

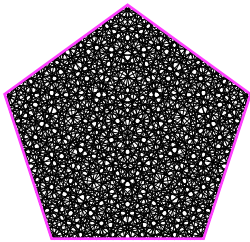
Billiards and number theory

Curtis T McMullen
Harvard University

Leutbecher, Cohen, Wolfart, Borho, Rosenberger,
Veech, Masur, Möller, Viehweg, Davis, Lelievre, Smillie, Ulcigrai,
Everett, Lin, Mager

Periodic billiards

Billiards in a regular pentagon

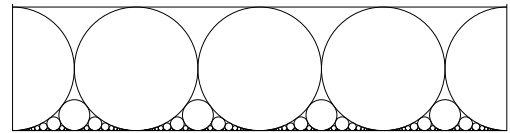
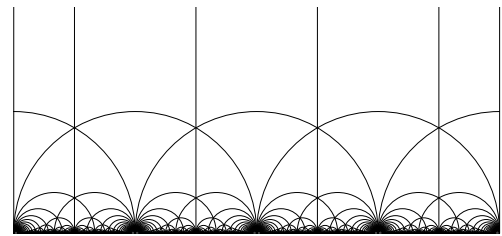


A dense set of slopes are periodic.

Which ones?

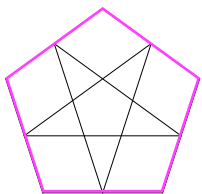
How do the periodic trajectories behave?

$SL_2(\mathbb{Z})$ and its cusps

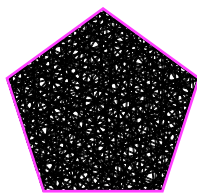


p/q 0 1

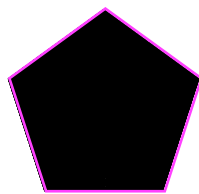
Slopes and lengths



s
 $L(s) = 5$



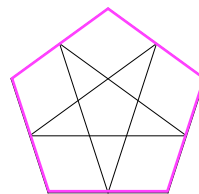
$4s$
 $L(s) = 469$



$20s$
 $L(s) = 2338$

$6765s$
 $L(6765s) = 1.734 \times 10^{25}$

Slopes, lengths and heights



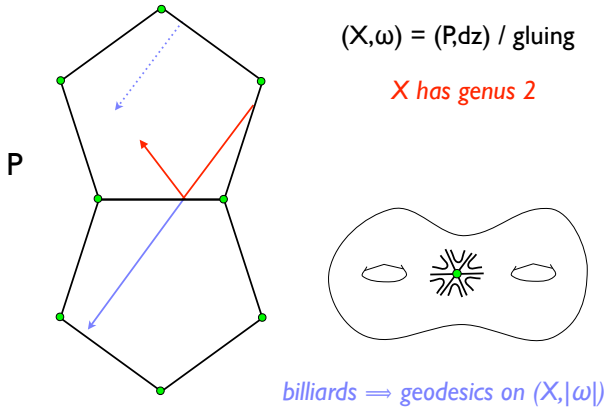
s

Theorem

The periodic slopes coincide with $\mathbb{Q}(\sqrt{5})s$,
and $\log L(xs) = O(h(x)^2)$.

exponent 2 is sharp

Pentagon \rightarrow X of genus 2



Fields and Rings

$$\zeta_n = \exp(2\pi i/n)$$

$$\epsilon_n = \zeta_n + \zeta_n^{-1}$$

$$L = \mathbb{Q}(\zeta_n) \quad \mathcal{O}_L = \mathbb{Z}(\zeta_n)$$

$$K = \mathbb{Q}(\epsilon_n) \quad \mathcal{O}_K = \mathbb{Z}(\epsilon_n)$$

$$\mathbb{Q} \quad \mathbb{Z}$$

Renormalization Group

thin $\Delta_n = \langle T, U \rangle \subset \text{SL}_2(\mathcal{O}_K)$ *arithmetic*

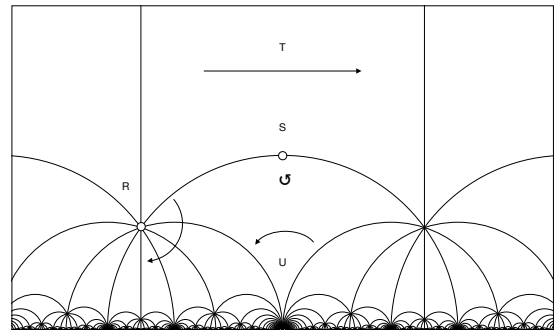
$$T = \begin{pmatrix} 1 & 2 + \epsilon_n \\ 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$R = UT$ has order n

Renormalization group for the pentagon

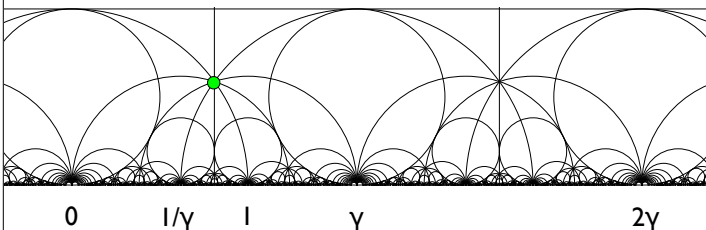
Δ_5



Theorem

For $n=5, 8, 10, 12$:
 the cusps of Δ_n coincide with $\mathbb{P}^1(K)$,
 and satisfy quadratic height bounds.

5 packing



Continued fractions

$$\gamma = (1 + \sqrt{5})/2$$

Every $s \in \mathbb{Q}(\gamma)$ can be expanded as a finite golden continued fraction,

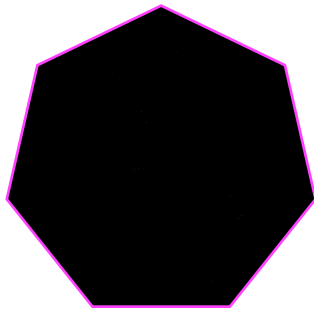
$$s = [a_1, a_2, a_3, \dots, a_N] = a_1\gamma + \frac{1}{a_2\gamma + \frac{1}{a_3\gamma + \dots + \frac{1}{a_N\gamma}}}$$

with $a_i \in \mathbb{Z}$.

Height bounds: length N and a_i are $O(1+h(s))$.

Open problem

Regular 7-gon

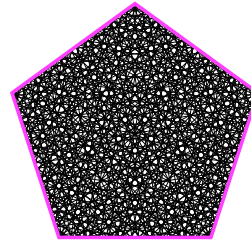


(i) Which slopes are periodic?

(ii) How long do we have to wait to test periodicity?!

Is there any algorithm at all?!!

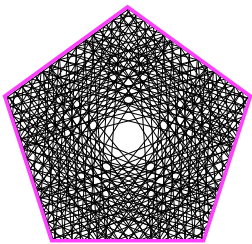
Distribution of trajectories



Every trajectory is periodic or uniformly distributed.

How are the periodic trajectories distributed?

Distribution of trajectories



Every trajectory is periodic or uniformly distributed.

How are the periodic trajectories distributed?

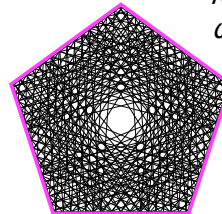
Davis-Lelievre: Not always uniformly!

Cantor set of measures?

Limit Measures

Theorem

For each periodic slope s , the limit measures M_s form a countable set, homeomorphic to $\omega^\omega + 1$.

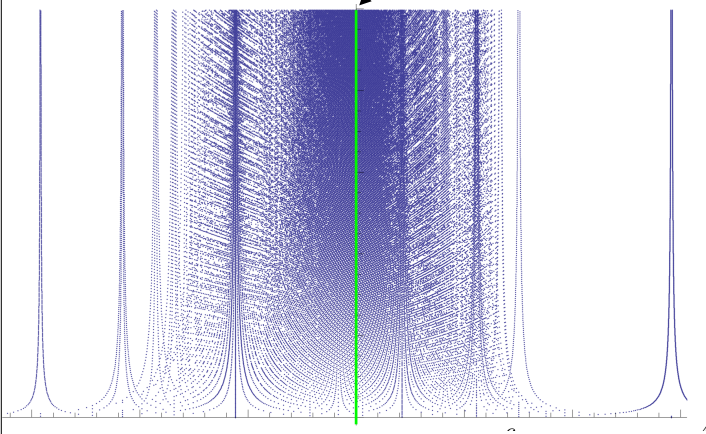


describes scarring & closure of ergodic measures

Limit Measures M_0 for the pentagon

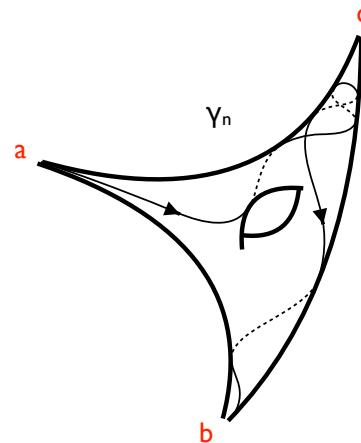
form a semigroup!

uniform measure



Modular symbols

(nonabelian)

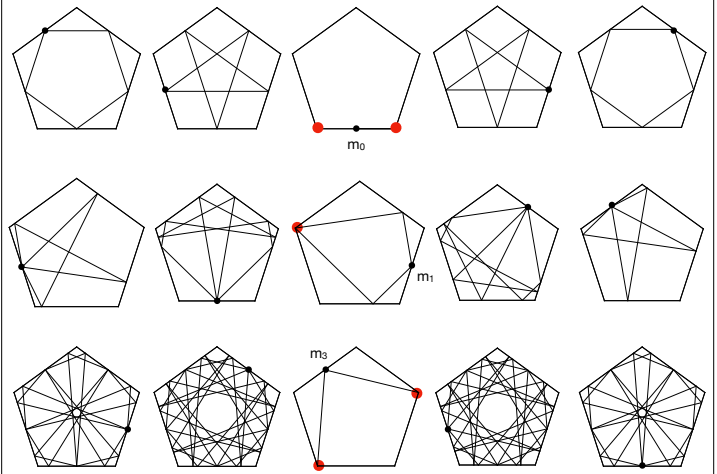


$$\gamma_n \rightarrow \delta_1 * \delta_2$$

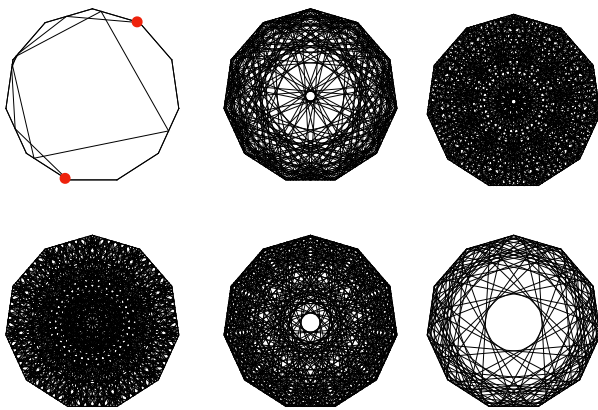
$$a \xrightarrow{\delta_1} c \xrightarrow{\delta_2} b$$

Cusps, congruence and chaos

Midpoint trajectories



n=11, midpoint trajectories



$D[\text{QtoI}[[6, 4, -1, -2, -2, -2, -2, -2, -2, -2, -2], 11], 11]$

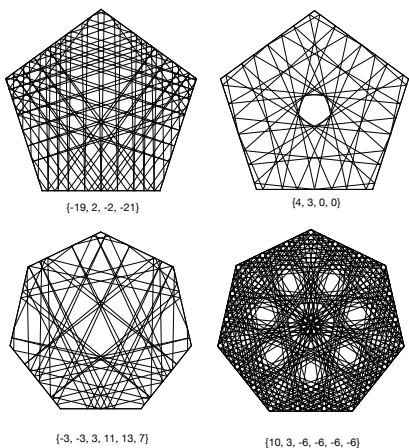
Congruence invariants

n odd; s in L , periodic direction

Theorem:

$$T_s(m_k) \text{ is a vertex connection} \\ \Leftrightarrow [s]_2 = [\zeta_n^k]_2 \in \mathbb{P}^1(\mathcal{O}_K/2)$$

Dihedral v. Reflection symmetry



Dihedral v. Reflection symmetry

$n=p$ prime; $\pi = 1 - \zeta_p$

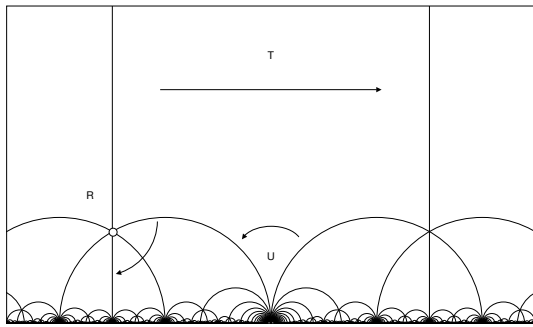
Theorem:

Trajectories $T_s(m_i)$ have dihedral symmetry

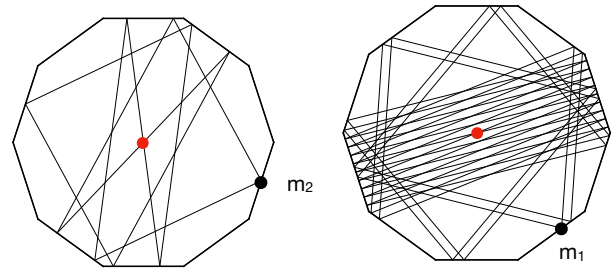
$$\Leftrightarrow v_\pi(s) = 0 \pmod{2}.$$

Renormalization group:
two cusps for n even

Δ_6

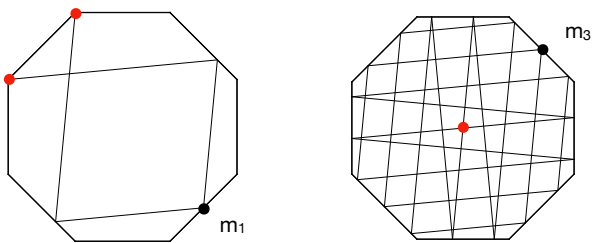


n=10: Central connections



(every slope has a central or vertex connection)

n=8: Central and vertex connections



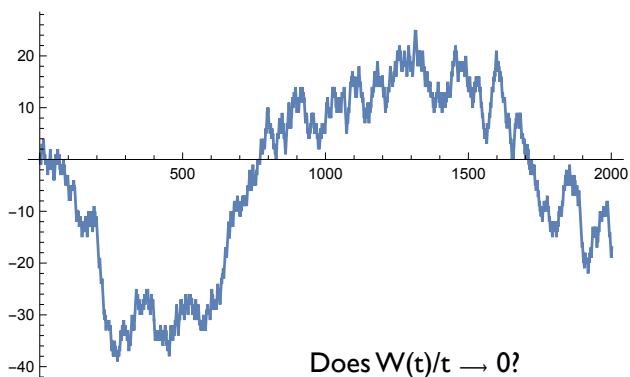
(every slope has both central and vertex connections, or none)

Central and vertex connections

Theorem

For $n = 4g+2$ or n a power of 2, the locations of the central and vertex connections are congruence invariants of s .

Chaos for n=12



Adelic perspective

$$[\mathrm{SL}_2(\mathcal{O}_K) : \Delta_n] = \infty$$

$$N = [\mathrm{SL}_2(\widehat{\mathcal{O}_K}) : \overline{\Delta}_n] \text{ is finite}$$

Theorem

If $N < 5000$ for $n=12$, then the location of the central trajectory is not a congruence invariant.

Curves on Hilbert modular varieties

Curves on a Hilbert modular surface

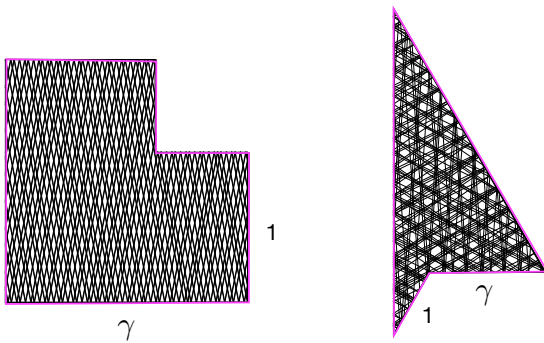
Theorem

The cusps of every geodesic curve $V = \mathbb{H}/\Gamma$ on X_K coincide with $\mathbb{P}^1(K)$, and satisfy quadratic height bounds.

Corollary

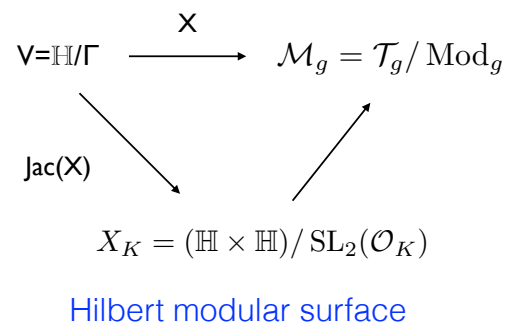
Results on billiards follow.

Applies to almost all families of optimal billiards

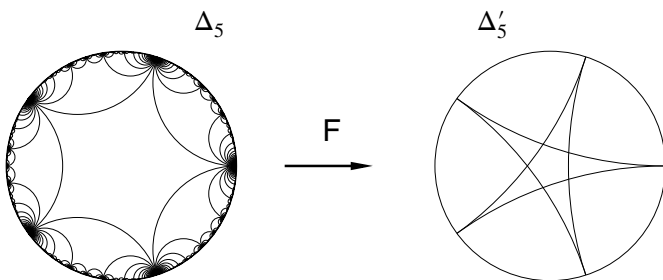


...since these are quadratic: Eskin - Filip - Wright

Factorization through X_K



Holomorphic pentagon-to-star map



$V \rightarrow X_K$ covered by $\mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$
via $x \rightarrow (x, F(x))$.

Proof of height bounds

Classical height on $\mathbb{P}^n(K)$

$$H(x) = H(x_0 : x_1 : \dots : x_n) = \prod_v \max_i |x_i|_v \geq 1$$

comparable to

$$\tilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v, \quad [a_0 : \dots : a_n] = [x],$$

(a_i are integers)

only requires knowledge of integers and infinite places

The projective line $\mathbb{P}_A^1(K)$

$$K \subset \text{End}(A) \otimes \mathbb{Q}$$

$$H_1(A, \mathbb{Q}) \cong K^2$$

$$\mathbb{P}_A^1(K) = \text{space of K-lines in } H_1(A, \mathbb{Q})$$

Height $H_A(x)$ on $\mathbb{P}_A^1(K)$

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

Hodge norm at v

$$x \in \mathbb{P}_A^1(K)$$

$$C \in H_1(X, \mathbb{Z})$$

$$x = [K \cdot C]$$

$$|C|_v = \left| \int_C \omega_v \right|^{1/g}$$

Why a height?

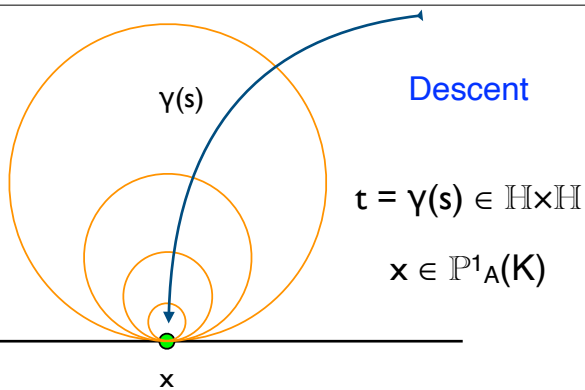
$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$$\tilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v$$

Theorem. Given a linear isomorphism

$$\iota : \mathbb{P}_A^1(K) \rightarrow \mathbb{P}^1(K)$$

we have $H(\iota(x)) \asymp H_A(x) \geq C_A > 0$.



$$H(x, A_t) \rightarrow 0$$

$A_t = \text{Abelian surface}$

$\Rightarrow x$ is a cusp of Γ

Geodesic curves

- Every geodesic curve V in M_g has a cusp.
- 'Every' geodesic curve V on a Hilbert modular surface X_k has a cusp.
- Does 'every' geodesic curve V in a Hilbert modular variety X_k have a cusp?

Theorem

There exists a compact geodesic curve V on X_k , $\dim=6$.

Not contained in any compact Shimura sub variety

Spectral Gap

Theorem

For all but finitely many $\Delta(p,q,r)$,
spherical and # hyperbolic places are about the same.

about 1/3 spherical

Cor (Takeuchi)

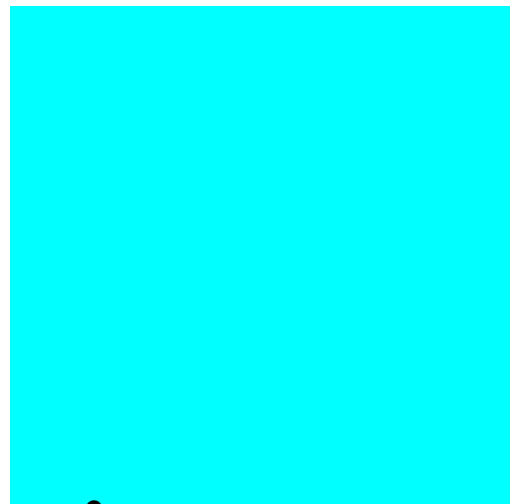
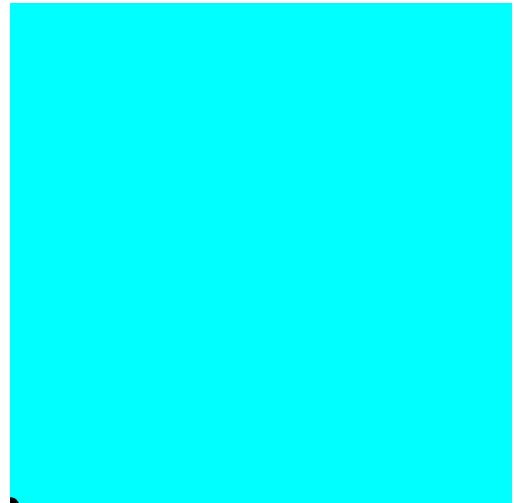
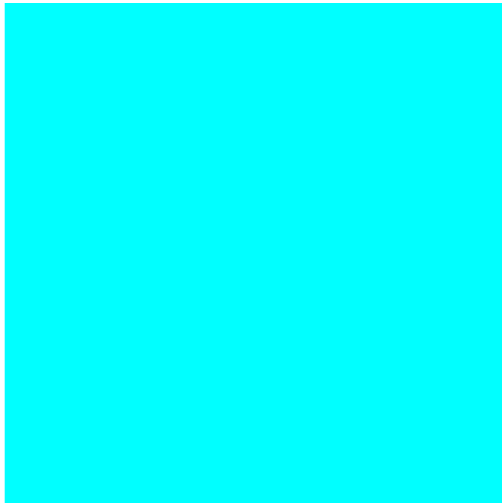
There are only finitely many arithmetic
triangle groups.

Cor (Waterman-Maclachlan)

There are only finitely many
purely hyperbolic triangle groups.

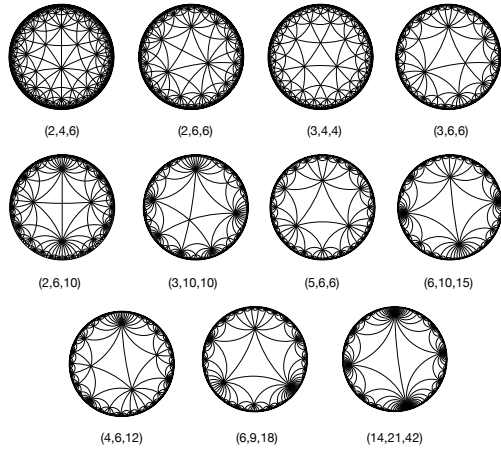
The moving tablecloth game

Key step in finiteness proofs



11 known purely hyperbolic triangle groups

M, Maclachlan-Wateman



Theorem

$\Delta(14,21,42)$ virtually embeds in SL_2 (its inv. trace field K).

There are only finitely many such examples.

Conjecture

This is the only example.

From $(14,21,42)$ to an exotic curve

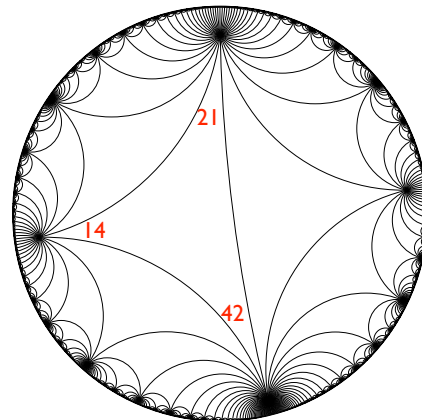
$K = \mathbb{Q}(\cos \pi/21)$ degree 6

$V = \mathbb{H}/\Delta' \rightarrow X_K$ geodesic curve on a 6D Hilbert modular variety

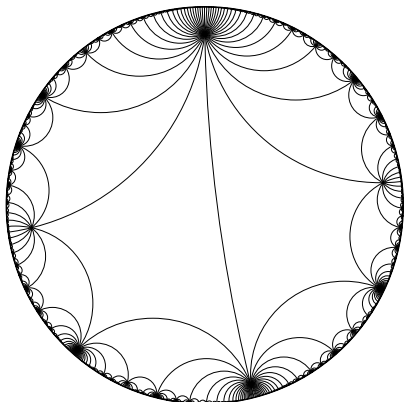
Theorem

V is a compact geodesic curve, but there is no compact Shimura variety with $V \subset S \subset X_K$.

Construction: Start with $\Delta(14,21,42)$



Pass to index 2



$V = \mathbb{H}/\Delta(14,21,42)'$

Construct 5 maps \mathbb{H} to \mathbb{H}

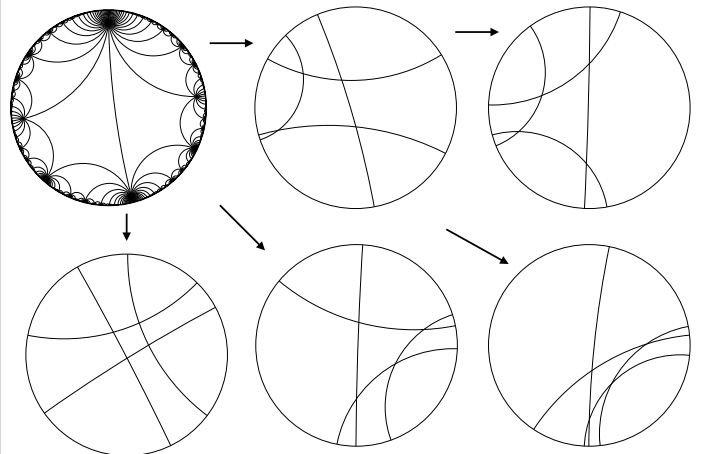
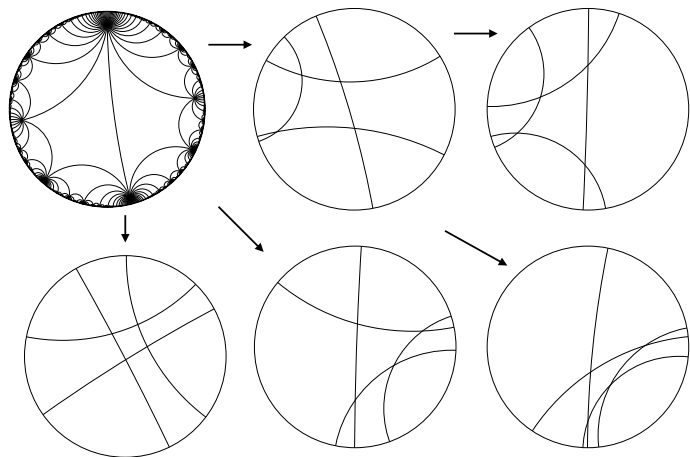


Image covers exotic V in X_K



Conclusion

There exists a compact geodesic curve V on X_K , $\dim=6$.

Problem

Are there more examples of exotic curves?

For example, with $\dim X_K = 3$?

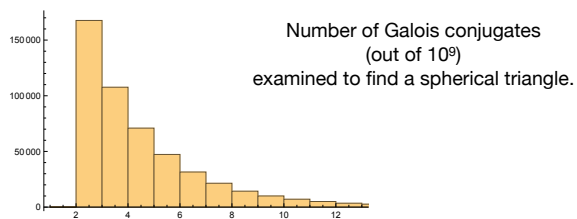
Problem

Are there more examples of purely hyperbolic triangle groups?

Experimental Evidence

There are no other purely hyperbolic triangle groups with $(p,q,r) < 5000$.

Recently verified using 5,000 cores running in parallel from 1-30 mins.



Total execution time 587 hours

References

Modular symbols for Teichmüller curves

Billiards and the arithmetic of non-arithmetic groups

Galois orbits in the moduli space of all triangles

Triangle groups and Hilbert modular varieties

Triangle groups: Cusps, congruence and chaos

www.math.harvard.edu/~ctm/papers