of the endomorphism $T|A$ (see [Shi, §5.5A]) is determined by the condition $\mathrm{Im}\, q > 0$. This information determines A itself, up to isogeny, yielding Table 10. ∎

Genus 3 curves branched over 4 points. For added perspective we describe some details in one of the simplest cases, namely genus 3 curves X of the form

$$y^4 = (x - b_1)(x - b_2)(x - b_3).$$

1. Eigenspaces. The most interesting eigenspace here is $H^1(X)_q$ for $q = \sqrt{-1} = \zeta_4^{-3}$. This eigenspace satisfies $H^1(X)_q \cong H^{1,0}(X)_q$; its signature is $(2, 0)$, and it is spanned by the holomorphic 1-forms dx/y^3 and $x\, dx/y^3$.

2. Factors of $\mathrm{Jac}(X)$. By Theorem 8.3 we have an isogeny

$$\mathrm{Jac}(X) \to E \times A \times A, \tag{8.1}$$

where $A \cong \mathbb{C}/\mathbb{Z}[\zeta_4]$ is the square torus and E is an elliptic curve that depends on X. Composing with the inclusion $X \to \mathrm{Jac}(X)$, we obtain a dominant map $p_0 : X \to E$ and a pair of dominant maps $p_i : X \to A$, $i = 1, 2$.

3. Elliptic differentials. These maps can be seen directly as follows. First, observe that X is branched over $B' = B \cup \{\infty\}$, and that $\mathrm{Aut}(\widehat{\mathbb{C}}, B') \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Applying a Möbius transformation to B', we can normalize so that its symmetry group is generated by $x \mapsto -x$ and $x \mapsto 1/x$; then the defining equation for X takes the form

$$X \,:\, y^4 = x^4 + bx^2 + 1. \tag{8.2}$$

Now consider the elliptic curves and holomorphic 1-forms defined by

$$\begin{aligned} E \,&:\, z^2 = x^4 + bx^2 + 1, \quad \omega_E = dx/z, \quad \text{and} \\ A \,&:\, y^4 = u^2 + bu + 1, \quad \omega_A = du/y^3. \end{aligned}$$

Since A has an order 4 automorphism $(u, y) \mapsto (u, \sqrt{-1}y)$, it is evidently a square torus. Setting $z = y^2$ and $u = x^2$, we obtain degree two maps $p_0 : X \to E$ and $p_1 : X \to A$. Composing with an automorphism of X, we define $p_2(x, y) = p_1(1/x, y/x)$.

We claim the triple of maps (p_0, p_1, p_2) induces the desired isogeny (8.1). To see this, just note that the pullbacks of the forms $(\omega_E, \omega_A, \omega_A)$ under (p_0, p_1, p_2) respectively yield a basis

$$= (dx/y^2,\ dx/y^3,\ -x\,dx/y^3)$$

for $\Omega(X)$ consisting of elliptic differentials.

4. Action of the braid group. The braid group acts on $\mathbb{P}H^1(X)_q$ as the octahedral group $\Gamma \cong S_4 \subset \mathrm{PU}(2)$, preserving the spherical metric. The generating twists τ_1, τ_2 act by rotations $R_1, R_2 \in \Gamma$ of order 4 about orthogonal axes, corresponding to a pair of elliptic differentials as above.

5. The period map. The period map determines an embedding

$$f_q : \mathcal{M}^*_{0,3} \to (\mathcal{H}^{2,0} \cong \mathbb{P}^1)/S_4$$

whose image omits just one point. The domain is the $(2, 3, \infty)$ orbifold and the target is the $(2, 3, 4)$ orbifold; and f_q sends the cusp to the cone point of order 4. These facts can be verified by showing the metric completion of $\mathcal{M}_{0,4}$ in the induced spherical metric is an orbifold; or by explicitly computing the period map, using the elliptic forms above.

Remark: Shimura–Teichmüller curves. Equation (8.2) determines a remarkable curve $V \subset \mathcal{M}_3$ (parameterized by b). Möller shows V is the only subvariety of any moduli space $\mathcal{M}_g$, $g \geq 2$ which is both a Shimura curve and a Teichmüller curve [Mo].

Irregular covers of elliptic curves. The preceding example was particularly simple because the maps $p_i : X \to A$ were Galois coverings. To give the flavor of a more typical example, we describe a map $p : X \to A$ in the case $(n, d) = (3, 6)$.

In this case X is a genus 4 curve, presented as a covering of $\widehat{\mathbb{C}}$ branched over 4 points. We can change the x-coordinate by a Möbius transformation so the defining equation for X becomes

$$X\ :\ y^6 = (x^3 - 3x - b)(x - 2)^3$$

for some $b \in \mathbb{C}$ (because the cross-ratio of the roots of the polynomial on the right takes on all possible values). Now consider the elliptic curve defined by

$$A\ :\ z^6 = (b - 2)^4 w(w - 1)^3.$$

Because of its order 6 symmetry $S(w, z) = (w, \zeta_6 z)$, we have $A \cong \mathbb{C}/\mathbb{Z}[\zeta_3]$; and the formula

$$(w, z) = \left(\frac{x^3 - 3x - b}{2 - b}, y(x + 1)\right) \tag{8.3}$$

defines an irregular, degree 3 map $p : X \to A$. This map shows A occurs as one of the rigid factors of $\mathrm{Jac}(X)$.

This calculation is motivated by the observation that the desired map $p : X \to A$ induces a map

$$\overline{p} : X/\langle T\rangle \to A/\langle S\rangle$$

between a pair of orbifolds with signatures $(2, 6, 6, 6)$ and $(2, 3, 6)$ respectively. Since both orbifolds have genus zero, we can regard $\overline{p}$ as a rational map of the form $w = \overline{p}(x)$. The signatures of these orbifolds suggest there exists such a map with $\deg(\overline{p}) = 3$, sending the 3 points of order 6 in the domain to the unique point of order 6 in the range. Any such map would be totally ramified over the singular point of order 3, so in suitable coordinates it would be a cubic polynomial; and we are led to formula (8.3).

Remark. In the case $(n, d) = (3, 10)$, the projection to the rigid factor $\mathrm{Jac}(X) \to A \cong \mathbb{C}^2/\mathbb{Z}[\zeta_5]$ is *not* induced by a map $p : X \to Y$ with $\mathrm{Jac}(Y) \cong A$. This can be established by showing there is no map from the $(10, 10, 10, 10)$ orbifold to the $(2, 5, 10)$ orbifold which lifts to the required $\mathbb{Z}/10$ covering spaces. (It is, however, induced by a correspondence.)

9 Definite integrals

In this section we present another consequence of finiteness of the representation of the braid group.

Theorem 9.1 *Suppose $n \geq 3$ and $0 < \mu < 1$. Then*

$$I(b_1, b_2, \ldots, b_n) = \int_{b_1}^{b_2} \frac{dx}{((x - b_1)(x - b_2)\cdots(x - b_n))^{\mu}}$$

is an algebraic function of $(b_1, \ldots, b_n)$ iff $\rho_q(B_n)$ is finite for $q = \exp(-2\pi i\mu)$, and $\mu \neq 1/n$.

Here the integral is initially defined for distinct points $b_i \in \mathbb{R}$, and then extended to $\mathbb{C}$ by analytic continuation. Using Theorem 8.1 to check the finiteness of $\rho_q(B_n)$, we obtain:

Corollary 9.2 *The definite integral $I(b_1, \ldots, b_n)$ is algebraic exactly for the 17 values of (n, μ) given in Table 11.*

n	3								
μ	2/3	1/4	3/4	1/6	5/6	1/10	3/10	7/10	9/10

n	4			5				6
μ	3/4	1/6	5/6	1/4	3/4	1/6	5/6	5/6

Table 11. Algebraic definite integrals.

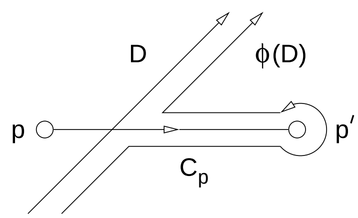

Figure 12. The cycle D is modified by a loop around p'.

The integral for the case $(n, \mu) = (4, 3/4)$ will be evaluated explicitly at the end of this section.

Meromorphic differentials. To set the stage for the proof, let ω be a meromorphic 1-form on a compact Riemann surface X with a finite set of poles $P \subset X$. Then ω determines a cohomology class $[\omega] \in H^1(X - P)$. If $\mathrm{Res}(\omega, p) = 0$ for every $p \in P$, then ω is traditionally called a differential of the *second kind*; it gives a well-defined cohomology class

$$[\omega] \in H^1(X) \subset H^1(X - P).$$

Now let $\phi \in \mathrm{Mod}(X, P)$ be a mapping-class stabilizing P, and assume ϕ is isotopic to the identity on X (it is in the kernel of the natural map $\mathrm{Mod}(X, P) \to \mathrm{Mod}(X)$). Then in the course of the isotopy, each $p \in P$ traces out a relative cycle $C_p \in H^1(X, P)$ joining p to $p' = \phi(p)$. It is then easy to compute the action of ϕ on $H_1(X - P)$: it satisfies (with suitable orientations)

$$[\phi(D)] = [D] + \sum_{p \in P} (C_p \cdot D)[U_{p'}],$$

where $U_{p'}$ is a small loop around p' (see Figure 12).

This computation facilitates the study of periods of general meromorphic 1-forms (which may have residues); in particular, it shows that

$$\int_D \phi^*(\omega) = \int_D \omega + \sum_P (C_p \cdot D) \operatorname{Res}(\omega, p'). \tag{9.1}$$

Proof of Theorem 9.1. The singularities of $I(b) = I(b_1, \dots, b_n)$ along the loci $b_i = b_j$ in $\mathbb{C}^n$ are algebraic iff μ is rational, so we may restrict attention to this case.

Consider a generic configuration of distinct points $B = \{b_1, \dots, b_n\}$. Let $\mu = k/d$ in lowest terms, let $q = \zeta_d^{-k}$, let X be the curve defined by $y^d = (x - b_1) \cdots (x - b_n)$ as usual, and let $\omega = dx/y^k$. Then $T^*(\omega) = q\omega$. The form ω potentially has poles on the set $P = \widetilde{\infty}$.

Let $L \subset X - P$ denote a lift of the segment $[b_1, b_2]$ to an arc connecting the corresponding points of $\widetilde{B}$, chosen so that $\omega | L$ agrees with the chosen branch of the integrand $I(b)$. Observe that $D = L - T(L)$ is a cycle on $X - P$, and that

$$\begin{aligned} I(b) &= \int_L \omega = (1-q)^{-1} \int_L \omega - T^*(\omega) = (1-q)^{-1} \int_D \omega \\ &= (1-q)^{-1} \langle D, \omega \rangle. \end{aligned}$$

Thus $I(b)$ simply measures one of the periods of ω.

Now suppose b is allowed to vary, keeping its coordinates distinct. Then $I(b)$ can be analytically continued, along many different paths, from b to any point in $S_n \cdot b$ (where the coordinates of b are permuted). The set of possible values for $I(\sigma \cdot b)$ so obtained are given, up to a factor of $(1-q)$, by

$$J(b) = \{ \langle \widetilde{\phi}(D), \omega \rangle \ : \ \phi \in \operatorname{Mod}(\mathbb{C}, B) \cong B_n \}. \tag{9.2}$$

Since $I(b)$ has algebraic singularities, it is a (multivalued) algebraic function of b iff $J(b)$ is finite.

To evaluate $|J(b)|$, first assume $k/d \neq 1/n$. Then ω is a differential of the second kind. To see this, let $e = \gcd(n, d)$. Then T^e fixes every $p \in P$, and satisfies

$$\operatorname{Res}(\omega, p) = \operatorname{Res}((T^e)^*\omega, p) = q^e \operatorname{Res}(\omega, p);$$

so the residues of ω vanish if $e < d$. On the other hand, if $e = d$ then $k/d \geq 2/n$, which implies ω is actually a holomorphic 1-form on the whole of X.

Since ω is of the second kind, it defines a cohomology class in $H^1(X)$. Thus the pairing in (9.2) depends only on the class of $\widetilde{\phi}(D) \in H_1(X)$, and therefore

$$J(b) = \langle D, \rho_q(B_n) \cdot \omega \rangle.$$

Clearly $J(b)$ is finite if $\rho_q(B_n)$ is finite. We also note that

$$0 \neq [D] \in H^1(X)^*_q, \tag{9.3}$$

because there exist configurations b such that $I(b) \neq 0$ (e.g. when all $b_i \in \mathbb{R}$). So conversely, if $J(b)$ is finite, then one of the 'matrix entries' of $\rho_q : B_n \to \mathrm{U}(H^1(X)_q)$ assumes only finitely many values; but ρ_q is irreducible (Corollary 5.3), so this implies $\rho_q(B_n)$ is finite as well.

It remains to show that $J(b)$ is infinite when $k/d = 1/n$. In this case ω has simple poles along P and $|P| = d$. Write $P = \{p_0, \ldots, p_{d-1}\}$ with $p_i = T(p_0)$. Since ω is an eigenform, it satisfies

$$\mathrm{Res}(\omega, p_i) = q^i \,\mathrm{Res}(\omega, p_0) \neq 0.$$

The idea is to show that $I(b)$ can be made to take on infinitely many different values by pushing the path of integration repeatedly through infinity, so it picks up a residue of ω. Note that (9.3) continues to hold, e.g. by considering one of the Galois conjugates of q.

Choose a cycle $C \in H_1(X, \mathbb{Z})$ such that

$$0 \neq E = \sum_{i=1}^{d} q^i T^i(C) \in H_1(X)_{\overline{q}} \cong H^1(X)^*_q. \tag{9.4}$$

(Most cycles have this property). Represent C by a closed loop on X that begins and ends at p_0, and otherwise avoids $P \cup \widetilde{B}$. Then the projection of C to $\widehat{\mathbb{C}}$ gives a loop $\gamma \in \pi_1(\widehat{\mathbb{C}} - B, \infty)$.

Let $\phi \in \mathrm{Mod}_c(\mathbb{C}, B)$ be a mapping class that pushes ∞ once around γ, and maps to the identity in $\mathrm{Mod}(\widehat{\mathbb{C}}, B)$. Then its canonical lift $\widetilde{\phi} \in \mathrm{Mod}(X - P)$ pushes p_i once around the cycle C_i, where $C_0 = C$ and $C_i = T^i(C_0)$; and $\widetilde{\phi}$ maps to the identity in $\mathrm{Mod}(X)$. In particular, by (9.1) we have

$$\begin{aligned} \langle \widetilde{\phi}(D), \omega \rangle &= \langle D, \omega \rangle + \sum_1^d (C_i \cdot D) \,\mathrm{Res}(\omega, p_i) \\ &= \langle D, \omega \rangle + \mathrm{Res}(\omega, p_0)(E \cdot D). \end{aligned}$$

Similarly, we have

$$\langle \widetilde{\phi}^k(D), \omega \rangle = \langle D, \omega \rangle + k \,\mathrm{Res}(\omega, p_0)(E \cdot D).$$

Thus $J(b)$ is infinite provided $E \cdot D \neq 0$, or more generally if $E \cdot \widetilde{\psi}(D) \neq 0$ for some $\psi \in \mathrm{Mod}_c(\mathbb{C}, B)$. But if these intersection numbers are all zero, then $E = 0$ or $D = 0$, since B_n acts irreducibly on $H_1(X)_q$. This contradicts equations (9.3) and (9.4). ∎

Genus 3 revisited. Here is an explicit formula for one of the algebraic integrals provided by Theorem 9.1. We have normalized so that $(b_1, b_2, b_3, b_4) = (0, 1, a, b)$.

Theorem 9.3 *For $1 < a < b < \infty$, the definite integral*

$$I(a,b) = \int_0^1 \frac{dx}{(x(x-1)(x-a)(x-b))^{3/4}}$$

satisfies

$$I(a,b)^4 \quad = \quad \frac{16\pi^2\Gamma(1/4)^4}{\Gamma(3/4)^4} \cdot \frac{c(1+c-2ac)^4}{a^2(a-1)^2(a-b)^3(c^2-1)^4},$$

where $c > 1$ is the largest root of the quadratic equation

$$(a-b)c^2 + (4ab - 2a - 2b)c + (a-b) = 0.$$

Eliminating c, we obtain the expression:

$$I(a,b)^4 = \frac{\pi^2\Gamma(1/4)^4}{\Gamma(3/4)^4} \cdot \frac{((2a-1)\sqrt{b(b-1)} + (2b-1)\sqrt{a(a-1)})^4}{a^2b^2(a-1)^2(b-1)^2(a+b-2ab-2\sqrt{ab(a-1)(b-1)})^3},$$

which exhibits the solution's symmetry in a and b.

Proof of Theorem 9.3. As in §8, we use the fact that the configuration $B = \{0, 1, a, b\} \subset \widehat{\mathbb{C}}$ has an order-two Möbius symmetry $g(x)$ satisfying $g(0) = 1$ and $g(a) = b$. (The point c has a natural meaning: it is the unique fixed point of g between a and b.) Thus by changing coordinates so the fixed points of g become $x = 0$ and $x = \infty$, we can transform $I(a,b)$ into an integral of the form:

$$\int_{-1}^1 \frac{(Ax + C)\,dx}{((x^2-1)(x^2-d^2))^{3/4}}.$$

The integrand is now a sum of elliptic differentials for the square torus, as discussed in §8. The term Ax can be ignored because it is odd; evaluating the remaining integral in terms of Γ-functions, we obtain the formula given in Theorem 9.3. ∎

Remark. If we change the exponent $\mu = 3/4$ to $1/2$, we essentially obtain the complete elliptic integral of the first kind,

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}.$$

This is a transcendental function of k, defined classically by the relation $\mathrm{sn}(K,k) = 1$ [WW, §22.3].

Notes and references. The values (α, β, γ) such that

$$I(b) = \int_0^1 x^\alpha (x-1)^\beta (x-b)^\gamma \, dx$$

is an algebraic function of b were determined by Schwarz in 1873 [Sch, §VI]; see [CW2] for a generalization to multivariable hypergeometric integrals, which includes Theorem 9.1 as a special case. The failure of algebraicity when ω has nonzero residues (e.g. when $\mu = 1/n$ in Theorem 9.1) is discussed by Klein in [Kl, §49], which corrects a statement of Riemann.

10 Complex hyperbolic geometry

In this section we discuss the cases where moduli space acquires a complex hyperbolic metric of finite volume. For example, we will see:

Theorem 10.1 *For $n = 4, 5, 6, 8$ and 12, the period map*

$$f_q : \mathcal{M}_{0,n} \to \mathbb{CH}^{n-3}/\rho_q(B_n)$$

at $q = \zeta_n^{-2}$ presents moduli space as the complement of a divisor in a finite-volume, arithmetic, complex-hyperbolic orbifold.

(This divisor corresponds to certain Dehn twists whose images under ρ_q have finite order.) Since $\mathcal{M}_2 \cong \mathcal{M}_{0,6}$ we also have:

Corollary 10.2 *The moduli space $\mathcal{M}_2$ can be completed to a complex hyperbolic orbifold of finite volume.*

Note that the complex hyperbolic structure on $\mathcal{M}_2$ comes from Hodge structures on surfaces of genus *four* (obtained as triple covers of $\widehat{\mathbb{C}}$ branched over six points).

These results are special cases of the work of Deligne–Mostow and Thurston, which we recall below. We will also give a self-contained proof of:

Theorem 10.3 *If* $\rho_q(B_n) \subset \mathrm{U}(1,s)$ *is discrete, then it is a lattice.*

Corollary 10.4 *Whenever* $\mathrm{U}(\Lambda_{n,q}) \subset \mathrm{U}(1,s)$ *is a lattice, so is* $\rho_q(B_n)$*.*

Table 9 furnishes 24 values of (n,q) with $n \geq 4$ where this Corollary applies, including the 5 cases covered by Theorem 10.1 above.

Hypergeometric functions and shapes of polyhedra. Let $\widetilde{\mathcal{M}}_{0,n}$ denote the moduli space of *ordered* points on $\widehat{\mathbb{C}}$ (an S_n-covering space of $\mathcal{M}_{0,n}$). Let P_n denote the pure braid group of n points on the sphere. There is a natural surjective map

$$P_n \to \pi_1(\widetilde{\mathcal{M}}_{0,n}).$$

Following [DM1] and [DM2], let $\mu = (\mu_1, \ldots, \mu_n)$ be a sequence of rational weights satisfying

$$0 < \mu_i < 1 \quad \text{and} \quad \sum \mu_i = 2.$$

Given $b = (b_1, \ldots, b_n) \in \mathbb{C}^{\langle \mathrm{n} \rangle}$, consider the multivalued algebraic 1-form

$$\omega_b = \frac{dx}{(x-b_1)^{\mu_1} \cdots (x-b_n)^{\mu_n}}.$$

Note that

$$K = (\omega_b) = -\sum \mu_i b_i$$

can be regarded as a generalized canonical divisor on $\widehat{\mathbb{C}}$, since $\deg(K) = -2$. The periods $\int_{b_i}^{b_j} \omega_b$ are multivariable hypergeometric functions, whose ratios become well-defined on the universal cover of $\widetilde{\mathcal{M}}_{0,n}$. These period coordinates give $\widetilde{\mathcal{M}}_{0,n}$ a complex hyperbolic metric g_μ, whose holonomy determines a representation

$$\xi_\mu : P_n \to \mathrm{Isom}(\mathbb{CH}^{n-3}) \cong \mathrm{PU}(1,n).$$

Alternatively, following [Th2], one can consider $|\omega_b|$ as a flat metric on $\widehat{\mathbb{C}}$ with cone angles $2\pi(1-\mu_i)$ at each point b_i. The space $(\widehat{\mathbb{C}}, |\omega_b|)$ is then isometric to a convex polyhedron in $\mathbb{R}^3$ of total area $A = \int_{\mathbb{C}} |\omega_b|^2$. By considering A as a quadratic form, Thurston obtains the same complex hyperbolic metric g_μ on $\mathcal{M}_{0,n}$ as Deligne and Mostow.[1]

The orbifold condition. The weights μ satisfy the *orbifold condition* if whenever $s = \mu_i + \mu_j < 1$, either

[1]The work [Th2] grew out of a discussion of problem 3 of the 27th Mathematical Olympiad (July 9, 1986).

- $(1-s)^{-1} \in \mathbb{Z}$, or
- $\mu_i = \mu_j$ and $2(1-s)^{-1} \in \mathbb{Z}$.

A list of the 94 different weights μ satisfying this condition, with $n \geq 4$, is given in the Appendix to [Th2]. This condition allows us to formulate following two important results.

Theorem 10.5 (Deligne–Mostow) *If the weights (μ_i) satisfy the orbifold condition, then the image $\xi_\mu(P_n)$ of the pure braid group is a lattice in* $\mathrm{PU}(1, n-3)$.

Theorem 10.6 (Thurston) *The metric completion of $(\widetilde{\mathcal{M}}_{0,n}, g_\mu)$ is a complex hyperbolic cone manifold of finite volume. It is an orbifold iff the weights (μ_i) satisfy the orbifold condition.*

See [DM1], [Mos1], and [Th2, Thm. 0.2], as well as [Par, Lemma 3.5], which clarifies the relationship between the two results.

Equal weights. It is straightforward to see that when $\mu_1 = \cdots = \mu_n = 2/n$, the constructions of Deligne–Mostow and Thurston give the same complex hyperbolic structure on

$$\mathcal{M}_{0,n} \cong \widetilde{\mathcal{M}}_{0,n}/S_n$$

as that furnished by the period mapping

$$f_q : \mathcal{T}_{0,n} \to \mathcal{H}^{1,n-3}$$

of §6 at $q = \zeta_n^{-2}$ (see Corollary 6.5). Similarly, if $\mu_1 = \cdots = \mu_{n-1} = k/d \neq \mu_n$, then $k/d \in (1/(n-1), 2/(n-1))$ and the complex hyperbolic metric on

$$\mathcal{M}^*_{0,n-1} \cong \widetilde{\mathcal{M}}_{0,n}/S_{n-1}$$

agrees with that furnished by the period mapping at $q = \zeta_d^{-k}$ (see Corollary 6.3). In particular, the images of the holonomy maps $\xi_\mu(P_n)$ and $\rho_q(B_n)$ are commensurable, and hence Theorem 10.5 provides a criterion for $\rho_q(B_n)$ to be a lattice.

Arithmetic lattices. To complement the results above, we present a short proof that $\rho_q(B_n) \subset \mathrm{U}(1,s)$ is a lattice whenever it is discrete.

Proof of Theorem 10.3. Assume $\rho_q(B_n) \subset \mathrm{U}(1,s)$ is discrete. Let $M = \mathcal{M}^*_{0,n}$ or $\mathcal{M}_{0,n}$ depending on whether $q = \zeta_d^{-2}$ or not, and let $Z = \mathbb{CH}^s/\rho_q(B_n)$. As we have seen in §6, the period map defines an analytic local homeomorphism of orbifolds $f_q : M \to Z$. By the Schwarz lemma,

f_q is distance-decreasing from the Teichmüller metric on M to (a suitable multiple of) the complex hyperbolic metric on Z.

Let $\overline{M}$ be the Deligne-Mumford compactification of M by stable curves with marked points, and let $\partial M = \overline{M} - M$. There is a unique maximal analytic continuation of f_q to an open set $\overline{M}_0$ with $M \subset \overline{M}_0 \subset \overline{M}$.

A local model $(U, \partial U)$ for $(M, \partial M)$ near any $x \in \overline{M}$ is given by a finite quotient of $(\Delta^a \times (\Delta^*)^b, 0 \times \Delta^b$ near $(0,0)$. Since the puncture of Δ^* is a cusp, every element of $\pi_1(U)$ is represented by a short loop in M. Thus every element of $H = f_*(\pi_1(U)) \subset \pi_1(Z)$ is represented by a short loop in Z.

If H is finite, then f_q lifts to a map $\widetilde{f}_q$ from a finite cover of U into the bounded domain $\mathbb{CH}^s$, and hence f_q itself extends analytically to a neighborhood of x. That is, $x \in \overline{M}_0$. Otherwise, there is an $h \in H$ which has no fixed point in $\mathbb{CH}^s$ but whose minimal translation distance is zero. Then by the Schwarz lemma, $f_q(U)$ is contained in the thin part of Z, and hence f_q tends to infinity at x.

It follows that $f_q : \overline{M}_0 \to Z$ is proper. In particular, $f_q(\overline{M}_0) = Z$ and $\mathrm{vol}(Z) = \mathrm{vol}(f_q(M))$ since $\dim \partial M < \dim M = \dim Z$. Since f_q is a contraction, the volume of $f_q(M) \subset Z$ is no more than a constant multiple of the volume of M in the Teichmüller metric, which is finite (see e.g. [Mc1][Thm. 8.1]). Thus $\mathrm{vol}(Z) < \infty$, and hence $\pi_1(Z) = \rho_q(B_n)$ is a lattice. ■

Note that Corollary 10.4 to Theorem 10.3 also follows from Theorem 10.5 and the following observation regarding the 36 entries in Table 9 with hyperbolic signature:

Proposition 10.7 *Whenever* $\mathrm{U}(\Lambda_{n,q}) \subset \mathrm{U}(1,s)$ *is a complex hyperbolic lattice, the corresponding weights* μ *satisfy the orbifold condition.*

Proof of Theorem 10.1. For these values of n, the entry $k/d = 2/n$ appears in Table 9, so we may apply Theorem 10.6. ■

Arithmetic hyperbolic structures for $n \le 12$. An examination of Table 9 also shows there is at least one arithmetic complex hyperbolic structure on $\mathcal{M}_{0,n}^*$ for every value of n with $3 \le n \le 11$. (For $n > 12$, hyperbolic signature no longer occurs.)

When $n = 3$, $\mathcal{M}_{0,3}^*$ can be completed to give the hyperbolic orbifold $\mathbb{H}/\Gamma(2,3,p)$ for any $p > 6$; this completion is arithmetic for 12 values of p. There are 6 arithmetic hyperbolic structures on $\mathcal{M}_{0,4}^*$ besides the one induced from the covering map $\mathcal{M}_{0,4}^* \to \mathcal{M}_{0,5}$; the case $n = 4$, $q = \zeta_3^{-1}$

was considered by Picard in 1883 [Pic]. There are also several arithmetic hyperbolic structures on $\mathcal{M}_{0,n}^*$ for $n = 5, 6, 7$, but only one for each $n = 8, 9, 10, 11$, and none for $\mathcal{M}_{12}^*$.

Non-arithmetic examples. Non-arithmetic lattices in $\mathrm{U}(1, s)$, $s > 1$, were first discovered in [Mos1]. For the braid group, Theorem 10.6 shows there is a unique case where $\rho_q(B_n)$ is nonarithmetic but $(\mathcal{M}_{0,n}^*, g_\mu)$ is still a complex hyperbolic orbifold: namely $n = 4$ and $\mu = (7, 7, 7, 7, 8)/18$. In this example $q = \zeta_{18}^{-7}$ and $\rho_q(B_n) \subset \mathrm{U}(1, 2)$.

Other values of μ giving non-arithmetic lattices are enumerated in [Th2, Appendix]. Only a finite number of non-arithmetic lattices in $\mathrm{U}(1, s)$ are presently known (up to commensurability).

Modular embeddings. Let us consider the non-arithmetic lattice coming from $n = 4$ and $d = 18$ in more detail. In this case the equation $y^{18} = x(x-1)(x-a)(x-b)$ determines a 2-dimensional family of curves X of genus $g = 25$. As in §7, we can regard $\Delta = \mathrm{Sp}_{2g}(\mathbb{Z})^T | \operatorname{Ker} \Phi_d(T^*)$ as an irreducible, arithmetic lattice in the product of unitary groups

$$\prod_{\Phi_d(q)=0, \operatorname{Im} q<0} U(H^1(X)_q) \cong U(1,2)^2 \times U(2)^2.$$

Because of the $\mathrm{U}(2)$ factor, Δ has no unipotents, so it determines a compact, 4-dimensional Shimura variety

$$S = (\mathbb{CH}^2 \times \mathbb{CH}^2)/\Delta.$$

Now let $\Gamma = \rho_q(B_n) \subset \mathrm{U}(1, 2)$ where $q = \zeta_{18}^{-7}$. Then Γ is a non-arithmetic lattice. Let $Z = \mathbb{CH}^2/\Gamma$ and let $r = \zeta_{18}^{-5}$ (so $H^1(X)_q$ and $H^1(X)_r$ both have signature $(1, 2)$). Using the period mappings at these two roots of unity, we obtain a holomorphic map $F : Z \to S$ making the diagram

$$\begin{array}{ccc} \mathcal{M}_{0,4}^* & & \\ \big\downarrow & \searrow^{(f_q, f_r)} & \\ \mathbb{CH}^2/\Gamma & \xrightarrow{F} & (\mathbb{CH}^2 \times \mathbb{CH}^2)/\Delta \end{array}$$

commute. When suitably normalized on the universal covers of domain and range, F determines a *modular embedding*

$$\widetilde{F} : \mathbb{CH}^2 \to \mathbb{CH}^2 \times \mathbb{CH}^2$$

of the form $\widetilde{F}(z) = (z, \widetilde{F}_2(z))$. The map $\widetilde{F}_2 : \mathbb{CH}^2 \to \mathbb{CH}^2$ is a transcendental function which intertwines the action of Γ with its Galois conjugate $\Gamma' =$

$\rho_r(B_n)$. The locus $F(Z) \subset S$ is a Kobayashi geodesic subvariety, reminiscent of the Teichmüller curves on Hilbert modular surfaces studied in [Mc2] and [Mc3].

For more details and further examples, see [CW3]. A similar construction can be carried out for nonarithmetic triangle groups [CW1].

11 Lifting homological symmetries

For brevity of notation, let $\mathrm{Sp}(X) = \mathrm{Sp}(H^1(X,\mathbb{Z})) \cong \mathrm{Sp}_{2g}(\mathbb{Z})$, let $\mathrm{Sp}(X)^T$ denote the centralizer of T^*, and let

$$\mathrm{Sp}(X)^T_d = \mathrm{Sp}(X)^T | \operatorname{Ker} \Phi_d(T^*),$$

where $\Phi_d(x)$ is the dth cyclotomic polynomial. As in §7, we have natural maps

$$B_n \to \mathrm{Mod}(X)^T \to \mathrm{Sp}(X)^T \to \mathrm{Sp}(X)^T_d.$$

We say a group homomorphism $\phi : G_1 \to G_2$ *almost onto* if $[G_2 : \phi(G_1)]$ is finite.

In this section we study the following purely topological question:

Question 11.1 *When is the natural map $B_n \to \mathrm{Sp}(X)^T_d$ almost onto?*

For example, by [A'C] we have:

Theorem 11.2 *When $d = 2$, the natural map $B_n \to \mathrm{Sp}(X)^T_d \cong \mathrm{Sp}(X)$ is almost onto for all $n \geq 3$.*

We will show that additional cases of Question 11.1 can be resolved using arithmeticity of lattices.

Proposition 11.3 *The following are equivalent:*

1. *The map $B_n \to \mathrm{Sp}(X)^T_d$ is almost onto.*
2. *The map $\mathrm{Mod}(X)^T \to \mathrm{Sp}(X)^T_d$ is almost onto.*
3. *There is a primitive dth root of unity q such that $\rho_q(B_n)$ has finite index in $\mathrm{U}(\Lambda_{n,q})$.*

Proof. Up to isotopy, any mapping-class $[\phi]$ commuting with T can be represented by a homeomorphism commuting with T (e.g. we may assume T is conformal and take ϕ to be a Teichmüller mapping). This easily implies that $B_n \to \mathrm{Mod}(X)^T$ is almost onto and hence (1) and (2) are equivalent. We have also seen, in §7, that the projection of $\mathrm{Sp}(X)^T_d$ to $\mathrm{U}(\Lambda_{n,q})$ is almost onto, so (1) is equivalent to (3). ∎

Theorem 11.4 *Suppose q is a primitive dth root of unity, and $\Gamma = \rho_q(B_n) \subset \mathrm{U}(1, s)$ is a lattice. Then $B_n \to \mathrm{Sp}(X)^T_d$ is almost onto if and only if Γ is arithmetic.*

Proof. If $\Gamma' = \mathrm{U}(\Lambda_{n,q})$ is discrete then the hypotheses imply both it and Γ are arithmetic lattices, by Theorem 10.3, and hence $[\Gamma' : \Gamma] < \infty$. Otherwise Γ' is indiscrete and $[\Gamma' : \Gamma] = \infty$. ∎

Corollary 11.5 *The map $B_3 \to \mathrm{Sp}(X)^T_d$ is almost onto if and only if the Fuchsian $(2, 3, p)$ triangle group is finite or arithmetic, where p is the order of $(d - 2)$ in $\mathbb{Z}/2d$.*

(Explicitly, using Table 9, we find $B_3 \to \mathrm{Sp}(X)^T_d$ is almost onto iff $d = 2, 3, 4, 5, 6, 7, 9, 10, 14, 15, 16, 18, 22$ or 24.)

Proof. We may assume $d = 2$ or $d \geq 7$, since otherwise $\mathrm{Sp}(X)^T_d$ is finite. Let q be the primitive dth root of unity closest to -1 with $\mathrm{Im}\, q \leq 0$. Then $\rho_q(B_3) \subset \mathrm{U}(1, 1)$ is a lattice, namely the discrete $(2, 3, p)$ triangle group (with $p = \infty$ when $d = 2$), and we may apply the Theorem above. ∎

Corollary 11.6 *The map $B_n \to \mathrm{Sp}(X)^T_d$ is almost onto whenever d and n appear in Table 9 with associated signature $(r, s) = (0, s)$ or $(1, s)$.*

Proof. In these cases either $\mathrm{Sp}(X)^T_d$ is finite or $\rho_q(B_n) \subset \mathrm{U}(1, s)$ is arithmetic by Corollary 10.4. ∎

Corollary 11.7 *The map $B_n \to \mathrm{Sp}(X)^T$ is almost onto for $d = 3$ and $n = 3, 4, 5, 6$; $d = 5$ and $n = 3, 4, 5$; and $d = 7$, $n = 3$.*

Proof. These cases appear in Table 9 as required, and d is prime, so $\mathrm{Sp}(X)^T_d = \mathrm{Sp}(X)^T$. ∎

On the other hand, since $\rho_q(B_4) \subset \mathrm{U}(1, 2)$ is a non-arithmetic lattice for $q = \zeta_{18}^{-7}$ (see §10), we find:

Corollary 11.8 *The image of B_4 has infinite index in $\mathrm{Sp}(X)^T_{18}$.*

Notes and references. It is well-known that for any closed orientable surface Σ_g of genus g, the natural map

$$\mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(\Sigma_g)$$

is surjective [MKS, Thm. N13] (see also [Bu, §7], [FM, Thm. 7.3]). One can then ask, which subgroups $H \subset \mathrm{Sp}(\Sigma_g)$ lift to $\mathrm{Mod}(\Sigma_g)$? This is a homological form of the Nielsen realization problem; cf. [Ker]. Question 11.1 is related to certain special cases of this question, where H is an abelian group of the form $\langle S, T\rangle$.

A version of question 11.1 for finite abelian coverings of closed surfaces is addressed in [Lo1].

12 The hyperelliptic case

In this section we discuss the hyperelliptic case, where X takes the form

$$y^2 = (x - b_1) \cdots (x - b_n) \qquad (12.1)$$

and $\omega = dx/y$. A special feature of this case is that the symmetry $T^*\omega = -\omega$ is preserved by the Teichmüller geodesic flow. Consequently, the period map

$$f_q : \mathcal{T}^*_{0,n} \to \mathcal{H}^{g,g}$$

is compatible with a natural action of $\mathrm{SL}_2(\mathbb{R})$. We will use this compatibility to analyze f_q and its image, and (in the next section) to describe f_q explicitly for the Teichmüller curves coming from regular polygons [V1].

The period domain. In the hyperelliptic case we have $d = 2$, $q = -1$, and $H^1(X)_q = H^1(X)$. Thus the period domain

$$\mathcal{H}^{g,g} \subset \mathbb{P}H^1(X) \cong \mathbb{P}\mathbb{C}^{g,g}$$

coincides with the space of all positive lines in the cohomology of X (§6). The real points

$$R_g \cong \mathbb{RP}^{2g-1}$$

of its ambient projective space $\mathbb{P}H^1(X)$ correspond to lines of the form $[\mathbb{C}v]$, $v \in H^1(X,\mathbb{R})$. Since $\langle v, v\rangle = 0$ for $v \in H^1(X,\mathbb{R})$, we have $R_g \subset \partial\mathcal{H}^{g,g}$. By restricting our attention to automorphisms that preserve this real structure, we may regard the period domain as the homogeneous space

$$\mathcal{H}^{g,g} = \mathrm{Sp}_{2g}(\mathbb{R})/(\mathrm{SO}(2,\mathbb{R}) \times \mathrm{Sp}_{2g-2}(\mathbb{R})). \qquad (12.2)$$

Secant lines. We say a line $L \subset \mathbb{P}H^1(X)$ is a *secant* if it joins a pair of real points $p, q \in R_g$. Every secant has the form $L = \mathbb{P}V$, where $V \subset H^1(X)$ is invariant under complex-conjugation and has signature $(1,1)$. The intersections $S = L \cap \mathcal{H}^{g,g}$ form the leaves of the *secant foliation* $\mathcal{S}$ of $\mathcal{H}^{g,g}$. Each leaf is a totally geodesic copy of $\mathbb{CH}^1$ (up to the sign of its metric).

The secant foliation can also be defined by observing that $\mathrm{SL}_2(\mathbb{R})$ acts $\mathbb{C} \cong \mathbb{R}^2$, and hence on $H^1(X) = \mathrm{Hom}(\pi_1(X), \mathbb{C})$; the orbits of this action project to the leaves of $\mathcal{S}$.

Weierstrass forms. Now assume $n = 2g+1$ is odd. Let $\mathcal{H}_g$ denote the moduli space of hyperelliptic Riemann surfaces of genus g. Using (12.1), each point configuration $[B] = [\{b_1, \ldots, b_n\}]$ in $\mathcal{M}^*_{0,n}$ determines an $X \in \mathcal{H}_g$ and a line $\mathbb{C}\omega \subset \Omega(X)$, where $\omega = dx/y$. We refer to ω as a *Weierstrass form*, since its divisor $(\omega) = (2g-2)\widetilde{\infty}$ is supported on a single Weierstrass point of X. The map $[B] \mapsto [(X, \omega)]$ gives a bijection

$$\mathcal{M}^*_{0,2g+1} \cong \mathbb{P}\Omega\mathcal{H}_g(2g-2) \cong \widetilde{\mathcal{H}}_g, \tag{12.3}$$

where $\widetilde{\mathcal{H}}_g$ is the degree $(2g+2)$ covering space of $\mathcal{H}_g$ parameterizing Riemann surfaces with a chosen Weierstrass point.

Each Weierstrass form determines a quadratic differential ω^2 on $\mathbb{C} - B$ hence a complex Teichmüller geodesic $W \cong \mathbb{H}$ through $([\mathbb{C}, B])$. These geodesics form the leaves of the *Weierstrass foliation* $\mathcal{W}$ of $\mathcal{T}^*_{0,2g+1}$. Under the isomorphism (12.3), the leaves of $\mathcal{W}$ correspond to the orbits of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{H}_g(2g-2)$ (see e.g. [KZ]). Since the actions of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{H}_g$ and $H^1(X)$ are compatible, we find:

Theorem 12.1 *The period map $f_q : \mathcal{T}^*_{0,2g+1} \to \mathcal{H}^{g,g}$ sends the Weierstrass foliation $\mathcal{W}$ to the secant foliation $\mathcal{S}$. Its restriction to each leaf is an isometry.*

Ergodicity. For $g = 1$, f_q simply gives the standard homeomorphism between $\mathcal{T}^*_{0,3}$ and $\mathcal{H}^{1,1} \cong \mathbb{H}$.

For $g > 1$ and $n = 2g+1$, the period map is still an equivariant local homeomorphism (§6). But the braid group now acts very differently on its domain and its range: while the orbits of B_n in Teichmüller space are discrete, a typical orbit in the period domain is *dense*. In fact:

Proposition 12.2 *The action of $\rho_q(B_n)$ on $\mathcal{H}^{g,g}$ is ergodic for $g > 1$.*

Proof. Equation (12.2) gives $\mathcal{H}^{g,g} = G/F$, where F is a closed subgroup of $G = \mathrm{Sp}_{2g}(\mathbb{R})$; and $\Gamma = \rho_q(B_n)$ is a lattice in G by Theorem 11.2. But for

$g > 1$, F is noncompact, so Γ acts ergodically on G/F by the Howe–Moore theorem [HM]. ∎

Thus the image of f_q is an open set of full measure, and f_q is *not* a covering map to its image (else $\rho_q(B_n)$ would have discrete orbits).

Proposition 12.3 *The group* $\operatorname{Ker}\rho_q$ *has infinite orbits on* $\mathcal{T}^*_{0,2g+1}$*, and hence the period map* f_q *is infinite-to-one for* $g > 1$*.*

Proof. The square of a Dehn twist about a loop enclosing an odd number of points of B lifts to a Dehn twist in $\operatorname{Mod}(X)$ which acts trivially on homology. ∎

For more on 'isoperiodic forms', see [Mc6, §9].

The image of the period map. Although the period map has a complicated structure, its image is reasonably tame. Indeed, the results of [Kap] easily imply:

Theorem 12.4 *For* $g > 1$*, the complement of the image of the period map*

$$f_q : \mathcal{T}^*_{0,2g+1} \to \mathcal{H}^{g,g}$$

is a nonempty, countable union of secant lines.

Proof. Let $G = \operatorname{Sp}_{2g}(\mathbb{R})$, $H = \operatorname{Sp}_2(\mathbb{R}) \times \operatorname{Sp}_{2g-2}(\mathbb{R})$, $\Gamma = \rho_q(B_{2g+1})$. By Theorem 11.2, Γ is a lattice in G. The image of the period map, modulo the secant foliation, is a nonempty, open, Γ-invariant subset $U \subset \mathcal{H}^{g,g}/\mathcal{S} \cong G/H$. By Ratner's theorem, for any $x \in G$ there is a connected Lie group J with $H \subset J$ such that $\overline{\Gamma x H} = \Gamma x J$ and $\Gamma \cap J_x$ is a lattice in $J_x = xJx^{-1}$ [Rat].

We will show $G/H - U$ is countable. Suppose $[xH] \not\in U$. Then $\Gamma x H$ is not dense in G, and hence $J \neq G$. As shown in [Kap, §3], this implies $J = H$. Thus $\Gamma_x = \Gamma \cap H_x$ is a lattice H_x. There are only countably many possibilities for Γ_x, since it is a finitely-generated subgroup of Γ; and H_x is the Zariski closure of Γ_x, so there are only countably many possibilities for H_x. But $N_G(H)/H$ is finite (in fact trivial for $g > 2$), so H_x determines $[xH]$ up to finitely many choices.

Thus the complement of the image of f_q is a countable union of leaves of the secant foliation. The complement is nonempty by [Kap, §1]: a cohomology class $p_* : \pi_1(X) \to \mathbb{Z}[i]$ determined by a degree one map $p : X \to \mathbb{C}/\mathbb{Z}[i]$ can never be realized by a holomorphic 1-form. ∎

The case $n = 2g + 2$: relative periods and quadratic differentials. Similar results hold for the period map $f_q : \mathcal{T}^*_{0,2g+2} \to \mathcal{H}^{g,g}$. In this case $\omega = dx/y$ has a pair of distinct zeros at the points $\{P, Q\}$ lying over ∞ in X, and f_q is a submersion with 1-dimensional fibers. The fibers form the leaves of the *absolute period foliation* $\mathcal{A}$ of $\mathcal{T}^*_{0,2g+2}$. Along any fiber A, one can normalize ω so its absolute periods are constant; then the (locally well-defined) relative period function

$$z = \int_P^Q \omega$$

determines a (globally well-defined) holomorphic quadratic differential $q = dz^2$ on A. The natural foliation of $\mathcal{T}^*_{0,2g+2}$ by complex Teichmüller geodesics (analogous to $\mathcal{W}$) is transverse to $\mathcal{A}$, so it gives rise to a homeomorphism map $\phi : A \to A'$ whenever $f_q(A)$ and $f_q(A')$ are joined by a secant; and in fact ϕ is a Teichmüller mapping, with dilatation proportional to $\overline{q}/|q|$. See [Mc6, §8] for a more detailed discussion, in the case of genus two.

13 Polygons and Teichmüller curves

Finally we describe certain totally geodesic *Teichmüller curves* in moduli space from the perspective of braid groups and polyhedra.

Regular polygons. We continue in the hyperelliptic setting, with $g > 0$ and $n = 2g + 1$. The Weierstrass foliation of $\mathcal{T}^*_{0,2g+1}$ descends to a foliation of $\mathcal{M}^*_{0,2g+1}$, which we also denote by $\mathcal{W}$.

Let

$$\alpha = \tau_1\tau_3\ldots\tau_{2g-1} \quad \text{and} \quad \beta = \tau_2\tau_4\ldots\tau_{2g} \tag{13.1}$$

be the elements of B_n obtained by grouping its even and odd generators together. The generators in each product commute, so their ordering is immaterial. Note that $\alpha\beta$ is a natural lifting of the Coxeter element of $S_n = W(A_{n-1})$ to the corresponding Artin group B_n. Thus the subgroup $\langle \alpha, \beta \rangle \subset B_n$ is a sort of halo of the Coxeter element.

Now let $B \subset \mathbb{C}$ be the vertices of a regular polygon with n sides, and let $V \subset \mathcal{M}^*_{0,n}$ be the leaf of the Weierstrass foliation through $[(\mathbb{C}, B)]$. Although most leaves of $\mathcal{W}$ are dense, Veech showed the special leaf V is *closed* [V1]. More precisely, using [Lei] or [Mc4] for the statement on π_1, we have:

Theorem 13.1 *The leaf V is a rational, totally geodesic algebraic curve in moduli space; and the image of the inclusion*

$$\pi_1(V) \hookrightarrow \pi_1(\mathcal{M}^*_{0,n})$$

is the $(2, n, \infty)$ *triangle group generated by the mapping-classes* $[\alpha]$ *and* $[\beta]$.

We remark that the triangle group relations follow from the identity

$$(\alpha\beta)^n = ((\alpha\beta)^g \alpha)^2 = \sigma^n$$

in the braid group (recall that the mapping class $[\sigma^n]$ is trivial).

As shown in [Loch] (see also [Mc5, §5]), the point configurations arising in V are given explicitly by

$$B(t) = \{\zeta + t\zeta^{-1} \,:\, \zeta^n = 1\},$$

where $t \in \mathbb{C}$ and $t^n \neq 1$. (Note that $B(t)$, $B(1/t)$ and $B(\zeta_n t)$ all represent the same point in moduli space.)

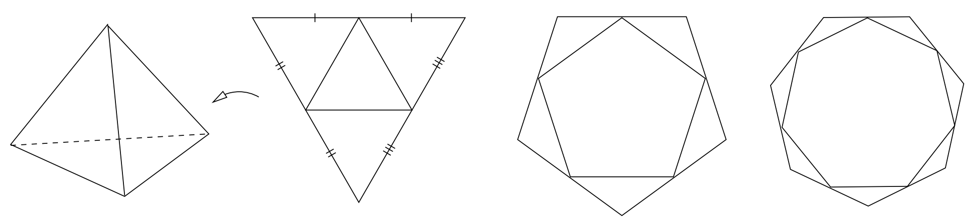

Figure 13. The sphere with the flat metric $|dx|/|x^n - 1|^{1/2}$, $n = 3, 5, 7$.

Polyhedra. Following [Th2], one can associate to each point configuration $B \subset \mathbb{C}$ the abstract Euclidean polyhedron $Q = (\widehat{\mathbb{C}}, |\omega|)$ determined by the flat metric

$$|\omega| = |dx/y| = |p(x)|^{-1/2}\,|dx|,$$

where $p(x) = \prod_B (x - b)$. This polyhedron has a cone angle of π at the points $x \in B$, and a cone angle of $(n-2)\pi$ at $x = \infty$.

Folded polygons and immersed disks. As $B(t)$ moves along V, one obtains a family of polyhedra related by Teichmüller mappings. These polyhedra take on particularly simple shapes at the orbifold points of V, which come from $t = 0$ and $t = -1$.

At $t = 0$ we have $p(x) = x^n - 1$, and Q can be obtained from a regular n-gon by folding its edges together at their midpoints, so that all n vertices are identified (see Figure 13). In the case $g = 1$ the result is a regular tetrahedron, but for $g > 1$ there is negative curvature at the Weierstrass point at infinity, so the polyhedron is no longer convex.

For $t = -1$, it is convenient to set $B = (\sqrt{-1}/2)B(t)$; then

$$p(x) = \prod_B (x-b) = \prod_{k=-g}^{g} \left(x - \sin\left(\frac{2\pi k}{n}\right)\right)$$

has real zeros $x_1 < x_2 < \cdots < x_n$. The Schwarz-Christoffel formula

$$f(z) = \int_0^z \frac{dt}{\sqrt{p(t)}}$$

yields a conformal immersion $f : \mathbb{H} \to \mathbb{C}$, which bends the real axis by 90 degrees at each point x_i, and $|\omega|$ is nothing more than the pullback of the Euclidean metric on $\mathbb{C}$ under f.

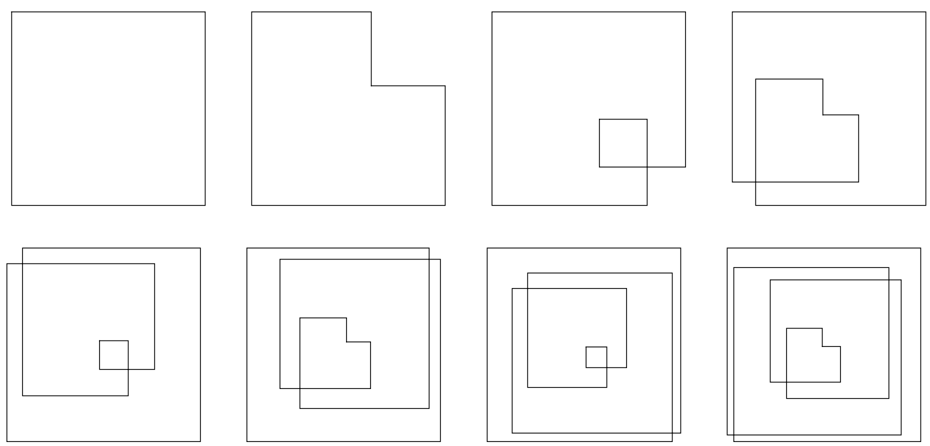

Figure 14. Immersed disks for $1 \le g \le 8$.

Thus Q can be visualized as the *double* of the immersed disk $D = f(\mathbb{H})$ across its boundary. The disk D is a square for $g = 1$, an L-shaped region for $g = 2$, and an immersed right-angled polygon for $g \ge 3$ (see Figure 14.) The immersion f for $g = 4$ is shown in more detail in Figure 15, where it has been factored as the composition $f(z) = h(z)^2$ of an embedding $h : \mathbb{H} \to \mathbb{C}$ with the squaring map.

Periods and eigenvectors. The lengths of the edges of D are given explicitly by the absolute values of periods of ω, namely

$$L_i = \int_{x_i}^{x_{i+1}} |p(x)|^{-1/2}\, dx.$$

These lengths can also be determined algebraically.

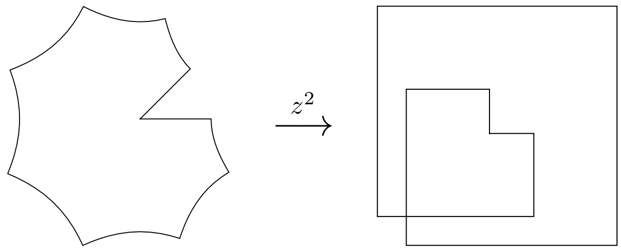

Figure 15. The immersion f, factored through z^2, for $g = 4$.

Theorem 13.2 *The lengths $(L_1, \ldots, L_{n-1})$ form a positive eigenvector for the adjacency matrix (C_{ij}) of the A_{n-1} Coxeter diagram.*

Here $C_{ij} = 1$ if $|i - j| = 1$ and 0 otherwise.

Proof. Let (h_i) be a positive eigenvector for C_{ij}, and let X be the hyperelliptic surface defined by $y^2 = p(x)$. The generators $[\tau_i]$ of $\mathrm{Mod}^*_{0,n}$ lift to Dehn twists about simple closed curves on X satisfying $S_i \cdot S_j = C_{ij}$. By a construction of Thurston, this implies there is a holomorphic quadratic differential q on X such that the loops (S_i) are represented by a cylinders of heights (h_i) on $(X, |q|)$, and the lifts of α and β are realized by affine automorphisms [Th1]; cf. [Mc4, §4]. In the case at hand, it is easy to see that q is a multiple of ω^2, and hence (h_i) is proportional to (L_i). ■

Remark. The fact that the ratios L_i/L_j are algebraic reflects the fact that the Jacobian of the curve $y^2 = p(x)$ admits complex multiplication by $\mathbb{Q}(i, \zeta_n + \zeta_n^{-1})$.

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