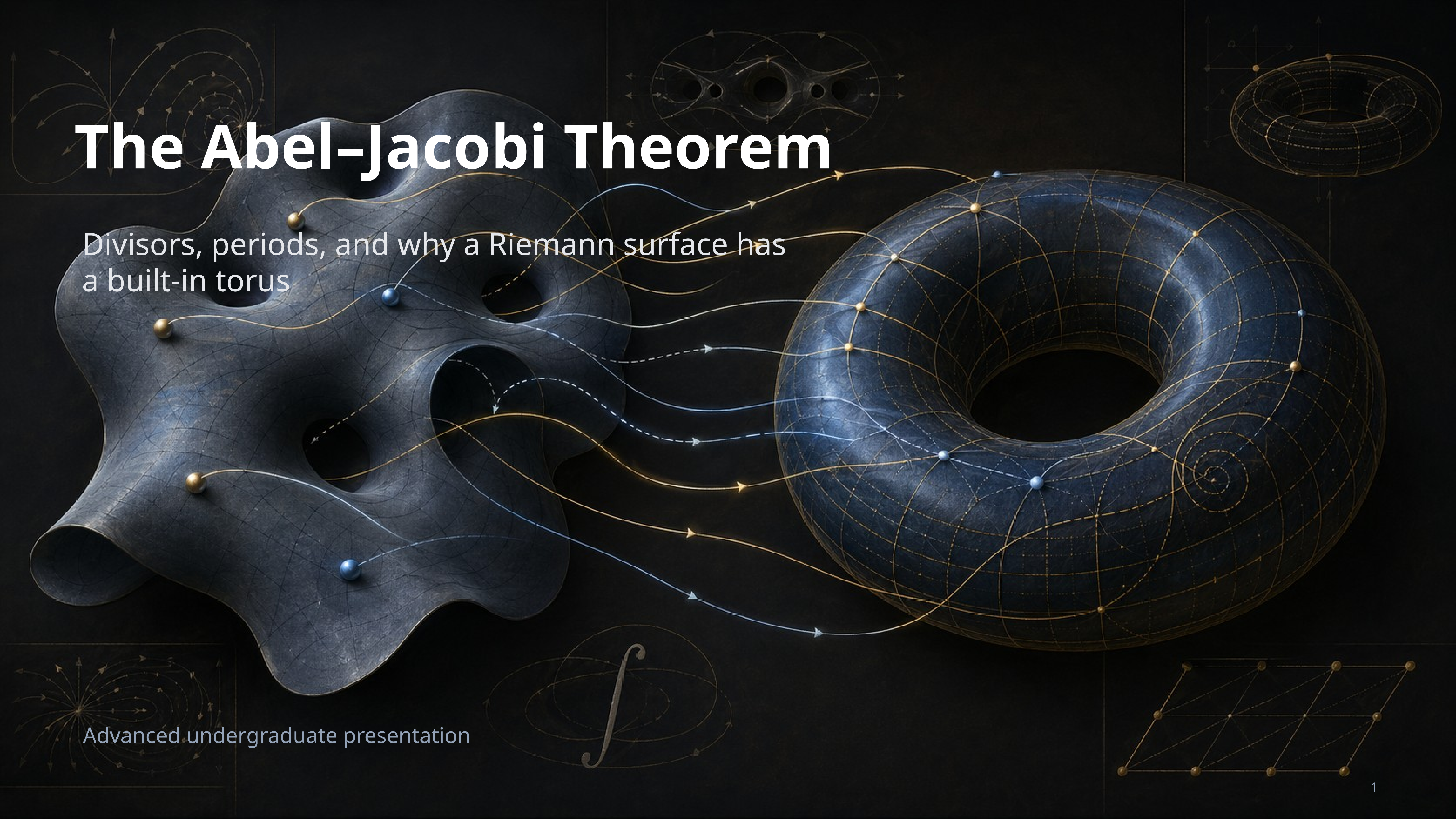
The Abel–Jacobi Theorem

Divisors, periods, and why a Riemann surface has a built-in torus



Advanced undergraduate presentation

ROADMAP

The theorem as a bridge

Question on X

When is a formal sum of points the divisor of a meromorphic function?

↓

Answer in Jac(X)

Integrate holomorphic 1-forms from a basepoint; the result lives modulo periods.

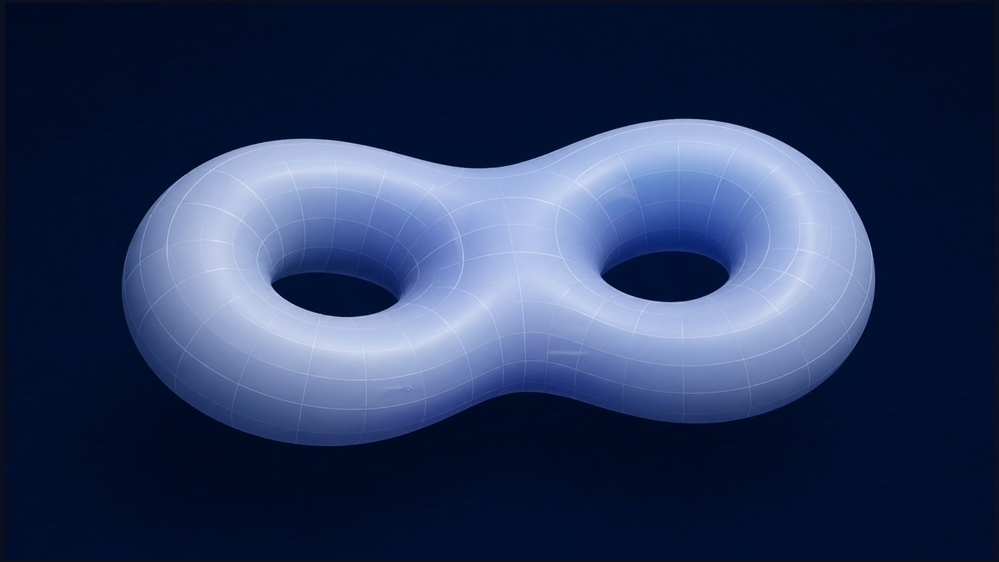
Punchline

Zero obstruction ⇔ the divisor is principal.

OBJECT 1

Riemann surfaces: where the story takes place

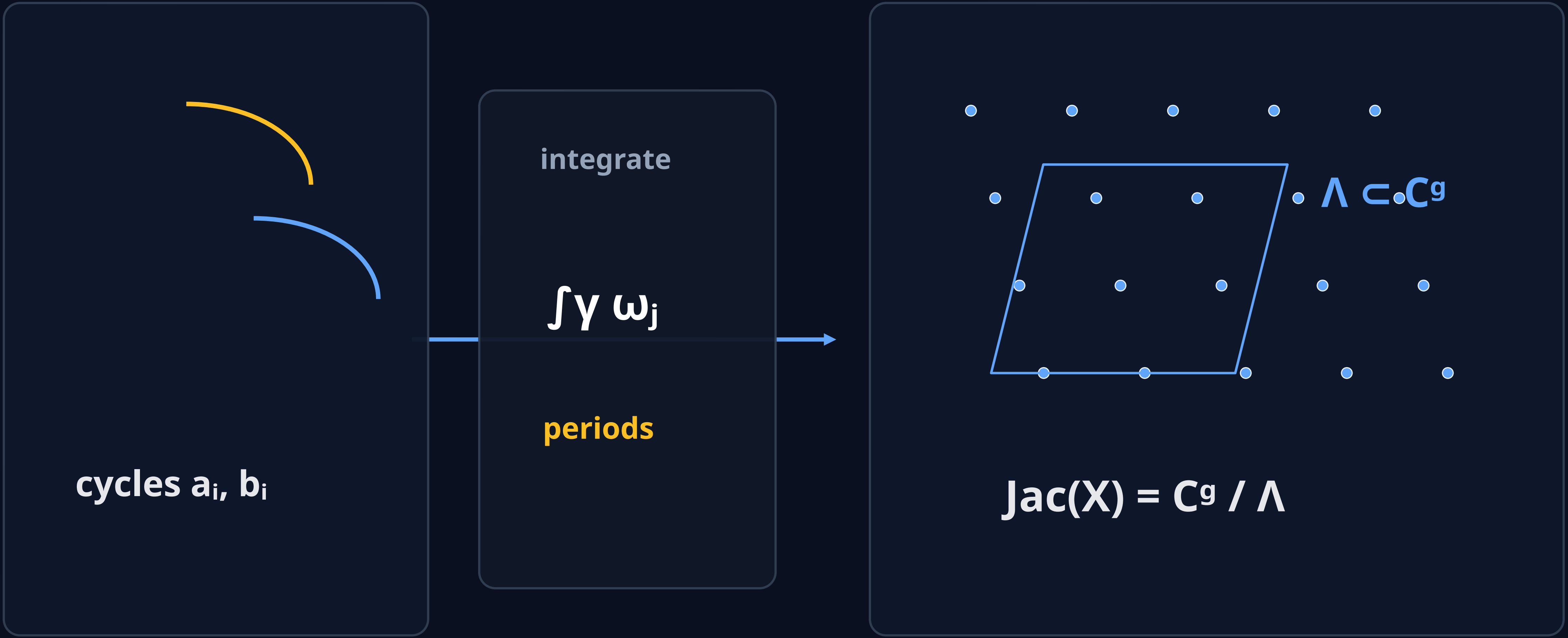
• A compact Riemann surface X is a 1-dimensional complex manifold.
• Its genus g counts the number of handles.
• There are exactly g independent holomorphic 1-forms.



Think of holomorphic 1-forms as “global measuring devices.”
They let us integrate paths on X into complex coordinates.

OBJECT 2

Periods make a lattice



OBJECT 3

Divisors: weighted configurations of points

$D = 2P - Q - R$

degree(D) = sum of coefficients

Degree zero is the stage for Abel-Jacobi.

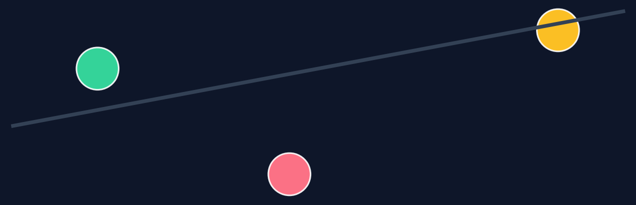
- Zeros count positively.
- Poles count negatively.
- A principal divisor is div(f): zeros minus poles of one meromorphic function.

THE CONSTRUCTION

The Abel-Jacobi map

Input: $D = \Sigma\, n_p\, p$

with $\Sigma\, n_p = 0$



choose paths from p_0

Output: a point of Jac(X)

$\Phi(D) = \Sigma\, n_p \int_{p_0}^{p} (\omega_1,\ldots,\omega_g) \bmod \Lambda$

mod periods

THE STATEMENT

The Abel-Jacobi theorem

For a compact Riemann surface X:

D is principal $\Leftrightarrow$ $\Phi(D) = 0$ in Jac(X)

$\mathrm{Div}^0(X)$ → **Jac(X)**

degree-zero divisors | complex torus

kernel = principal divisors

Why one direction is believable

- Suppose D = div(f). Then D records zeros and poles of f.
- The logarithmic differential df/f has residues equal to those coefficients.
- Riemann bilinear relations imply the relevant integrals land in the period lattice.
- So Φ(D) = 0 in the quotient Jac(X).

local picture

df/f measures winding of f around 0

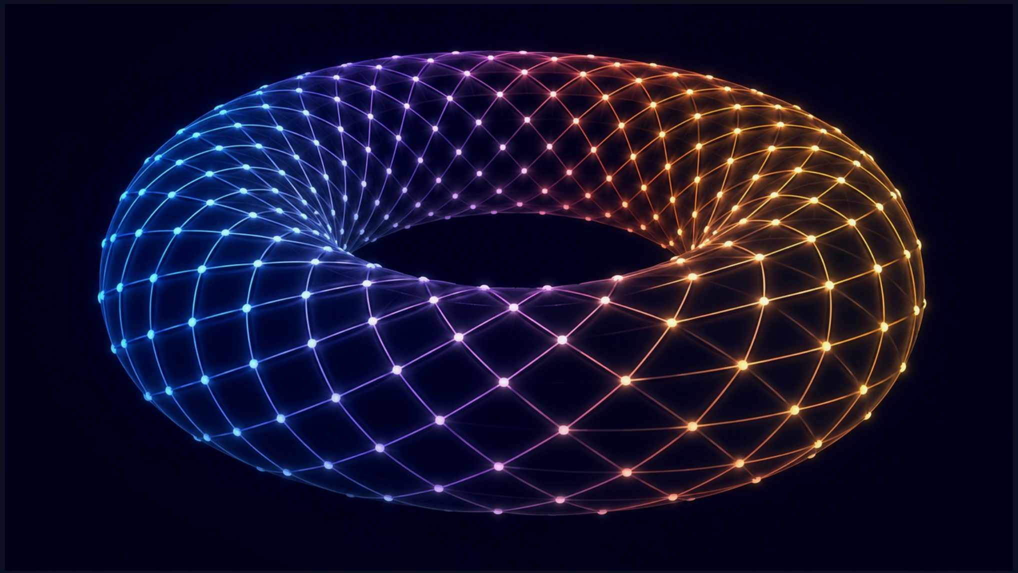
PROOF IDEA II

Why the converse is deep

If $\Phi(D) = 0$, the obstruction vanishes.

- Construct a meromorphic object with prescribed zeros and poles.
- The only possible failure is multi-valuedness around cycles.
- The Abel-Jacobi condition cancels that monodromy.
- Theta functions provide the standard construction.

zero obstruction ⇒ single-valued function f



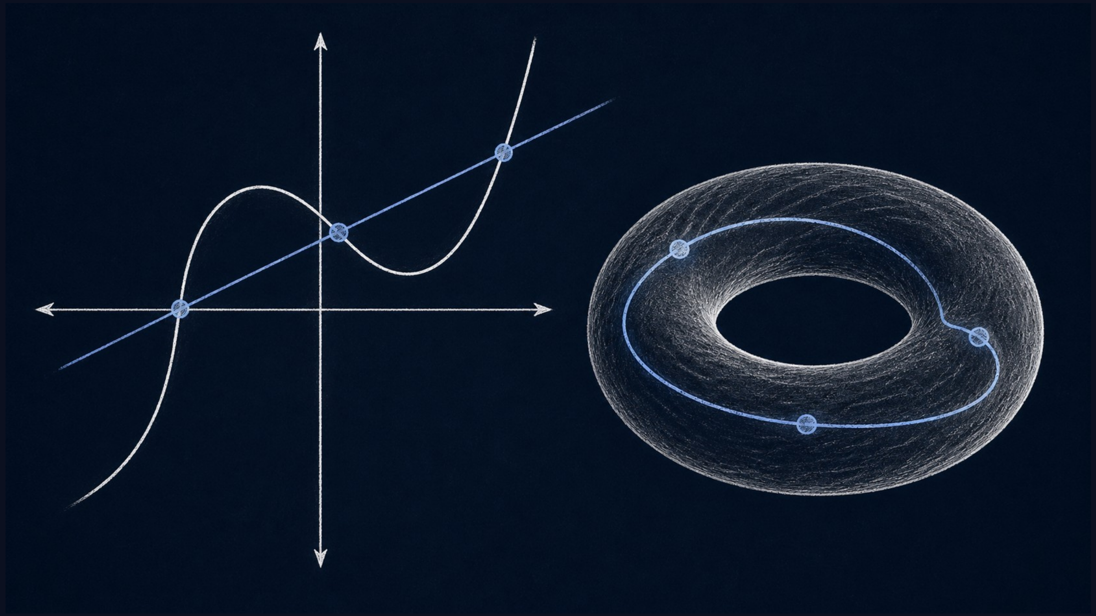
TEST CASE

Example: genus 1 and the elliptic curve group law

When g = 1:

• Jac(E) is naturally the elliptic curve E itself.
• A divisor P + Q + R − 3O is principal exactly when P + Q + R = O in the group law.
• A line cutting a cubic at P, Q, R gives this relation.

Abel-Jacobi recovers the familiar chord-and-tangent rule.



SIGNIFICANCE

What the theorem buys you

Classification

$\mathrm{Pic}^0(X) \cong \mathrm{Jac}(X)$

Line bundles of degree zero become points on a torus.

Geometry

Periods encode shape

Complex analysis turns topological cycles into algebraic information.

Modern reach

Theta, moduli, arithmetic

The same language appears in algebraic geometry, integrable systems, and number theory.

WRAP-UP

Three takeaways

1. Periods turn X into a torus: $Jac(X) = C^g / \Lambda$.

2. Divisors ask for functions with prescribed zeros and poles.

3. Abel-Jacobi says the only obstruction is $\Phi(D)$.

Study question: for genus 2, what information does the Jacobian remember that the surface itself hides?