Math 213BR Solution Set 1
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Problem 1

Let $X = \mathbb{C} \setminus \{0, 1\}$. Give a pair of closed differential forms on X that furnish a basis for the deRham cohomology group $H^1(X, \mathbb{C}) \cong \mathbb{C}^2$.

Proof. Consider the differential forms dz/z and $dz/(z-1)$. Observe that these forms are holomorphic and therefore closed. Integrating dz/z and $dz/(z-1)$ along small loops about 0 and 1 (respectively) yields $2\pi i$. It follows that neither form is exact, and moreover, that no linear combination is exact. Thus, dz/z and $dz/(z-1)$ furnish a basis for $H^1(X, \mathbb{C}) \cong \mathbb{C}^2$. □

Problem 2

Let $f(z) = z(e^z - 1)$. Prove there exists an analytic function $h(z)$ defined near $z = 0$ such that $f(z) = h(z)^2$. Find the first 3 terms in the power series expansion $h(z) = \sum a_n z^n$. Does $h(z)$ extend to an entire function on $\mathbb{C}$?

Proof. Since f has a zero of order 2 at the origin, we can write $f(z) = z^2 g(z)$ for an analytic function $g(z)$ with $g(0) \neq 0$. Choose a well-defined branch of the square root of $g(z)$ in a neighborhood of zero. Define $h(z) = z\sqrt{g(z)}$ such that $h(z)^2 = f(z)$.

We compute the first three terms in the power series expansion $h(z) = \sum a_n z^n$ as follows. From the power series expansion of e^z, we see that

$$f(z) = z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} + \cdots .$$

Via coefficient matching, we have that $a_0^2 = 0$ implies $a_0 = 0$; then $a_1^2 = 1$ implies $a_1 = 1$; then $2a_1a_2 = 1/2$ implies $a_2 = 1/4$; and finally, $2a_1a_3 + a_2^2 = 1/6$ implies $a_3 = 5/96$. That is,

$$h(z) = z + \frac{z^2}{4} + \frac{z^3}{96} + \cdots .$$

Observe that $g(z)$ vanishes whenever $f(z)$ has a zero of order 1 (i.e., at $z = 2\pi i$). Thus, $h(z)$ cannot be extended to an entire function on $\mathbb{C}$. □

Problem 3

Let $B \subset \mathbb{C}$ be the union of the circles of radius 1 centered at ± 1.

1. Describe the universal cover of B as a topological space.
2. Draw a connected covering space B' of B with deck group S_3.
3. Draw a picture of $B'' = B'/A_3$. What is the deck group B''/B?

Proof. In Figure 1, the universal cover is shown on the left; two possible spaces with deck group S_3 are shown in the middle; and the result of quotienting each possibility by the action of A_3 is shown on the right. The deck group B''/B is $\mathbb{Z}/2$. □

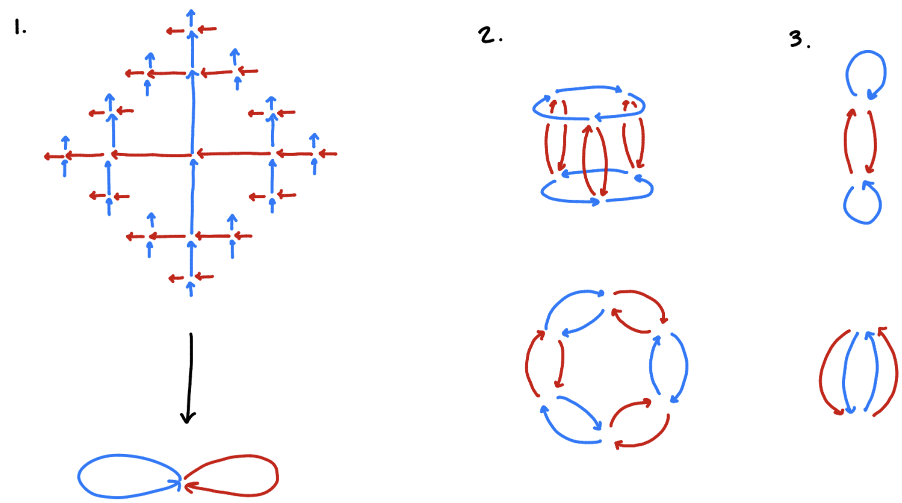

Figure 1: Covering spaces in Problem 3.

Problem 4

Show that the orbit of the unit circle $S^1 \subset \hat{\mathbb{C}}$ under $\mathrm{SL}_2(\mathbb{Z}[i])$ is a discrete subset of the space $\mathcal{C}$ of all circles on $\hat{\mathbb{C}}$. (Hint: think of $\mathcal{C}$ as a space of Hermitian forms.)

Proof. Recall that we can understand the space of (oriented) circles as the space of normalized Hermitian forms of signature $(1, 1)$, where the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

corresponds to the unit circle. (See page 6 of the course notes.) Under this correspondence, the action of $\mathrm{SL}_2\mathbb{C}$ on circles in $\hat{\mathbb{C}}$ can be transported to the space of Hermitian forms. We see that acting by $A \in \mathrm{SL}_2\mathbb{C}$ gives $H \mapsto A^*HA$. Now, note that the orbit of M under the restriction of this action to $\mathrm{SL}_2(\mathbb{Z}[\mathrm{i}])$ contains only matrices with $\mathbb{Z}[i]$-valued coefficients, and since $\mathrm{SL}_2(\mathbb{Z}[\mathrm{i}]) \subset \mathrm{SL}_2\mathbb{C}$ is discrete, so is the orbit of M. □

Problem 5

Let S be a closed smooth surface of genus two. Let us say a surface T is *large* if it arises as a connected regular covering space of S with an infinite deck group. Describe as many different large surfaces as you can (hint: five), and prove no two surfaces on your list are diffeomorphic to one another.

Proof. See Figure 2; the spaces are distinguished by the number of ends and whether the ends are planar or non-planar. To understand how they cover S, either

- divide the space into pairs of pants;
- or find a cover of the torus and add a "handle" to each fundamental domain. □

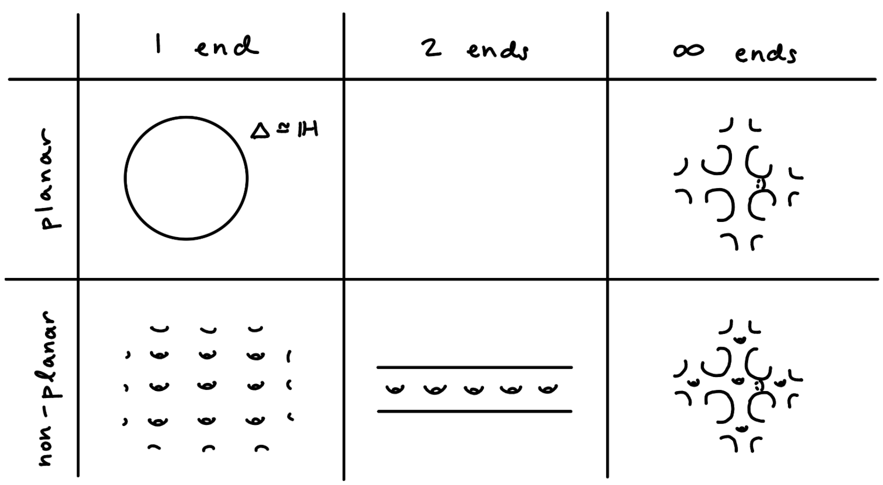

Figure 2: Covering spaces in Problem 5.