

# A PROOF OF BELYI'S THEOREM

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## 1. BACKGROUND

In this note, we provide a proof of the following theorem due to Belyi.

**Theorem 1.1** (Belyi). *A compact Riemann surface  $X$  is an algebraic curve over  $\overline{\mathbb{Q}}$  if and only if there exists a non-constant meromorphic function  $h : X \rightarrow \hat{\mathbb{C}}$  such that all branch points lie in the set  $\{0, 1, \infty\}$ .*

We begin with several preliminary definitions. Let  $X$  be a Riemann surface and let  $h : X \rightarrow \hat{\mathbb{C}}$  be a non-constant meromorphic function. Near each point  $x \in X$ , the map  $h$  can be expressed in local coordinates as  $z \mapsto z^m$  for a unique positive integer  $m$ . If  $m \geq 2$ , then  $x$  is called a *critical point* and  $h(x)$  is called a *critical value* or a *branch point*.

Suppose that  $h$  has branch points  $B = \{p_1, \dots, p_n\} \subset \hat{\mathbb{C}}$ . Then the restriction of  $h$  to  $X \setminus h^{-1}(B) \rightarrow \hat{\mathbb{C}} \setminus B$  is a covering map. We therefore say that  $X$  is a *branched cover* of  $\hat{\mathbb{C}}$ , and the map  $h$  is *branched over  $B$* . We record the branching structure of a meromorphic function as follows.

**Definition 1.2.** Let  $h : X \rightarrow \hat{\mathbb{C}}$  be a meromorphic function. The *branch data* of  $h$  consists of the branch points  $p_1, \dots, p_n \in \hat{\mathbb{C}}$  together with integer partitions  $P_i$  of  $\deg(h)$  describing the multiplicity of  $h$  at each point in  $h^{-1}(p_i)$ .

**Example 1.3.** Consider the map  $h : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  given by  $x \mapsto x^2(x-1)^3$ . The branch points of  $h$  are  $0, 1$ , and  $\infty$  and the corresponding partitions are  $\{2, 3\}$ ,  $\{2, 3\}$ , and  $\{5\}$ , respectively.

Riemann surfaces can also be viewed as algebraic curves via the following theorem.

**Theorem 1.4.** *Every compact Riemann surface  $X$  is an algebraic curve over  $\mathbb{C}$ . In other words, there exists an embedding  $X \hookrightarrow \mathbb{P}^n$  such that the image of  $X$  is the zero set of a collection of homogeneous polynomials  $f_1, \dots, f_k$  over  $\mathbb{C}$  in  $n+1$  variables.*

Belyi's theorem thus provides an algebraic description of Riemann surfaces which are branched covers of  $\hat{\mathbb{C}}$  branched over exactly three points. Note that the choice of  $0, 1, \infty$  is arbitrary because  $h$  can be post-composed with a Möbius function sending  $0, 1, \infty$  to any three points in  $\hat{\mathbb{C}}$ .

## 2. BELYI'S THEOREM

In this section, we provide a proof of Theorem 1.1, following pieces of those given by DeMarco [2], Wolfart [7], and Kock [4].

**2.1. The “only if” direction.** Suppose first that  $X \subseteq \mathbb{P}^n$  is a smooth algebraic curve defined over  $\overline{\mathbb{Q}}$ . Then on a Zariski open subset,  $X$  is the zero set of  $n$  homogeneous polynomials  $f_1, \dots, f_{n-1}$  in  $n+1$  variables  $x_1, \dots, x_{n+1}$  with coefficients in  $\overline{\mathbb{Q}}$ . Let  $p : X \rightarrow \hat{\mathbb{C}}$  be the meromorphic function given by projecting onto the first coordinate. Without loss of generality, suppose that  $p$  is non-constant.

We first show that the critical values of  $p$  belong to  $\overline{\mathbb{Q}} \cup \{\infty\}$ . Note that  $x \in X$  is a critical point of  $p$  precisely when  $T_x X \subseteq 0 \times \mathbb{P}^{n-1}$ , where

$$TX_x = \{v \in \mathbb{P}^n \mid df_1(x)v = \cdots = df_{n-1}(x)v = 0\}.$$

We claim that  $T_x X \subseteq 0 \times \mathbb{P}^{n-1}$  if and only if  $dx_1, df_1(x), \dots, df_{n-1}(x)$  are linearly dependent.

Suppose first that  $dx_1, df_1(x), \dots, df_{n-1}(x)$  are linearly independent. Then there exists a solution  $v$  to the linear system given by the equations  $df_1(x)v = \cdots = df_{n-1}(x)v = 0$  and  $dx_1 v = 1$ . Note that  $v \in T_x X$  and  $v \notin 0 \times \mathbb{P}^{n-1}$ ; thus,  $T_x X \not\subseteq 0 \times \mathbb{P}^{n-1}$ . Conversely, suppose that  $dx_1, df_1(x), \dots, df_{n-1}(x)$  are linearly dependent. Choose  $\lambda_1, \dots, \lambda_{n-1}$  such that

$$dx_1 = \lambda_1 df_1(x) + \cdots + \lambda_{n-1} df_{n-1}(x).$$

Evaluating this equation at any tangent vector  $v \in T_x X$  yields  $dx_1 v = 0$ . Thus,  $T_x X \subseteq 0 \times \mathbb{P}^{n-1}$ , as desired.

We have shown that  $x \in X$  is a critical point if and only if  $dx_1, df_1(x), \dots, df_{n-1}(x)$  are linearly dependent. Equivalently, the critical points of  $p$  are precisely the solutions to the algebraic equations  $f_1(x) = \cdots = f_{n-1}(x) = 0$  and

$$\det \begin{bmatrix} | & | & | & \cdots & | \\ dx_1 & df_1 & df_2 & \cdots & df_{n-1} \\ | & | & | & \cdots & | \end{bmatrix} = 0.$$

It follows that the set of critical points of  $p : X \rightarrow \hat{\mathbb{C}}$  is finite and fixed under elements of  $\text{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$ . Thus, the coordinates of the critical points must lie in  $\overline{\mathbb{Q}}$ . Since  $p$  is a projection onto the first coordinate, the critical values of  $p$  must belong to  $\overline{\mathbb{Q}} \cup \{\infty\}$ .

We will use  $p$  to construct a non-constant meromorphic function  $X \rightarrow \hat{\mathbb{C}}$  with rational (or infinite) critical values. To do so, we require the following lemma.

**Lemma 2.1.** *Suppose that  $P : X \rightarrow \hat{\mathbb{C}}$  is meromorphic with critical values in  $\overline{\mathbb{Q}} \cup \{\infty\}$ . Let  $m$  be the maximum algebraic degree of a finite critical value of  $P$ . Then there exists a non-constant meromorphic function  $Q : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  such that the finite critical values of  $Q \circ P$  have algebraic degree at most  $m - 1$ .*

*Proof.* Suppose that  $P$  has  $k$  critical values with algebraic degree  $m$ , and call one such critical value  $\alpha_0$ . Let  $Q_0 \in \mathbb{Z}[x]$  be the minimal polynomial of  $\alpha_0$ , and consider the composition  $Q_0 \circ P$ . The chain rule implies that

$$\text{CV}(Q_0 \circ P) = \text{CV}(Q_0) \cup Q_0(\text{CV}(P)).$$

For any  $\alpha \in \text{CV}(P) \subset \overline{\mathbb{Q}}$ , note that  $Q_0(\alpha) \in \mathbb{Z}[\alpha]$ , so  $\deg \alpha \geq \deg Q_0(\alpha)$ . Then since  $Q_0(\alpha_0) = 0$ , the set  $Q_0(\text{CV}(P))$  contains at most  $k - 1$  elements of degree  $m$ . Furthermore, the elements of  $\text{CV}(Q_0)$  are images under  $Q_0$  of roots of  $Q'_0$ , and therefore have algebraic degree at most  $m - 1$ . Thus,  $\text{CV}(Q_0 \circ P)$  contains at most  $k - 1$  elements of degree  $m$ .

Now successively define  $Q_1, Q_2, \dots, Q_j$  where  $Q_i$  ( $1 \leq i \leq j$ ) is the minimal polynomial of a degree  $n$  critical value of  $Q_{i-1} \circ \cdots \circ Q_0 \circ P$  and  $Q = Q_j \circ \cdots \circ Q_0$  has no critical values of degree  $m$ . By construction, the critical values of  $Q \circ P$  have degree at most  $m - 1$ , as desired.  $\square$

Let  $m = \max\{\deg y \mid y \in \text{CV}(p)\}$ . By Lemma 2.1, there exists a non-constant meromorphic function  $P_{m-1}$  such that the critical values of  $P_{m-1} \circ p : X \rightarrow \hat{\mathbb{C}}$  have algebraic degree at most  $m - 1$ . Continuing inductively, we obtain a non-constant meromorphic function  $P_1 \circ P_2 \circ \cdots \circ P_{m-1} \circ p : X \rightarrow \hat{\mathbb{C}}$  such that all critical values have degree 1; that is, they belong to  $\mathbb{Q} \cup \{\infty\}$ .

For simplicity, we apply a dilation  $D(z) = rz$  such that the critical values of

$$D \circ P_1 \circ \cdots \circ P_{m-1} \circ p$$

belong to  $([0, 1] \cap \mathbb{Q}) \cup \{\infty\}$ . We now compose with a series of polynomials in  $\mathbb{Q}[z]$  which send each critical value into the set  $\{0, 1, \infty\}$ . For any  $a, b \in \mathbb{Z}_{>0}$ , consider the polynomial

$$g_{a,b}(z) = \frac{(a+b)^{a+b}}{a^a b^b} z^a (1-z)^b.$$

By taking the derivative, we see that the critical points of  $g_{a,b}$  are 0, 1, and  $a/(a+b)$ . Thus, the critical values lie at  $g_{a,b}(0) = g_{a,b}(1) = 0$  and  $g_{a,b}(a/(a+b)) = 1$ . Since  $g_{a,b}$  attains its maximum at  $a/(a+b)$  and is positive on  $(0, 1)$ , we must have that  $g_{a,b}([0, 1]) = [0, 1]$ . Moreover,  $g_{a,b}(\infty) = \infty$  and  $g_{a,b}(\mathbb{Q}) \subseteq \mathbb{Q}$ . Thus, the subsets  $\{0, 1, \infty\}$  and  $([0, 1] \cap \mathbb{Q}) \cup \{\infty\}$  are invariant under  $g_{a,b}$ .

Note that every element of  $([0, 1] \cap \mathbb{Q}) \cup \{\infty\}$  other than 0, 1, and  $\infty$  can be expressed in the form  $a/(a+b)$  for some  $a, b \in \mathbb{Z}_{>0}$ . Thus, we can successively compose  $D \circ P_1 \circ \cdots \circ P_{m-1} \circ p$  with polynomials of the form  $g_{a,b}$  to send the critical values one by one into the set  $\{0, 1, \infty\}$ . That is, there exist  $g_{a_1, b_1}, \dots, g_{a_l, b_l} \in \mathbb{Q}[z]$  such that  $a_i/(a_i + b_i)$  is a critical value of the composition  $g_{a_{i-1}, b_{i-1}} \circ \cdots \circ g_{a_1, b_1} \circ D \circ P_1 \circ \cdots \circ P_{m-1} \circ p$  and

$$g_{a_l, b_l} \circ \cdots \circ g_{a_1, b_1} \circ D \circ P_1 \circ \cdots \circ P_{m-1} \circ p$$

has critical values in  $\{0, 1, \infty\}$ , as desired.  $\square$

**2.2. The “if” direction.** We now prove the converse. Let  $X$  be a compact Riemann surface. Suppose that there exists a non-constant meromorphic function  $h : X \rightarrow \hat{\mathbb{C}}$  with critical values in  $\{0, 1, \infty\}$ .

**Definition 2.2.** A *Belyi pair*  $(X, h)$  consists of a Riemann surface  $X$  together with a non-constant meromorphic function  $h : X \rightarrow \hat{\mathbb{C}}$  which is unramified outside of  $\{0, 1, \infty\}$ .

Recall that  $X$  is an algebraic curve over  $\mathbb{C}$  ([3], Theorem 1.9). That is, there exists an embedding  $X \hookrightarrow \mathbb{P}^n$  such that the image of  $X$  is the zero set of homogenous polynomials  $f_1, \dots, f_{n-1}$  in the variables  $x_1, \dots, x_{n+1}$ . It follows from Corollary 1.23 (iii) in [6] that the function  $h : X \rightarrow \hat{\mathbb{C}}$  is a rational function in  $x_1, \dots, x_{n+1}$ . Define  $\mathcal{P}(X, h)$  to be the ordered list consisting of the coefficients of  $f_1, \dots, f_{n-1}$  and the coefficients of  $h$ .

For any element  $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ , define  $\mathcal{P}^\sigma(X, h) := \sigma \cdot \mathcal{P}(X, h)$ . We claim that  $\mathcal{P}^\sigma(X, h)$  describes a Belyi pair  $(X^\sigma, h^\sigma)$  of the same degree and genus as  $(X, h)$  and with the same branch data. To see that this is true, first observe that  $h^\sigma = \sigma \circ h \circ \sigma^{-1}$ , where (by a slight abuse of notation)  $\sigma^{-1}$  acts on elements of  $X^\sigma$  coordinate-wise. Thus,  $\deg(h) = \deg(h^\sigma)$ . Now let  $x$  be a critical point of  $h$ , and note that  $\sigma(x)$  is a critical point of  $h^\sigma$  of the same degree. Furthermore, the critical value  $h(x) \in \{0, 1, \infty\}$  is fixed under  $\text{Gal}(\mathbb{C}/\mathbb{Q})$ , so  $h^\sigma(\sigma(x)) = \sigma \circ h(x) = h(x)$ . It follows that the branch data of  $(X, h)$  is preserved under  $\sigma$ . Now recall that the degree of  $X$  is defined as the number of intersections with a general hypersurface  $H$ . Since  $H$  is defined by linear equations, the image  $\sigma(H)$  is also a hypersurface. Clearly, the number of intersections of  $X$  and  $H$  is preserved under  $\sigma$ , so the degrees of  $X$  and  $X^\sigma$  are equal. Lastly, the genus of  $X$  is preserved under  $\sigma$  by the Riemann-Hurwitz formula.

**Definition 2.3.** The Belyi pairs  $(X, h)$  and  $(X', h')$  are *isomorphic* if there exists a conformal isomorphism  $i : X \rightarrow X'$  such that  $h' \circ i = h$ .

Note that if  $(X, h)$  and  $(X', h')$  are isomorphic via  $i$ , then  $(X^\sigma, h^\sigma)$  and  $(X'^\sigma, h'^\sigma)$  are isomorphic via  $i^\sigma = \sigma \circ i \circ \sigma^{-1}$ . This is because

$$i^\sigma(X^\sigma) = \sigma \circ i \circ \sigma^{-1}(X^\sigma) = \sigma \circ i(X) = \sigma(X') = X'^\sigma$$

and

$$h'^{\sigma} \circ i^{\sigma} = (\sigma \circ h' \circ \sigma^{-1}) \circ (\sigma \circ i \circ \sigma^{-1}) = \sigma \circ h' \circ i \circ \sigma^{-1} = \sigma \circ h \circ \sigma^{-1} = h^{\sigma}.$$

Thus,  $\text{Gal}(\mathbb{C}/\mathbb{Q})$  acts on the isomorphism classes of Belyi pairs. Recall that there are finitely many branched covers with identical branch data over the critical values  $\{0, 1, \infty\}$ . (This is a consequence of the Riemann Existence Theorem, as there are finitely many possible monodromy homomorphisms for any fixed branch data.) It follows that  $S := \text{Stab}([(X, h)])$  has finite index in  $\text{Gal}(\mathbb{C}/\mathbb{Q})$ .

**Definition 2.4.** Given a variety  $X$  over  $\mathbb{C}$  and a meromorphic function  $h : X \rightarrow \hat{\mathbb{C}}$ , the *moduli field*  $M(X, h)$  is the fixed field of the subgroup  $S := \text{Stab}([(X, h)]) \subseteq \text{Gal}(\mathbb{C}/\mathbb{Q})$ .

We claim that  $M(X, h)$  is a finite extension of  $\mathbb{Q}$ . View  $\text{Gal}(\mathbb{C}/\mathbb{Q})$  as a topological group under the Krull topology. Denote the topological closure of  $S$  by  $\bar{S}$ , and note that

$$[\text{Gal}(\mathbb{C}/\mathbb{Q}) : \bar{S}] \leq [\text{Gal}(\mathbb{C}/\mathbb{Q}) : S] < \infty.$$

Thus, the complement of  $\bar{S}$  in  $\text{Gal}(\mathbb{C}/\mathbb{Q})$  is a finite union of cosets  $\sigma\bar{S}$  for  $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ . It follows that  $\bar{S}$  is open, so by the Fundamental Theorem of infinite Galois Theory, the fixed field  $\mathbb{C}^{\bar{S}}$  is a finite subextension of  $\mathbb{Q}$ . Finally, Theorem 4.7(4) in [1] implies that  $M(X, h) = \mathbb{C}^S = \mathbb{C}^{\bar{S}}$ , so  $M(X, h)$  is a number field, as desired.

We aim to show that  $X$  is defined over a finite extension of  $M(X, h)$ , and thus over  $\bar{\mathbb{Q}}$ . (See Theorem 2.2 in [4] for a more general version of this statement.)

Let  $Q \in \mathbb{Q} \setminus \{0, 1, \infty\}$ , and fix  $P \in h^{-1}(Q)$ . Denote the genus of  $X$  by  $g$ , and define the divisor  $D := (g + 1)[P]$ . Then the Riemann-Roch Theorem implies that there exists a non-constant meromorphic function on  $X$  with a pole only at  $P$ . Choose one such function  $f : X \rightarrow \hat{\mathbb{C}}$  for which the order of the pole at  $P$  is minimal. Then  $f$  is uniquely determined up to constant addition and multiplication. Note that the meromorphic function  $\phi = h - Q$  gives a chart in a neighborhood of  $P$  such that  $\phi(P) = 0$ . We may thus assume that  $f$  is normalized such that the Laurent expansion in the chart  $\phi$  has leading coefficient 1 and constant coefficient is 0.

Denote the field of meromorphic functions on  $X$  by  $K(X)$ . The remainder of the proof relies on the relationship between meromorphic functions and function field extensions; we refer the reader to [5] for additional background. In this case,  $f$  and  $h$  correspond to the field extensions  $K(X)/\mathbb{C}(f)$  and  $K(X)/\mathbb{C}(h)$ , respectively. Note that  $\mathbb{C}(f, h)$  is a subextension of  $K(X)/\mathbb{C}(f)$  and  $K(X)/\mathbb{C}(h)$ . Since  $h$  is unramified at  $P$  but  $f$  is totally ramified at  $P$ , the meromorphic function corresponding to the extension  $K(X)/\mathbb{C}(f, h)$  must be both unramified and totally ramified at  $P$ , and is thus an isomorphism. It follows that  $K(X) = \mathbb{C}(f, h)$ .

The remainder of this proof is dedicated to showing that the minimal polynomial of  $f$  over  $\mathbb{C}(h)$  is defined over a finite extension of  $M(X, h)$ . Because the minimal polynomial defines  $X$ , this implies the “if” direction of Belyi’s Theorem.

Let  $P^{\sigma}$  be the point on  $X^{\sigma}$  obtained by applying  $\sigma$  to every coordinate of  $P$ . Define  $S_P \subseteq S$  to be the subset containing all  $\sigma$  for which there exists a conformal isomorphism  $i_{\sigma} : X \rightarrow X^{\sigma}$  such that  $h^{\sigma} \circ i_{\sigma} = h$  and  $i_{\sigma}(P) = P^{\sigma}$ . Note that  $i_{\sigma}$  is uniquely determined. In particular, for any conformal isomorphism  $i'_{\sigma}$  satisfying the conditions, the composition  $i'_{\sigma} \circ i_{\sigma}^{-1} \in \text{Aut}(h)$  fixes  $P$ ; since nontrivial Deck transforms have no fixed points,  $i'_{\sigma} \circ i_{\sigma}^{-1}$  must be the identity.

We can thus define an action of  $S_P$  on  $K(X)$  via  $\sigma : g \mapsto g^{\sigma} \circ i_{\sigma}$ . Note that this action is semilinear; that is, for any  $g_1, g_2 \in K(X)$  and  $\lambda \in \mathbb{C}$ , we have

$$\begin{aligned} \sigma(g_1 + g_2) &= (g_1 + g_2)^{\sigma} \circ i_{\sigma} = g_1^{\sigma} \circ i_{\sigma} + g_2^{\sigma} \circ i_{\sigma} = \sigma(g_1) + \sigma(g_2), \\ \sigma(g_1 g_2) &= (g_1 g_2)^{\sigma} \circ i_{\sigma} = (g_1^{\sigma} g_2^{\sigma}) \circ i_{\sigma} = (g_1^{\sigma} \circ i_{\sigma})(g_2^{\sigma} \circ i_{\sigma}) = \sigma(g_1)\sigma(g_2), \end{aligned}$$

and

$$\sigma(\lambda g_1) = (\lambda g_1)^\sigma \circ i_\sigma = \sigma(\lambda) \cdot (g_1^\sigma \circ i_\sigma).$$

It follows that the action of  $S_P$  preserves additive and multiplicative inverses. Note also that for constant functions, this action agrees with the standard Galois action of  $S_P$  on  $\mathbb{C}$ , and therefore fixes  $\mathbb{Q}$ .

Let  $\sigma \in S_P$  and note that  $\sigma(h) = h^\sigma \circ i_\sigma = h$ . That is,  $h$  is fixed under the action of  $S_P$ . Thus, for an expression of the form  $\lambda(h - Q)^{-n}$  where  $\lambda \in \mathbb{C}$  and  $n \in \mathbb{N}$ , we can write

$$\sigma(\lambda(h - Q)^{-n}) = \sigma(\lambda)(h - Q)^{-n}.$$

It follows that the Laurent expansion of  $\sigma(f)$  at  $P$  in the chart  $\phi = h - Q$  satisfies the same conditions as  $f$ . Since  $f$  is determined uniquely by these conditions, we must have  $\sigma(f) = f$ . Thus, the minimal polynomial of  $f$  over  $\mathbb{C}(h)$  is fixed under  $S_P$ . Its coefficients therefore lie in  $k(h)$ , where  $k \subset \mathbb{C}$  is the fixed field of  $S_P$ .

We now show that  $S_P$  has finite index in  $S$ . By an analogous infinite Galois argument to the above, it follows that  $k$  is a finite extension of  $M(X, h)$ .

For any  $\sigma \in S \setminus S_P$ , choose  $i_\sigma : X \rightarrow X^\sigma$  to be any map such that  $h = h^\sigma \circ i_\sigma$ . We can then define an action of  $S$  on  $h^{-1}(Q)/\text{Aut}(h)$  via  $[P] \mapsto [i_\sigma^{-1}(P^\sigma)]$ . We see that this map is well-defined as follows. Let  $P_1, P_2 \in h^{-1}(Q)$ , and suppose that there exists  $\phi \in \text{Aut}(h)$  such that  $\phi(P_1) = P_2$ . Then consider the composition  $\psi = i_\sigma^{-1} \circ \sigma \circ \phi \circ \sigma^{-1} \circ i_\sigma$ , where  $\sigma$  acts coordinate-wise. Since  $\psi \in \text{Aut}(h)$  and  $\psi(i_\sigma^{-1}(P_1^\sigma)) = i_\sigma^{-1}(P_2^\sigma)$ , we have  $[i_\sigma^{-1}(P_1^\sigma)] = [i_\sigma^{-1}(P_2^\sigma)]$ .

We claim that that  $S_P$  is precisely the stabilizer of  $[P]$  under this action. It is clear that  $S_P \subseteq \text{Stab}([P])$ . For the other direction, let  $\sigma \in S$  and suppose that there exists  $\phi \in \text{Aut}(h)$  such that  $\phi(P) = i_\sigma^{-1}(P^\sigma)$ . Then the composition  $i_\sigma \circ \phi$  maps  $P \mapsto P^\sigma$ , so (by construction) we must have  $i_\sigma = i_\sigma \circ \phi$  and  $\sigma \in S_P$ . Since  $h^{-1}(Q)/\text{Aut}(h)$  is finite, it follows that  $S_P$  has finite index in  $S$ . (Note that in the special case when  $h$  is a Galois covering, we have  $S = S_P$ , so  $k = M(X, h)$ .)

We have thus shown that the minimal polynomial of  $f$  over  $\mathbb{C}(h)$  has coefficients in  $k(h)$  for  $k$  a finite extension of  $M(X, h)$ . As discussed above, it follows that  $X$  is an algebraic curve over  $k \subset \overline{\mathbb{Q}}$ , as desired.  $\square$

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