

The question mark function, welding, and complex dynamics

Curtis T. McMullen

11 February 2025

Contents

1	Introduction	1
2	The question mark function	8
3	Geometrically finite covering maps on S^1	17
4	Classification	24
5	Criteria for geometric finiteness	26
6	Construction of Minkowski pairs	32
7	Welding and Julia sets of polynomials	36
8	Dynamics from hypocycloids	40
9	$SL_2(\mathbb{Z})$ and the deltoid	42
10	Automorphisms and $PGL_2^+(\mathbb{Q})$	45
11	Welding is not closed under composition	48
A	Appendix: The modular group and quadratic polynomials . .	50

1 Introduction

In this paper we show that infinitely many cusped Julia sets arising in complex dynamics are essentially the same as the Jordan curve $W(?)$ defined using Minkowski's question mark function as a conformal welding homeomorphism. The proof of this universality rests on a general result in *real dynamics*: the function $?(x)$ arises from any topological conjugacy between parabolic and hyperbolic dynamical systems on S^1 (subject to the condition of geometric finiteness).

We also show that $\text{Aut}(?)$ is very rich; for example, it contains a copy of $\text{PGL}_2^+(\mathbb{Q})$. As a corollary we show that the set of welding maps on S^1 is not closed under composition.

The question mark function. Minkowski's function $?: \mathbb{R} \rightarrow \mathbb{R}$, introduced in [Min], is the unique homeomorphism satisfying $?(0) = 0$, $?(x+1) = ?(x) + 1$, and

$$?\left(\frac{p+r}{q+s}\right) = \frac{?(p/q) + ?(r/s)}{2}$$

for all pairs of rationals with $ps - qr = 1$. Its graph is shown in Figure 1.

By construction, $?(x)$ gives a bijection between $\mathbb{Q}$ and the ring of dyadic rationals, $\mathbb{Z}[1/2]$. Its analytic properties include the following:

1. The map $?(x)$ is Hölder continuous, of exponent $\alpha = \log(2)/\log(\gamma^2)$, $\gamma = (1 + \sqrt{5})/2$;
2. Its first derivative vanishes almost everywhere;
3. All derivatives of $?(x)$ vanish when $x \in \mathbb{Q}$;
4. In particular, $?^{-1}(x)$ is *not* Hölder continuous; and hence
5. The homeomorphism $?(x)$ is not quasimetric.

There are also interesting connections between $?(x)$, earthquakes, and Thompson's simple group T . These connections, developed in §2, will play a crucial role in the sequel (see especially Theorem 2.1).

Welding. By periodicity, the question mark function gives a homeomorphism $?: S^1 \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$, which we will denote by the same symbol. Although it is not quasimetric, this map is a *strong welding homeomorphism* (see §7), and thus it determines an essentially unique Jordan curve $W(?) \subset \widehat{\mathbb{C}}$. The curve $W(?)$ is characterized by the property that the boundary values of the Riemann mappings to the two components of its complement differ by $?: S^1 \rightarrow S^1$.

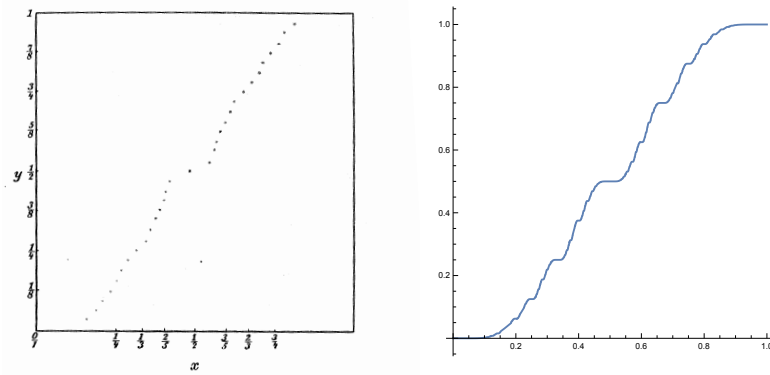


Figure 1. Plots of Minkowski's function $W : [0, 1] \rightarrow [0, 1]$. The left image is from [Min].

Complex dynamics. In this paper we will show that $W(?)$ is a canonical model for a broad range of Julia sets that emerge from matings between parabolic and hyperbolic dynamical systems.

To make this precise, let $\mathcal{J}$ denote the set of Jordan curves $J \subset \mathbb{C}$, subject to the equivalence relation $[J_1] = [J_2]$ if $J_1 = \phi(J_2)$ for some quasi-conformal map $\phi : \mathbb{C} \rightarrow \mathbb{C}$. In §7 and §8 we will show:

Theorem 1.1 *Suppose $P_d(z) = z^d + c$, $d \geq 2$, has a parabolic fixed point. Then its Julia set satisfies*

$$[J(P_d)] = [W(?)]$$

in the space of Jordan curves $\mathcal{J}$.

Theorem 1.2 *Let J_n , $n \geq 3$, be the boundary of the region obtained by unfolding a hypocycloid with n cusps, $P_n \subset \mathbb{C}$, via iterated Schwarz reflection. Then*

$$[J_n] = [W(?)] \in \mathcal{J}.$$

The same is true for its quotient $J_{n,k}$ by a rotation of order $k|n$.

The curves J_n and $J_{n,k}$ can also be described as Julia sets of correspondences (§8). See Figures 2 and 3 for examples of the cusped curves appearing in Theorems 1.1 and 1.2.

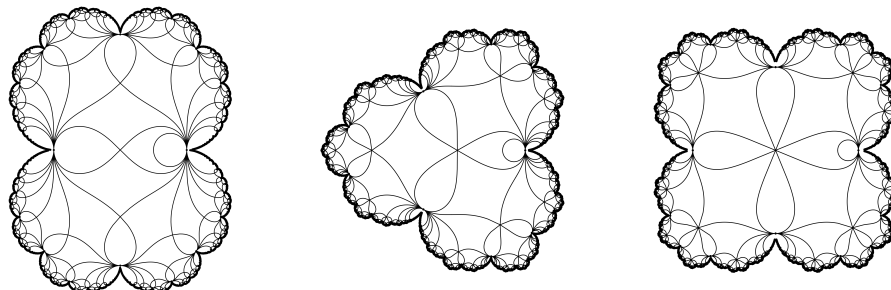


Figure 2. Julia sets of parabolic polynomials $z^d + c$, $d = 2, 3$ and 4 .

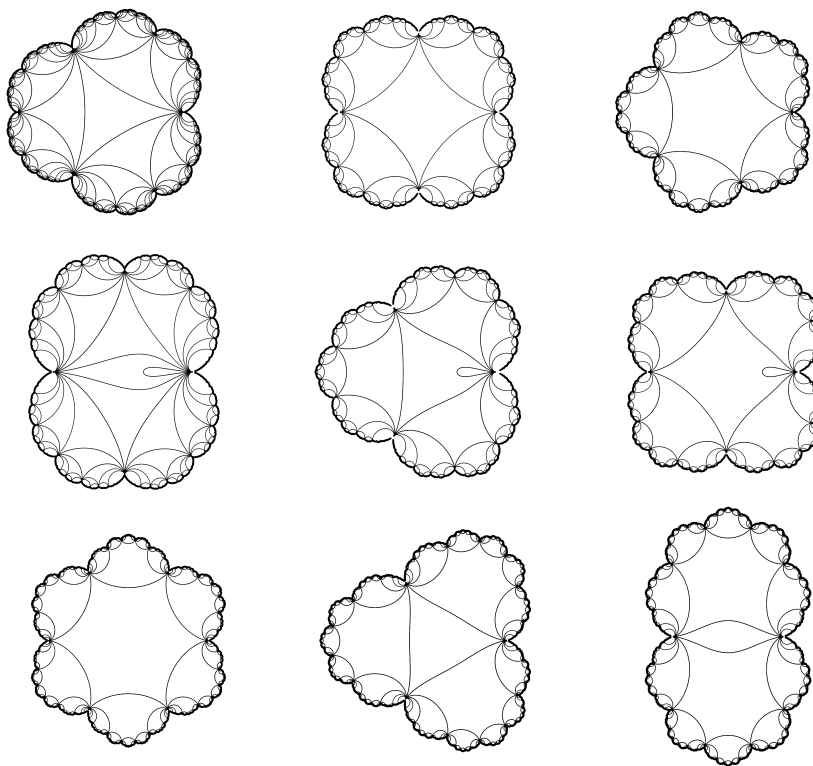


Figure 3. Julia sets from hypocycloids: J_n for $n = 3, 4$ and 5 ; $J_{n,n}$, $n = 3, 4$ and 5 ; and J_6 , $J_{6,2}$ and $J_{6,3}$.

As special cases, we have:

Corollary 1.3 *The Jordan curve $W(?)$ is quasiconformally equivalent to the cauliflower Julia set $J(z^2 + 1/4)$, as well as the deltoid Julia set $J_3 \subset \mathbb{C}$.*

Related statements for the matings introduced in [BP] appear in the Appendix; see also Figure 4.

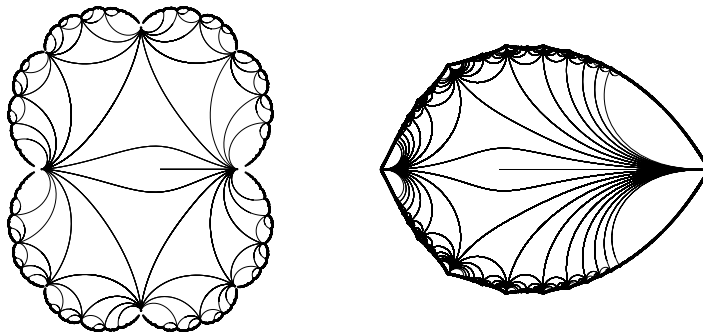


Figure 4. Matings of z^2 and $z^2 + 1/4$ with $\mathrm{SL}_2(\mathbb{Z})$. The first Julia set is equivalent to $W(?)$; the second is a quasicircle.

Dynamics on S^1 . We next state two rather general results in *real dynamics* that underly the proofs of Theorems 1.1 and 1.2 above.

To formulate the first we introduce, in §3, the notion of a *geometrically finite* covering map $f : S^1 \rightarrow S^1$. Almost no smoothness is required. Any two geometrically finite covering maps of the same degree are topologically conjugate, and in §3 we show:

Theorem 1.4 *A topological conjugacy $\phi : S^1 \rightarrow S^1$ between a pair of geometrically finite covering maps f and g is quasisymmetric $\iff \phi$ preserves parabolics.*

(This means that ϕ gives a bijection between the parabolic cycles of f and g .)

Classification results that follow are stated in §4, and criteria for geometric finiteness of piecewise smooth mappings are developed in §5.

The second result (§6) brings $?(x)$ into play and explains its unexpected universality. To state it, let

$$\mathrm{Möb} \subset \mathrm{QS} \subset \mathrm{Homeo}^+(S^1)$$

denote the groups of orientation-preserving Möbius transformations, quasymmetric maps, and homeomorphisms of the circle.

Theorem 1.5 *Let $\phi : S^1 \rightarrow S^1$ be a topological conjugacy between a pair of geometrically finite covering maps f and g . Suppose that f has at least one parabolic cycle, but g has none. Then*

$$[\phi] = [?]$$

in the moduli space $\mathcal{M} = \text{QS} \setminus \text{Homeo}^+(S^1) / \text{QS}$.

Corollary 1.6 *Under the same hypotheses, we have $[W(?)] = [W(\phi)]$ in the space of Jordan curves $\mathcal{J}$.*

The proof of Theorem 1.5 uses an explicit construction of *Minkowski pairs* (f, g) representing all possible parabolic behaviors.

Meta-Teichmüller theory. The discussion of $[?]$ above can be placed in a broader context, as follows. Let

$$\mathcal{T} = \text{QS} \setminus \text{Homeo}^+(S^1) \quad \text{and} \quad \mathcal{M} = \text{QS} \setminus \text{Homeo}^+(S^1) / \text{QS}$$

There is a natural map

$$\mathcal{J} \rightarrow \mathcal{M},$$

sending a Jordan curve $[J]$ to its welding homeomorphism $[\omega_J]$.

Every homeomorphism of S^1 , no matter how wild, determines a point in $\mathcal{T}$. Similarly, every Jordan curve determines a point in $\mathcal{J}$. In contrast, the traditional universal Teichmüller space

$$\mathcal{U} = \text{Möb} \setminus \text{QS}$$

maps to a single point in $\mathcal{T}$, and all quasicircles represent the same point in $\mathcal{J}$.

One can thus regard the study of $\mathcal{T}$ and $\mathcal{J}$ as instances of *meta-Teichmüller theory*: the classification of topological objects up to quasiconformal equivalence. In this setting, the role of the mapping-class group is played by

$$\text{Mod} = \text{QS},$$

and the analogue of moduli space itself is

$$\mathcal{M} = \mathcal{T} / \text{Mod}.$$

Our investigation of $W(?)$ represents a first foray into this rich territory beyond quasicircles.

Automorphisms. In traditional Teichmüller theory, the (conformal) automorphism group of a Riemann surface $X \in \mathcal{T}_g$ is the same as its stabilizer in the mapping-class group Mod_g . By the same token, we define the (quasisymmetric) automorphism group of $[\omega] \in \mathcal{T}$ by

$$\text{Aut}(\omega) = \text{Mod}^\omega = \{\phi \in \text{QS} : \omega\phi\omega^{-1} \in \text{QS}\}.$$

When $\omega = \omega_J$ is a strong welding homeomorphism for a Jordan curve $J \subset \mathbb{C}$, the group $\text{Aut}(\omega)$ is isomorphic to:

$$\text{Aut}(J) = \{\phi : J \rightarrow J : \phi \text{ extends to an element of } \text{QC}(\mathbb{C})\}.$$

Automorphisms of cusped curves. The fact that $[W(?)]$ arises as a Julia set in many different ways indicates that this curve does not *remember* any particular dynamical system that generates it. As a reflection of this flexibility, we expect $\text{Aut}(?)$ to be large.

Indeed, it is easy to see (from results in §2) that:

$\text{Aut}(?)$ contains the group PS of piecewise- $\text{SL}_2(\mathbb{Z})$ homeomorphisms of S^1 .

As a step towards displaying the full richness of $\text{Aut}(?)$, in §10 we will show:

Theorem 1.7 *The group $\text{Aut}(?)$ contains a copy of $\text{PGL}_2^+(\mathbb{Q})$.*

The proof involves piecewise linear versions of the congruence groups $\Gamma(n) \subset \text{SL}_2(\mathbb{Z})$. A consequence for continued fractions is explained at the end of §10. In §7 we will also show:

Theorem 1.8 *The group $\text{Aut}(?)$ preserves the rational points $\mathbb{Q}/\mathbb{Z} \subset S^1$.*

Corollary 1.9 *The group $\text{Aut}(J(z^2+1/4))$ preserves the cusps of the Julia set.*

Here a point in $J(z^2+1/4)$ is a *cusps* if its forward orbit lands on the parabolic fixed point at $p = 1/2$.

Welding is not closed under composition. Let $h : S^1 \rightarrow S^1$ be any piecewise-linear homeomorphism fixing $x = 0$, and satisfying, when $|x|$ is small,

$$h(x) = \begin{cases} 2x & \text{if } x > 0 \text{ and} \\ x & \text{if } x \leq 0. \end{cases} \quad (1.1)$$

As an unanticipated consequence of Theorem 1.7, in §11 we will show:

Theorem 1.10 *The set of welding maps in $\text{Homeo}^+(S^1)$ is not closed under composition. In fact $?h^{-1}$ is not a welding map, even though h and $?$ are.*

Another proof of the first statement, different in spirit, is given in [Rod]; an earlier example appears in [Vai, p.19].

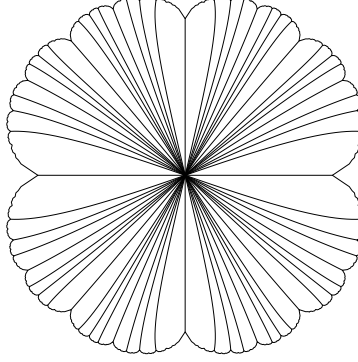


Figure 5. The Bers boundary of $\mathcal{T}_{1,1}$.

Cusped curves elsewhere. To conclude we draw attention to another setting where $W(?)$ seems to appear.

Conjecture 1.11 *The Bers boundary of any 1-dimensional Teichmüller space, ∂B_X , represents the same point as $W(?)$ in $\mathcal{J}$.*

Support for this conjecture is provided by Figure 5, taken from [KS], which shows the Bers embedding of $\mathcal{T}_{1,1}$ with basepoint the square torus.

Questions.

1. Is there an explicit dynamical system on $\widehat{\mathbb{C}}$ with Julia set *equal* to $W(?)$.
2. Can one analyze Julia sets with cusps on both sides, such as the pine tree Julia set in [LN, Figure 1.2]? (See also the end of §4 below.)
3. In view of Theorem 1.10, is there also a quasimetric ϕ such that $?^{-1}\phi?$ is not a welding map?

Notes and references. For background on complex dynamics, including parabolic Julia sets, see [Mil]. Useful references for the question mark map,

weldings, and quadrature domains include [CFP], [Bi] and [LM]; see also [ILRS].

The function $?(x)$ appears in [BP, eq. (1.3)] in relation to matings with $\mathrm{SL}_2(\mathbb{Z})$, and in later studies of the deltoid such as [LLMM, §5.4.2]. Nevertheless results such as $[J_n] = [W(?)]$ above are new, even for $n = 3$, since $?(x)$ is quite different from the welding map for J_n . For more on complex dynamics and matings, see also [BLS], [MM], [BLLM], [BL], [LLM], [LN] and references therein. See [LM] for another connection between Thompson's group and quasiconformal symmetries of Julia sets.

The author formulated a version of Conjecture 1.11 in 1983, with $W(?)$ replaced by $J(z^2 + 1/4)$, based on evidence provided by D. Wright's plot of the Maskit boundary of $\mathcal{T}_{1,1}$. For images of this cusped curve and much more, see [MSW].

Acknowledgments. I would like to thank S. Cantat, A. Rodriguez and Y. Luo for stimulating discussions and feedback, as well as Y. Komori and T. Sugawa for permission to use their image from [KS] in Figure 5. I would also like to thank D. Wright, S. Bullett, S. Rohde and N. Makarov for introducing me over the years to the Bers boundary, matings with $\mathrm{SL}_2(\mathbb{Z})$ and the deltoid.

Notation. As usual, $A = O(B)$ means $|A| \leq C|B|$ for some implicit constant C . If, in addition, $B = O(A)$, we write $A \asymp B$.

We let $|I| = |b - a|$ denote the length of an interval $I = [a, b]$ in $\mathbb{R}$ or $S^1 = \mathbb{R}/\mathbb{Z}$. Occasionally we also let $|E|$ denote the cardinality of a finite or countable set E .

The standard n th root of unity is given by

$$\zeta_n = \exp(2\pi i/n).$$

The punctured plane $\mathbb{C}^*$, the upper halfplane $\mathbb{H}$ and the unit disk Δ are the subsets of the complex plane $\mathbb{C}$ defined by the conditions $z \neq 0$, $\mathrm{Im}(z) > 0$ and $|z| < 1$ respectively; while $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. We let $\mathrm{QC}(U)$ denote the group of quasiconformal automorphisms of a domain $U \subset \widehat{\mathbb{C}}$.

2 The question mark function

In this section we develop properties of the question mark function, the Farey and dyadic tilings of $\mathbb{H}$, and the subgroups PS and PA in $\mathrm{Homeo}^+(S^1)$. Along the way we explain the analytic properties of $?(x)$ listed in §1.

Groups acting on $\mathbb{H}$ and S^1 . Let $\mathbb{Z}[1/2]$ denote the subring of $\mathbb{R}$ generated by $1/2$. Its unit group is $2^{\mathbb{Z}}$. A central role in the study of $?(x)$ is played by the matrix groups

$$S = \mathrm{SL}_2(\mathbb{Z})/(\pm I)$$

and

$$A = \mathrm{Aff}_2(\mathbb{Z}[1/2]) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in 2^{\mathbb{Z}}, b \in \mathbb{Z}[1/2] \right\}.$$

We identify S and A with the groups of Möbius transformations

$$S = \{f(z) = (az + b)/(cz + d) : ad - bc = 1, a, b, c, d \in \mathbb{Z}\}$$

and

$$A = \{g(z) = 2^a z + b : a \in \mathbb{Z}, b \in \mathbb{Z}[1/2]\}.$$

Piecewise diffeomorphisms. Let $S^1 = \mathbb{R}/\mathbb{Z}$, and let $f : S^1 \rightarrow S^1$ be a continuous map. We say f is a *piecewise diffeomorphism* if there is a covering of S^1 by finitely many *closed* intervals I_n such that $f|_{I_n}$ is a C^∞ diffeomorphism to its image for each n . To be precise, all derivatives of f must be continuous up to ∂I_n ; thus for each $k \geq 0$,

$$\sup_{S^1} |f^{(k)}(x)| < \infty \quad \text{and} \quad \inf_{S^1} |f'(x)| > 0. \quad (2.1)$$

We do not require that f is a bijection; indeed f may be a covering map.

Break points. The smallest set $B(f) \subset S^1$ such that $f|_{S^1 - B(f)}$ is smooth is the set of *break points* of f .

Similarly definitions apply for maps $f : \mathbb{R} \rightarrow \mathbb{R}$. In all cases, $B(f)$ is locally finite.

Piecewise projective and linear groups. A homeomorphism f belongs to the group of *integral piecewise projective maps*

$$PS \subset \mathrm{Homeo}^+(S^1)$$

if it lifts to a piecewise diffeomorphism $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ such that $B(\tilde{f}) \subset \mathbb{Q}$, and $\tilde{f}$ agrees with an element of S on each component of $\mathbb{R} - B(\tilde{f})$. The group of *dyadic piecewise linear maps*

$$PA \subset \mathrm{Homeo}^+(S^1)$$

is defined similarly: we require that $B(\tilde{f}) \subset \mathbb{Z}[1/2]$, and that $\tilde{f}$ agrees with an element of A on each component of $\mathbb{R} - B(\tilde{f})$.

Thompson's group. The group PA is the same as Thompson's group T ; it is an infinite, finitely-presented simple group. See [CFP] and references therein.

Covering maps. We let P_dS and P_dA denote the set of degree d *covering maps* $f : S^1 \rightarrow S^1$ that locally coincide with elements of PS and PA respectively. (For $d < 0$ we allow the natural orientation-reversing maps.)

Our study of $?(x)$ pivots on the following fundamental result, which we will prove below:

Theorem 2.1 *The question mark map on S^1 gives a bijection $f \mapsto g = ?f?^{-1}$ between P_dS and P_dA , for all $d \neq 0$.*

Corollary 2.2 (Thurston) *Thompson's simple group PA and the piecewise-projective group PS are isomorphic.*

See [CFP, §7].

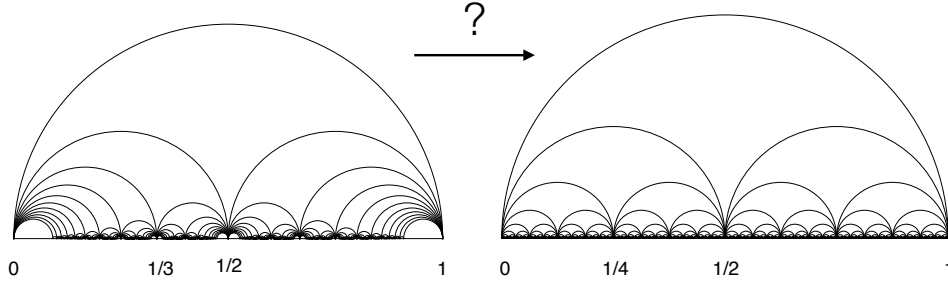


Figure 6. The map $?(x)$ relates the Farey and dyadic triangulations of $\mathbb{H}$.

Hyperbolic geometry. Let $T(a, b, c) \subset \mathbb{H}$ denote the ideal hyperbolic triangle with vertices $a, b, c \in \widehat{\mathbb{R}}$, and let $E(a, b) \subset \mathbb{H}$ denote the hyperbolic geodesic joining a to b . The question mark function relates the *Farey* and *dyadic* triangulations of $\mathbb{H}$; for the portions above $[0, 1]$, see Figure 6. We describe each of these in turn.

The Farey tiling. This is the ideal triangulation given by the orbit

$$\mathrm{SL}_2(\mathbb{Z}) \cdot T(0, 1, \infty);$$

see Figure 6(left). Its edges coincide with the geodesics joining $\mathbb{Z}$ to ∞ , together with the geodesics $E(p/q, r/s)$ joining pairs of rational numbers

satisfying

$$|ps - rq| = 1.$$

In the latter case $I = [p/q, r/s]$ is a *Farey interval*. We adopt the convention that p/q and r/s have positive denominators. Noting that

$$I = f \cdot [0, \infty], \quad \text{where } f(z) = (p + rz)/(q + sz) \in S, \quad (2.2)$$

we readily deduce:

Given any two Farey intervals $I, J \subset \mathbb{R}$, there is a unique $f \in S$ such that $f(I) = J$.

Subdivision. The *Farey midpoint* (or *mediant*) of a Farey interval $[a, b] = [p/q, r/s]$ is defined by

$$c = \text{mid}_S[a, b] = \frac{p + r}{q + s}.$$

It is characterized by the property that $T(a, b, c)$ is a Farey triangle, or equivalently, that $[a, c]$ and $[c, b]$ are also Farey intervals. By connectedness of the triangulation it is easy to see:

Every Farey interval can be obtained by starting with those of the form $[n, n + 1]$ and performing iterated Farey subdivision.

In particular:

Any two Farey intervals are nested or disjoint.

Clearly every rational number occurs as the endpoint of *some* Farey interval. By iterating subdivision until both a and b appear, we find:

Any interval $[a, b]$ with rational endpoints can be tiled by Farey intervals.

The dyadic tiling. The *dyadic intervals* in $\mathbb{R}$ are those of the form

$$[a, b] = 2^{-n}[p, p + 1] \quad \text{with } p, n \in \mathbb{Z} \text{ and } n \geq 0.$$

The edges of the *dyadic triangulation* of $\mathbb{H}$ consists of the geodesics $E(n, \infty)$, for all $n \in \mathbb{Z}$, and $E(a, b)$ for every dyadic interval $[a, b]$; see Figure 6(right). The dyadic midpoint of a dyadic interval $[a, b]$ is given simply by:

$$c = \text{mid}_A[a, b] = (a + b)/2.$$

It too is characterized by the fact that $T(a, b, c)$ is a dyadic triangle. The dyadic tiling is *not* invariant under the action of A on $\mathbb{H}$; however, it is easy to see:

For any pair of dyadic intervals I and J , there is a unique $g \in A$ such that $g(I) = J$.

It is also immediate that:

Any interval $[a, b]$ with endpoints in $\mathbb{Z}[1/2]$ can be tiled by dyadic intervals.

and

Every dyadic interval can be generated by starting with those of the form $[n, n + 1]$ and applying iterated dyadic subdivision.

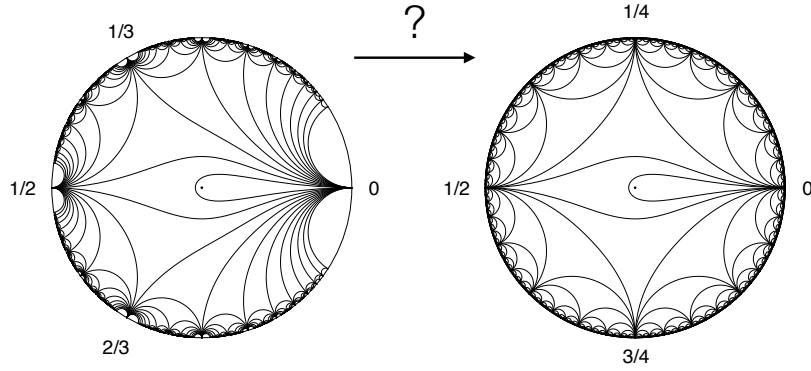


Figure 7. Farey and dyadic tilings of Δ^* .

Passage to S^1 . Since $g(z) = z + 1$ belongs to both S and A , the triangulations of $\mathbb{H}$ just described cover triangulations of $\Delta^* \cong \mathbb{H}/\mathbb{Z}$; see Figure 7. Similarly we obtain subdivisions of $S^1 = \mathbb{R}/\mathbb{Z}$ into Farey and dyadic intervals.

The question mark function. As in §1, the question mark function is defined as the unique homeomorphism $? : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $?(n) = n$ for all $n \in \mathbb{Z}$ and

$$?(\text{mid}_S[a, b]) = \text{mid}_A[?(a), ?(b)] \quad (2.3)$$

for all Farey intervals $[a, b]$. This equation implies:

The function $?(x)$ gives a bijection between $\mathbb{Q}$ and $\mathbb{Z}[1/2]$, sending Farey intervals to dyadic intervals.

Levels. A Farey or dyadic interval I has *level* k if it arises from an interval of the form $[n, n+1]$ after k subdivisions. In the dyadic case, $|I| = 2^{-k}$. The level of an interval is preserved by $?(x)$.

Earthquakes. It is natural to expect an extension of $?(x)$ to a map of $\mathbb{H}$, since we have a bijection

$$T(a, b, c) \rightarrow T(?(a), ?(b), ?(c))$$

between triangles in the Farey and dyadic tilings. This idea can be made precise using the language of earthquakes (cf. [Th], [SWW]):

Theorem 2.3 *The question mark map arises as the boundary values of a signed right earthquake along the edges of the Farey triangulation of $\mathbb{H}$. The shearing distance $\delta(a, b)$ for the edge $E(a, b)$, $a < b$, is given by*

$$\begin{cases} \delta(n, \infty) = \delta(n, n+1) = 0 & \text{for all } n \in \mathbb{Z}, \\ \delta(p/q, r/s) = \log 2 & \text{if } q < s, \text{ and} \\ \delta(p/q, r/s) = -\log 2 & \text{if } q > s. \end{cases}$$

Proof. There is a single isometry invariant $\lambda = \lambda(T, T') > 0$ that describes how a pair of ideal triangles T and T' meet along an edge E . It is characterized by the property that if $T = T(-1, 0, \infty)$ and $E = E(0, \infty)$, then $T' = T(0, \lambda, \infty)$.

Any two adjacent triangles T and T' in the Farey triangulation are related by reflection through their common edge E ; thus $\lambda(T, T') = 1$. For the dyadic triangulation, we also have $\lambda(T, T') = 1$ when $E = E(n, \infty)$ or $E(n, n+1)$; but in all other cases, $\lambda(T, T') = 1/2$ or 2 . In the latter case, a shear of distance $\pm \log 2$ is required to convert the pair of Farey triangles meeting along $E(a, b)$ to the dyadic pair meeting along $E(?(a), ?(b))$. To check the sign of the $\delta(a, b)$, use the fact that the denominator of $\text{mid}_S(a, b)$ is greater than that of a or b .

The action of this earthquake on $\mathbb{R}$ fixes $\mathbb{Z}$ and intertwines the operations mid_S and mid_A , so it agrees with $?(x)$. ■

Maps on S^1 . We now turn to an examination of the groups A , S , PA and PS .

Let I, J be intervals in $\mathbb{R}$. If $I = f(J)$ for some $f \in S$, then I is a Farey interval $\iff J$ is a Farey interval.

The dyadic case is slightly different: a given $g \in A$ preserves dyadic intervals only when they are sufficiently small. This is clear by inspecting the action of g on the dyadic triangulation.

Lemma 2.4 *Let $f : I \rightarrow J$ be a map between intervals in $\mathbb{R}$, and let $g = ?f?^{-1}$. Then the following are equivalent:*

1. *I and J are Farey intervals, and $f \in S$;*
2. *$?(I)$ and $?(J)$ are dyadic intervals, and $g \in A$.*

(More precisely, f is the restriction of an element of S to I , and similarly for g .)

Proof. Suppose the first condition holds. Then $?(I)$ and $?(J)$ are indeed dyadic intervals, and there is a unique $h \in A$ sending $?(I)$ to $?(J)$. The map h is characterized by the property that for any dyadic interval $[a, b] \subset I$ we have

$$\text{mid}_A[h(a), h(b)] = h \cdot \text{mid}_A[a, b].$$

We claim $g = ?f?^{-1}$ has the same property. Indeed, we can write $[a, b] = [?c, ?d]$ for some Farey interval $[c, d] \subset I$, and thus

$$\begin{aligned} \text{mid}_A[g(a), g(b)] &= \text{mid}_A[?f(c), ?f(d)] = ?f \text{mid}_S[c, d] \\ &= g? \text{mid}_S[c, d] = g \cdot \text{mid}_A[a, b]. \end{aligned}$$

Thus $g = h \in A$. The converse is similar. ■

Descent to S^1 . As in §1, we will also regard the question mark function as a homeomorphism

$$? : S^1 \rightarrow S^1 = \mathbb{R}/\mathbb{Z}.$$

This is the viewpoint adopted in the statement of Theorem 2.1.

Proof of Theorem 2.1. Let us first show that $?PS?^{-1} \subset PA$.

Let $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ be a pair of tilings of S^1 by cyclically ordered Farey intervals. Since any two Farey intervals in $\mathbb{R}$ are related by a unique element of S , there is a unique $f \in PS$ such that $f(A_i) = B_i$ and $B(f) \subset \bigcup \partial A_i$. Conversely, every $f \in PS$ arises in this way. To see this, use the fact that the break points of f occur at rational numbers, and intervals with rational endpoints can be subdivided into Farey intervals.

By the Lemma above, $g = ?f?^{-1}$ is a piecewise dyadic map relative to the tilings of S^1 by the dyadic intervals $(?A_i)$ and $(?B_i)$. In particular, $g \in PA$.

The proof that $?^{-1}PA? = PS$ is similar, taking care to choose the subdivision (A_i) of S^1 fine enough that both A_i and $B_i = g(A_i)$ are dyadic intervals. The generalization to P_dS and P_dA is immediate. ■

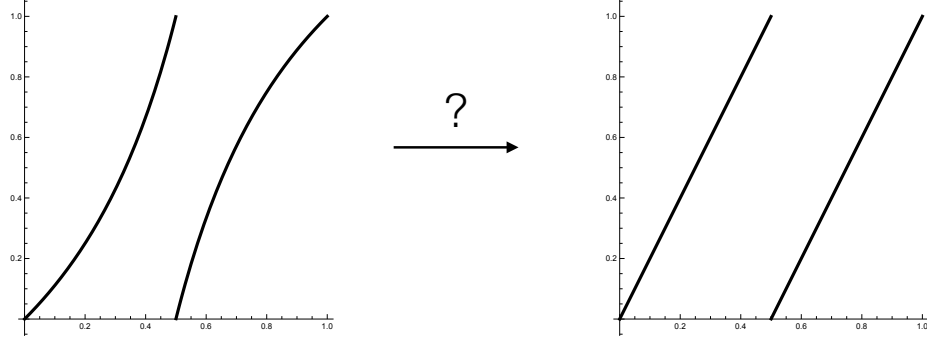


Figure 8. Conjugation by $?(x)$ transforms $f(x)$ into $\chi_2(x) = 2x$ on S^1 .

Covering maps of degree two. Here is an important example of Theorem 2.1. Define $f \in PS_2$ and $\chi_2 \in PA_2$ by $\chi_2(x) = 2x \bmod 1$ and by

$$f(x) = \begin{cases} x/(1-x) & \text{for } x \in [0, 1/2], \text{ and} \\ (2x-1)/x & \text{for } x \in [1/2, 1]. \end{cases} \quad (2.4)$$

Then

$$?f?^{-1} = \chi_2, \quad (2.5)$$

as is readily verified using Lemma 2.4; in fact:

The map $? : S^1 \rightarrow S^1$ is the unique topological conjugacy between $f(x)$ and $\chi_2(x) = 2x$.

See Figure 8. This provides an alternative definition of the map $?(x)$.

Quadratic irrationals. As observed by Minkowski:

$$\text{We have } ?(x) \in \mathbb{Q} \iff \deg(x/\mathbb{Q}) \leq 2.$$

This follows from fact that $\deg(x/\mathbb{Q}) \leq 2 \iff x$ is a fixed point of some nontrivial $f \in S$. For example, the golden ratio satisfies $?(\gamma) = 5/3$.

As stated in the Introduction, we have $?(x) \in C^\alpha$ with $\alpha = \log(2)/\log(\gamma^2)$ (and this is sharp). To see where α comes from, observe that there are elements $f(x) = (x-1)/(2-x) \in S$ and $g(x) = 4x-5 \in A$, conjugate by $?(x)$ near their respective fixed points γ and $5/3$, such that

$$f'(\gamma) = \gamma^4 \quad \text{and} \quad g'(5/3) = 4.$$

It follows readily that $\alpha = \log(4)/\log(\gamma^4)$ is an upper bound for the modulus of Hölder continuity of $?(x)$. For the lower bound, observe that $?$ preserves

levels and that the Farey intervals containing γ shrink more rapidly than those about any other point in $\mathbb{R}$. See [Sa, p.438].

Derivatives. The same reference shows:

We have $?'(x) = 0$ for almost every $x \in \mathbb{R}$.

See also remark (3) below. Equation (2.3) implies that $?(1/(n+1)) = 1/2^n$, and therefore all derivatives of $?(x)$ vanish at $x = 0$. Similar reasoning shows:

All derivatives of $?(x)$ vanish at every rational point in $\mathbb{R}$.

Remarks.

1. Any $f \in P_d S$ is C^1 , and all its rational periodic points are parabolic. This follows from the fact that $B(f) \subset \mathbb{Q}$, and that $g'(x) = 1$ whenever $g \in S$, $x \in \mathbb{Q}$ and $g(x) = x$.
2. For $x \in [0, 1]$, the binary expansion of $?(x) = 0.b_1b_2b_3\dots$ can be easily computed from its continued fraction expansion, as follows: if $x = 1/(a_1 + 1/(a_2 + \dots))$, then (b_i) begins with a run of $a_1 - 1$ zeros, then a_2 ones, then a_3 zeros, etc. For example, if $x = (\sqrt{5} - 1)/2 = 1/(1 + 1/(1 + \dots))$, then $?(x) = 0.101010\dots = 2/3$.
When x is rational, one should require that the last term in its continued fraction expansion is $a_n = \infty$. For example, if $x = 1/4 = 1/(4 + 1/\infty)$, then $?(x) = 0.00011111\dots = 0.001 = 1/2^3$.
3. For almost every $x = 1/(a_1 + 1/(a_2 + \dots)) \in [0, 1]$, the average value of a_n is infinite (see e.g. [CFS, §4, Theorem 4]). On the other hand, the expected length of a run of 0s or 1s in the binary expansion of a randomly chosen x is finite. This is one way to show that $?'(x) = 0$ almost everywhere.
4. The question mark function also occurs naturally in Conway's theory of numbers and games; see [Con, Fig. 8, p.84].
5. For a discussion of PL actions of *closed* surface groups on the circle, see [Gh, Theorem 2].

3 Geometrically finite covering maps on S^1

In this section we introduce the notion of a geometrically finite covering map on S^1 . These generalize expanding maps by allowing finitely many parabolic cycles and minimal smoothness. Many concrete examples are given in §5.

We then prove Theorem 1.4, which states:

A topological conjugacy ϕ between a pair of geometrically finite covering maps is quasimetric $\iff$ it preserves parabolics.

Definition. A topological covering map $f : S^1 \rightarrow S^1$ is *geometrically finite* if:

1. It is topologically expanding;
2. It has only finitely many parabolic points; and
3. There exists a $k \geq 1$ such that whenever f^n is injective on an interval I , either
 - $f^n|I$ is k -quasimetric, or
 - $f^i(I)$ is controlled by the thin part of f , for some $i < n$.

These terms will be explained below. The definitions are formulated so that the following result is immediate:

If f is geometrically finite, then so is any map $g = \phi f \phi^{-1}$ quasimetrically conjugate to f .

Quasimetric maps. Let $\phi : I \rightarrow J$ be a homeomorphism between a pair of intervals. Recall that ϕ is k -quasimetric, $k \geq 1$, if for any pair of adjacent intervals $A, B \subset I$ with $|A|/|B| = 1$, we have

$$1/k < |\phi(A)|/|\phi(B)| < k.$$

We say ϕ is quasimetric if it is quasimetric for some k .

The following equivalent definition will be useful below:

Proposition 3.1 *The map ϕ is quasimetric if and only if there exist $\alpha, \beta \geq 1$ such that any interval C can be split into a pair of adjacent intervals A, B satisfying*

$$1/\alpha \leq |A|/|B| \leq \alpha \quad \text{and} \quad 1/\beta \leq |\phi(A)|/|\phi(B)| \leq \beta.$$

A useful reference is [LV, Ch. 6].

Rulers. To define parabolic (and hyperbolic) periodic points, the following construction will be useful.

A *ruler* (I, R) consists of an interval $I = [a_0, b_0]$ and a countable closed set $R \subset I$ such that

$$R = \{a_0 < a_1 < a_2 < \cdots < p < \cdots < b_2 < b_1 < b_0\},$$

where $a_i \rightarrow p$ from below and $b_i \rightarrow p$ from above. The points (a_i) and (b_i) can be regarded as ‘rules’ or ‘tick marks’ along I . We let

$$\Delta a_i = |a_i - a_{i+1}| \quad \text{and} \quad \Delta b_i = |b_i - b_{i+1}|.$$

Two rulers (I, R) and (I', R') are *equivalent* if there is an orientation-preserving homeomorphism $\rho : I \rightarrow I'$ such that $\rho(R) = R'$. An equivalence may be Lipschitz, quasimetric, etc. according to the quality of ρ .

Rulers from dynamics. We say a ruler (I, R) as above is *associated* to a homeomorphism $h : I \rightarrow J \subset I$ if $h(p) = p$, $a_i = h^i(a_0)$ and $b_i = h^i(b_0)$. In this case p is topologically attracting, and the tick marks (a_i) and (b_i) record orbits of h converging to p from above and below.

For a local homeomorphism f with a topologically *repelling* fixed point p , the associated ruler is defined using $h = f^{-1}$.

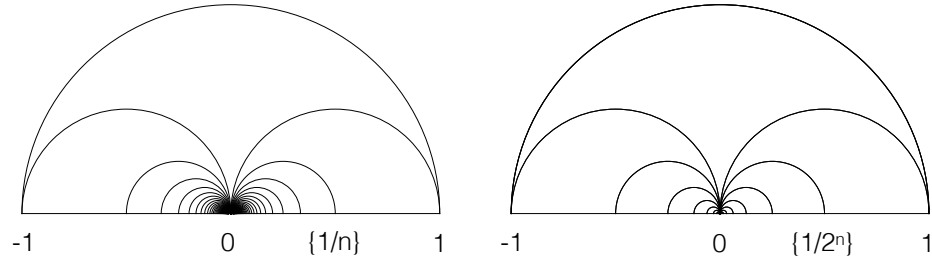


Figure 9. The model parabolic and hyperbolic rulers (I, R) .

Model rulers. The *model parabolic ruler* (I, R) is defined by $I = [-1, 1]$ and

$$R = \{0\} \cup \{\pm 1/n : n \geq 1\}. \quad (3.1)$$

The model *hyperbolic ruler* is defined similarly, with $I = [-1, 1]$ and

$$R = \{0\} \cup \{\pm 2^{-n} : n \geq 0\}.$$

These rulers are associated, respectively, to the fixed point $p = 0$ of the model parabolic and hyperbolic maps

$$T(x) = \frac{x}{1 + |x|} \quad \text{and} \quad A(x) = \frac{x}{2}. \quad (3.2)$$

See Figure 9.

Definition: Parabolic points. We say p is a *parabolic* fixed point of a local homeomorphism f if p is topologically attracting or repelling, and its associated rulers are all *quasisymmetrically* equivalent to the model parabolic ruler defined above.

This definition is local: it only depends on the orbits of f near p . We extend the definition to periodic points p by replacing f with an iterate such that f^n preserves orientation and fixes p . It is immediate that:

A quasisymmetric conjugacy preserves parabolics.

One can also readily verify that:

If p is a parabolic fixed point, then $f'(p) = 1$.

Exponent of parabolicity. To see that our definition includes more conventional notions of parabolicity, we observe:

Proposition 3.2 *Let $f(x)$ be a homeomorphism defined near $x = 0$ in $\mathbb{R}$ and satisfying, for some $e > 1$,*

$$f(x) - x \asymp |x|^{e-1}. \quad (3.3)$$

Then f has a parabolic fixed point at $x = 0$.

When $f(x)$ satisfies equation (3.3) in suitable local coordinates, we say its parabolic point has *exponent* e . For example, $f(x) = x/(1 - |x|)$ has a parabolic point of exponent 2 at $x = 0$. If f is C^∞ , then its exponent must be an odd integer $e \geq 3$.

Proof of Proposition 3.2. Let $h(x)$ be a local inverse for $f(x)$; then $p = 0$ is topologically attracting for h . Let (I_p, R_p) be one of its associated rulers, with $I_p = [a_0, b_0]$ and $R_p = \{a_i, b_i\} \cup \{p\}$ as above. Let (I, R) be the model parabolic ruler, and let $\rho : (I, R) \rightarrow (I_p, R_p)$ be the unique orientation-preserving homeomorphism that is linear on each component of $I - R$. Recall that $R = \{0\} \cup \{\pm 1/n\}$, so ρ sends $-1/(n+1)$ to a_n and $+1/(n+1)$ to b_n .

Now consider the case of exponent $e = 2$. Given $s > 0$, let

$$T_s(x) = \frac{x}{sx + 1} = x - sx^2 + O(s^2|x|^3).$$

Note that for $n \gg 0$ we have

$$T_s^n(x) = \frac{x}{snx + 1} \sim \frac{1}{sn}. \quad (3.4)$$

Since the map $h : [a_0, b_0] \rightarrow [a_1, b_1]$ satisfies

$$|h(x) - x| \asymp x^2, \quad (3.5)$$

we can choose $0 < s < u$ such that

$$0 < T_u(x) < h(x) < T_s(x)$$

for all $x \in [0, b_0]$. It follows by induction that

$$T_u^n(b_0) < b_n = h^n(b_0) < T_s^n(b_0)$$

for all n ; hence $b_n \asymp 1/n$ by equation (3.4), and

$$\Delta b_n \asymp b_n^2 \asymp \frac{1}{n^2} \asymp \Delta \frac{1}{n}.$$

by equation (3.5). Similar reasoning applies to the sequence a_n . Thus corresponding gaps in the rulers (I_p, R_p) and (I, R) have comparable size. It follows that $\rho'(x) \asymp 1$ wherever it is defined, and hence $\rho : I \rightarrow I_p$ is a bilipschitz homeomorphism (and therefore quasisymmetric).

To treat general parabolics, just note that conjugation by $\phi(x) = \text{sign}(x)|x|^\alpha$, $\alpha = 1/(e - 1)$, converts a parabolic fixed point of exponent e into one of exponent 2. Since $\phi(x)$ is quasisymmetric, we can reduce to the case of exponent 2 and apply the preceding argument to complete the proof. ■

Hyperbolic points. We define hyperbolic points in the same way as parabolic points, but using the model hyperbolic ruler. It is readily verified that:

The model hyperbolic and parabolic rulers are not quasisymmetrically equivalent;

hence a periodic point cannot be both parabolic and hyperbolic.

Definition: Topological expansion. We now turn to the global dynamics of covering maps.

Let $\chi_d : S^1 \rightarrow S^1$ be the standard linear covering map of degree d , given by

$$\chi_d(x) = dx \bmod 1.$$

A covering map $f : S^1 \rightarrow S^1$ is *topologically expanding* if for any interval $I \subset S^1$ there exists an n such that $f^n(I) = S^1$.

A *conjugacy* between a pair of covering maps f and g is an *orientation-preserving* homeomorphism $\phi : S^1 \rightarrow S^1$ such that $\phi f \phi^{-1} = g$; in other words, the diagram

$$f \circ S^1 \xrightarrow{\phi} S^1 \circ g$$

commutes. The following result is well-known (see [Sh] for the C^1 case):

Proposition 3.3 *Let $f : S^1 \rightarrow S^1$ be a covering map of degree d . Then f is topologically expanding $\iff |d| > 1$ and f is topologically conjugate to χ_d .*

Sketch of the proof. If f is topologically expanding, then clearly $|d| > 1$ and hence f has at least one fixed point p . Normalize so that $p = 0$, and lift f to a homeomorphism $\tilde{f}$ on the universal cover $\mathbb{R}$ of S^1 satisfying $\tilde{f}(0) = 0$. Then

$$\tilde{\phi}(x) = \lim_{n \rightarrow \infty} d^{-n} \tilde{f}^n(x) \tag{3.6}$$

converges exponentially fast to a lift of the desired conjugacy ϕ on S^1 . The converse is immediate. ■

Markings. A *marking* of f is a map ϕ conjugating f to χ_d as above. For each $p \in \text{Fix}(f)$ there is a unique marking such that $\phi(p) = 0$; thus f has $|d - 1|$ possible markings.

Rulers for covering maps. Let $f : S^1 \rightarrow S^1$ be a topologically expanding map of degree d . We associate to each fixed point p of f the *canonical* ruler (I_p, R_p) characterized by

$$\phi(I_p) = [-d^{-2}, d^{-2}] \quad \text{and} \quad \phi(R_p) = \{0\} \cup \{\pm 1/d^n : n \geq 2\}, \tag{3.7}$$

where $\phi : S^1 \rightarrow S^1$ is a marking such that $\phi(p) = 0$. Note that:

1. All points in R_p belong to the backwards orbit of p ;

2. The intervals I_p associated to the $(d-1)$ different fixed points of f are disjoint, and
3. The definition of (I_p, R_p) depends only on the topological dynamics of f .
4. In particular: *If a topological conjugacy sends p to q , then it sends (I_p, R_p) to (I_q, R_q) .*

To handle the case of a periodic point p , we first replace f with an iterate so that p becomes fixed and $\deg(f) > 0$.

Definition: The thin part of f . In the sequel we will exclusively use rulers for *parabolic points*.

An interval $J \subset S^1$ is *controlled* by a ruler (I, R) if

$$J \subset I \quad \text{and} \quad |J \cap R| \geq 3.$$

If J is controlled by one of the canonical rulers (I_p, R_p) associated to a parabolic point of f , we say J is controlled by the *thin part* of f .

Geometric finiteness. The definition of a geometrically finite covering map is now complete. To set the stage for the proof of Theorem 1.4, we begin with a special case.

Proposition 3.4 *A covering map f of degree $|d| > 1$ is geometrically finite, without parabolics, $\iff f$ is quasisymmetrically conjugate to χ_d .*

Proof. Suppose f is geometrically finite, without parabolics. Then $f^n|I$ is uniformly quasisymmetric whenever it is injective. Consequently the maps $\phi_n = d^{-n}f^n$ appearing on the right hand side of equation (3.6) are uniformly quasisymmetric. The limiting conjugacy ϕ to χ_d is therefore quasisymmetric as well. The converse is immediate. $\blacksquare$

Parabolic control. For the general case we will use rulers to control the parabolic behavior of f . Here is a preliminary remark.

Given a ruler (I, R) and an interval $A \subset I$, let $A[R]$ denote the largest subinterval of A with endpoints in R . It is elementary to check:

Proposition 3.5 *If J is controlled by the model parabolic ruler (I, R) , then there exists a pair of adjacent intervals such that $J = A \cup B$, $|A|/|B| \asymp 1$, $|A \cap R| \geq 2$ and $|B \cap R| \geq 2$.*

Moreover, whenever an interval $A \subset I$ satisfies $|A \cap R| \geq 2$, we have $|A| \asymp |A[R]|$.

Proof of Theorem 1.4. Let ϕ be a topologically conjugacy between a pair of geometrically finite covering maps f and g . Replacing f and g with suitable iterates, we can assume that every parabolic point is fixed.

Suppose ϕ preserves parabolics. We wish to show ϕ is quasimetric. Let I be any interval in S^1 , and let $I' = \phi(I)$. There are now two possibilities.

First, it may be that $f^n(I)$ is never controlled by the thin part of f . Then, since ϕ preserves parabolics, the same is true for $g^n(I')$. Write I as the union of two adjacent interval $A \cup B$ of equal length, and choose n as large as possible while keeping $f^n|I$ injective. Then $f^n(A)$ and $f^n(B)$ have definite size, so ϕ changes their lengths by at most a bounded factor. In addition, $f^n|I$ and $g^n|I'$ are uniformly quasimetric, by the definition of geometric finiteness. Consequently, we have:

$$1 = \frac{|A|}{|B|} \asymp \frac{|f^n(A)|}{|f^n(B)|} \asymp \frac{|\phi f^n(A)|}{|\phi f^n(B)|} = \frac{|g^n \phi(A)|}{|g^n \phi(B)|} \asymp \frac{|\phi(A)|}{|\phi(B)|},$$

which, by Proposition 3.1, verifies quasimetricity of ϕ for the interval I .

Otherwise, there exists an $n \geq 0$ such that $f^n|I$ is injective and $J = f^n(I)$ is controlled by the canonical ruler (I_p, R_p) associated to some parabolic fixed point p . Moreover, $q = \phi(p)$ is a parabolic fixed point for g , and $J' = g^n(I')$ is controlled by (I_q, R_q) .

By the definition of parabolicity, there is a quasimetric homeomorphism

$$\rho : I_p \rightarrow I_q$$

such that $\rho|_{R_p} = \phi|_{R_p}$ and $\rho(R_p) = R_q$. We can also assume that $f^i(J)$ is *not* controlled by the thin part of f for any $i < n$. Then, by the definition of geometric finiteness, $f^n|I$ and $g^n|I'$ are k -quasimetric, where k only depends on f and g .

In view of Proposition 3.5, and the quasimetricity of ρ , we can find a pair of adjacent intervals such that $J = A \cup B$, $|A| \asymp |B|$, $|A \cap R_p| \geq 2$, and $|B \cap R_p| \geq 2$. Moreover, we have $|A| \asymp |A[R_p]|$ and $|B| \asymp |B[R_p]|$. Throughout, the implicit constants only depend on f . Similar statements hold for $A' = \phi(A)$ and $B' = \phi(B)$ relative to R_q .

We now make critical use of observation (4) above: since ρ sends R_p to R_q , it also sends $A[R_p]$ to $A'[R_q]$ and $B[R_p]$ to $B'[R_q]$. Since ρ is quasimetric, this implies:

$$\frac{|A|}{|B|} \asymp \frac{|A[R_p]|}{|B[R_p]|} \asymp \frac{|A'[R_q]|}{|B'[R_q]|} \asymp \frac{|A'|}{|B'|} = \frac{|\phi(A)|}{|\phi(B)|}.$$

Thus ϕ behaves distorts the relative lengths of the intervals A and B only by a bounded amount.

Now let (A_0, B_0) be the preimages of (A, B) under $f^n : I \rightarrow J$. Since f^n and g^n are uniformly quasimetric, we conclude similarly that $I = A_0 \cup B_0$, $|A_0| \asymp |B_0|$, and

$$\frac{|A_0|}{|B_0|} \asymp \frac{|\phi(A_0)|}{|\phi(B_0)|}.$$

Since I was an arbitrary interval, this shows that ϕ is quasimetric.

Conversely, if ϕ is quasimetric, then it must preserve parabolics. ■

Remarks.

1. *Horocycles.* Let (I, R) denote the model parabolic ruler given by equation (3.1). The space of intervals $J \subset I = [-1, 1]$ can be identified with the space of points (x, y) in the upper halfplane, $y > 0$, satisfying $y + |x| \leq 1$. Under this identification, the intervals J controlled by the model ruler — those satisfying $|J \cap R| \geq 3$ — correspond approximately to the region above the parabola $y = x^2$. This parabola mimics the invariant horocycle defining the thin part for a Fuchsian group.
2. *Hyperbolic and parabolic points.* It can be shown that, in terms of the ruler definition above, every periodic point of a geometrically finite covering map is either hyperbolic or parabolic. In fact, the local dynamics is quasimetrically conjugate to one of the two models given in equation (3.2).

However, the *local definition* of parabolic and hyperbolic fixed points does not imply that such a conjugacy exists. One also needs uniform quasimetricity of suitable iterates of f , which is included in the definition of geometric finiteness.

4 Classification

In this brief section we record some consequences of Theorem 1.4.

Markings and invariants. Let f be a geometrically finite covering map of degree d . We let $P(f) \subset S^1$ denote its set of parabolic points, and $\text{Fix}(f)$ its set of fixed points.

Given $d \in \mathbb{Z}$ with $|d| > 1$, let $\mathcal{T}_d$ denote the Teichmüller space of *marked* geometrically finite covering maps of degree d . A point in $\mathcal{T}_d$ is specified

by a pair (f, ϕ) where $\phi f \phi^{-1} = \chi_d$. The action of the *mapping-class group* $\text{Mod}_d \cong \mathbb{Z}/(d-1)$ is generated by the transformation

$$(f, \phi(x)) \mapsto (f, \phi(x) + 1/(d-1)),$$

which simply composes $\phi(x)$ with the generator of the automorphism group of $\chi_d(x)$. The map to the quotient *moduli space*, $\mathcal{T}_d \rightarrow \mathcal{M}_d = \mathcal{T}_d / \text{Mod}_d$, forgets the marking ϕ . (For a related discussion, see [Mc1].)

We typically regard a point in $\mathcal{T}_d$ as a covering map f with an implicit marking ϕ , and simply denote it by f .

There is a *natural* conjugacy between any two points $f_1, f_2 \in \mathcal{T}_d$, given in terms of their markings by

$$\phi = \phi(f_1, f_2) = \phi_2^{-1} \circ \phi_1. \quad (4.1)$$

We write $[f_1] = [f_2]$ if ϕ is quasisymmetric.

Invariants. The combinatorial location of the parabolic points of f is recorded by the invariant

$$\Pi(f) = \phi(P(f)) \subset S^1.$$

Note that $\Pi(f)$ is a finite union of periodic cycles for χ_d . We let $\mathcal{T}_d(Q)$ denote the locus where $\Pi(f) = Q$.

Types. We say a geometrically finite covering map f has *type* (d, n) if $\deg(f) = d$, $|P(f)| = n$ and $P(f) \subset \text{Fix}(f)$. The last condition is always satisfied if we replace f with f^m for some m . Note that $n \leq |d-1|$.

Classification statements. We can now formulate some immediate consequences of Theorem 1.4.

Theorem 4.1 *The natural conjugacy between f_1 and f_2 in $\mathcal{T}_d$ is quasisymmetric $\iff \Pi(f_1) = \Pi(f_2)$.*

Corollary 4.2 *A geometrically finite covering map is determined, up to quasisymmetric conjugacy, by its degree and by the combinatorial location of its parabolic points.*

Corollary 4.3 *We have $[f_1] = [f_2]$ in $\mathcal{T}_d \iff f_1$ and f_2 belong to the same block of the partition $\mathcal{T}_d = \bigsqcup_Q \mathcal{T}_d(Q)$.*

Corollary 4.4 *There are only finitely many geometrically finite covering maps with $|\deg(f)| \leq d$ and $|P(f)| \leq p$, up to quasisymmetric conjugacy.*

Corollary 4.5 *Any two geometrically finite covering maps of type (d, n) , with $d > 1$ and $n = 0, 1, d - 2$ or $d - 1$, are quasimetrically conjugate. The conjugacy is unique if $n = 1$ or $d = 2$.*

Proof. In this case we can choose a marking such that $\Pi(f) = (0, 1, \dots, n - 1)/(d - 1) \subset S^1$. ■

From dynamics to moduli space. We remark that the function $\phi(f_1, f_2)$ determines a natural map

$$\Phi : \mathcal{T}_d \times \mathcal{T}_d \rightarrow \mathcal{M},$$

where $\mathcal{M} = \text{QS} \setminus \text{Homeo}^+(S^1)/\text{QS}$ as in §1. Each subproduct $\mathcal{T}_d(Q_1) \times \mathcal{T}_d(Q_2)$ is sent to a single point, which we will also denote by $\Phi(Q_1, Q_2)$. For example, the pine tree Julia set of [LN, Figure 1.2] corresponds to $\Phi(\{0\}, \{1/2\})$ for $d = 3$. Theorem 1.5 is equivalent to the assertion that:

The map Φ sends $\mathcal{T}_d(Q) \times \mathcal{T}_d(\emptyset)$ to $[?]$ whenever Q is nonempty.

Question. When does $\Phi(Q_1, Q_2) = \Phi(Q'_1, Q'_2)$? And is there cancellation? That is, do we have $\Phi(Q_1, Q_2) = \Phi(Q_1 - P, Q_2 - P)$ if $P \subset Q_1 \cap Q_2$?

5 Criteria for geometric finiteness

In this section we will prove:

Theorem 5.1 *Let $f : S^1 \rightarrow S^1$ be a covering map of degree d , $|d| > 1$. Assume:*

1. *The map f is a piecewise diffeomorphism;*
2. *Every break point $b \in B(f)$ is pre-periodic;*
3. *The map f^{2p} is C^1 at every break point of period p ;*
4. *The parabolic points $P(f)$ form a finite set; and,*
5. *The map f expands a smooth metric ρ on $S^1 - P(f)$.*

Then f is geometrically finite.

All the geometrically finite covering maps we will consider in the sequel satisfy these conditions.

Expansion. Here is an elaboration of the final condition. A *smooth metric* is given by $\rho = \rho(x)|dx|$ for some C^∞ function $\rho(x) > 0$; it is *expanded* by f if

$$|f'(x)|_\rho = \frac{f^*(\rho)}{\rho} = \frac{|f'(x)|\rho(f(x))}{\rho(x)} > 1$$

whenever both x and $f(x)$ lie in $S^1 - P(f)$. (If x is a break point, we require that the right and left derivatives of f satisfy the inequality above.)

We emphasize that the metric ρ is allowed to blow up at points in $P(f)$. Although one can frequently take $\rho = |dx|$, in §6 we will let ρ be the hyperbolic metric on $S^1 - P(f)$, which behaves like $|dx|/|x|$ at parabolic points.

Derivatives and parabolics. Recall that the condition that f is a piecewise diffeomorphism implies each derivative of f is uniformly bounded above, and $|f'(x)|$ is bounded below (see equation (2.1). Moreover, any parabolic fixed point p of f (as defined in §3) has the local form

$$f(p+x) = p + \begin{cases} x + A|x|^e + O(|x|^{e+1}) & \text{for } x \geq 0, \text{ and} \\ x - B|x|^e + O(|x|^{e+1}) & \text{for } x \leq 0, \end{cases}$$

for some $A, B > 0$ and integer $e \geq 2$; cf. Proposition 3.2.

Examples. Here are some representative applications of Theorem 5.1.

1. *Expanding maps.* Any smooth map $f : S^1 \rightarrow S^1$ with $|f'(x)| > 1$ for all x is geometrically finite. In this case $B(f) = P(f) = \emptyset$ and $\rho = |dx|$ is expanded by f .
2. *Expanding Blaschke products.* Let $F : \overline{\Delta} \rightarrow \overline{\Delta}$ be a Blaschke product of degree $d > 1$ with $F(0) = 0$. Then $f = F|_{S^1}$ is geometrically finite. Indeed, f is measure-preserving [Mar]; equivalently, $\sum_{f(z)=w} 1/|f'(z)| = 1$ for all $w \in S^1$. This implies $|f'(z)| > 1$ and hence we have a special case of (1).

For more on expanding Blaschke products, see e.g. [Mc1], [Mc2] [Mc3], and references therein.

3. *Parabolic Blaschke products.* There is a unique Blaschke product $F_d : \overline{\Delta} \rightarrow \overline{\Delta}$ of degree d , with a critical point of multiplicity $(d-1)$ at $z = 0$, such that $F_d(1) = 1$, $F_d'(1) = 1$ and $F_d''(1) = 0$. Explicitly,

$$F_d(z) = \frac{(d+1)z^d + (d-1)}{(d+1) + (d-1)z^d}. \quad (5.1)$$

One can check that $|F'_d(z)| > 1$ on $S^1 - \{1\}$. Thus $f_d = F_d|_{S^1}$ is geometrically finite, of type $(d, 1)$, with $P(f_d) = \{1\}$. Its unique parabolic point has exponent $e = 3$.

This map arises in connection with cauliflower Julia sets in the family $z^d + c$, such as $J(z^2 + 1/4)$ when $d = 2$.

4. *Continued fractions.* An example with a breakpoint is provided by the piecewise Möbius map $f : S^1 \rightarrow S^1$ defined by equation (2.4). Here f is geometrically finite of type $(2, 1)$, with $B(f) = \{0, 1/2\}$, $P(f) = \{0\}$, and $f(1/2) = 0$. The metric $\rho = |dx|$ is expanded by f on $S^1 - P(f)$, since $f'(x) > 1$ except at the parabolic fixed point of f (see Figure 8). Since $?f?^{-1} = \chi_2$, we can regard $?(x)$ as the unique *marking* of f .
5. *Reflections in a hyperbolic polygon.* Let $Q \subset \bar{\Delta}$ be an ideal hyperbolic polygon with $n \geq 3$ vertices $V \subset S^1$ and with 0 in the interior of Q . Each edge e of Q determines a hyperbolic reflection $\rho_E : \bar{\Delta} \rightarrow \bar{\Delta}$. Define $f : S^1 \rightarrow S^1$ by

$$f(z) = \rho_e(z) \quad \text{if the radial line from 0 to } z \text{ passes through } e.$$

This map is only piecewise smooth. Its fixed points coincide with $B(f) = P(f) = V$, and every parabolic point has exponent $e = 2$. The condition that e separates 0 from z implies $|f'(z)| > 1$ on $S^1 - P(f)$, and thus f is geometrically finite, of type $((1 - n), n)$.

The case $n = 3$ arises in connection with the deltoid, and $n > 3$ in connection with other domains of quadrature; see §8.

6. *Piecewise linear maps.* Let $f : S^1 \rightarrow S^1$ be a piecewise-linear covering map such that $f'(x) > 1$ where it is defined, and every branch point is pre-fixed. If f is C^1 at every fixed point, then f is geometrically finite.
7. *A counterexample.* On the other hand, the degree 3 covering map $f : S^1 \rightarrow S^1$ defined by

$$f(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \text{ and} \\ 4x & \text{for } x \in [-1/2, 0] \end{cases}$$

is *not* geometrically finite. Indeed, the ratio 2^n between the right and left derivatives of f^n at $x = 0$ grows without bound, so the iterates of f are not uniformly quasimetric.

Nonlinearity. The *nonlinearity* of a diffeomorphism between intervals, $f : I \rightarrow J$, is the 1-form defined by

$$Nf = d(\log |f'|) = f''(x)/f'(x) dx.$$

It satisfies the cocycle condition

$$N(f \circ g) = Ng + g^*(Nf). \quad (5.2)$$

Note that $Nf = 0 \iff f(x) = ax + b$ is linear. The quantity $|Nf|$ is naturally a smooth measure on I , and its total mass

$$\eta = \eta(I, f) = \int_I |Nf|$$

controls the variation of $f'(x)$; namely for any $x, y \in I$ we have

$$e^{-\eta} \leq |f'(x)|/|f'(y)| \leq e^\eta.$$

Thus for any pair of intervals $A, B \subset I$ we have

$$e^{-\eta}|A|/|B| \leq |f(A)|/|f(B)| \leq e^\eta|A|/|B|, \quad (5.3)$$

and therefore:

The map $f|I$ is k -quasisymmetric, where $k = \exp(\eta(I, f))$.

Dynamics. Now let $f : S^1 \rightarrow S^1$ be a piecewise diffeomorphism. Then $|Nf| = O(|dx|)$ where defined (see equation 2.1), and thus

$$\eta(I, f) = O(|I|) \text{ provided } I \cap B(f) = \emptyset.$$

Using the cocycle condition (5.2), we find similarly that for a composition

$$I_n \xrightarrow{f} I_{n-1} \xrightarrow{f} I_{n-2} \xrightarrow{f} \cdots \xrightarrow{f} I_0,$$

with all intervals disjoint from $B(f)$, the nonlinearity of the map $f^n : I_0 \rightarrow I_n$ satisfies

$$\eta(I_n, f^n) = O\left(\sum_{k=1}^n |I_k|\right). \quad (5.4)$$

This fact combined with expansion leads to control of large iterates of f .

Proof of Theorem 5.1. Let $f : S^1 \rightarrow S^1$ be a covering map of degree d , $|d| > 1$, satisfying the stated conditions. Let $|I|_\rho = \int_I \rho$ denote the length

of I in the expanding metric ρ on $S^1 - P(f)$. It is easy to verify that f is topologically expanding, since f expands ρ and its parabolic points are topologically repelling. Our main task is to show large iterates of f are uniformly quasimetric.

First suppose that $B(f) = P(f) = \emptyset$. Then by compactness, we have uniform expansion:

$$|f(I)|_\rho \geq \lambda \cdot |I|_\rho$$

for some $\lambda > 1$ depending only on f . Moreover, we have $|I| \asymp |I|_\rho$. Now suppose $f^n : I_n \rightarrow I_1$ is injective, and $I_k = f^{n-k}(I_n)$. Then

$$\sum_1^n |I_k| \asymp \sum_1^n |I_k|_\rho \leq \sum_1^n \lambda^{-k} |I_0|_\rho \leq \left(\sum_0^\infty \lambda^{-k} \right) |I_0|_\rho = O(1),$$

since the geometric series converges and $|I_0|_\rho \leq |S^1|_\rho$. It follows as in equation (5.4) that $\eta = \eta(I_n, f^n) = O(1)$ and hence $f^n|_{I_n}$ is k -quasimetric, where $k = e^\eta = O(1)$. Thus f is geometrically finite.

Continuing under the assumption $B(f) = \emptyset$, now suppose $P(f) = \{p\}$ and $\deg(f) > 0$. Then $f(p) = p$. Let (I_p, R_p) be the ruler for p (defined as in equation (3.7)), with

$$I_p = [a_0, b_0] \quad \text{and} \quad R_p = \{a_0, a_1, a_2, \dots, p, \dots, b_2, b_1, b_0\}$$

(so $f(a_i) = a_{i+1}$ and similarly for (b_i)). Let $A_n = [a_n, a_{n+1}]$ and let $B_n = [b_{n+1}, b_n]$. (Note we have shifted the initial index to 0.)

Our main task is to control the dynamics on small intervals that pass close to p . We first remark that for each $n, k \geq 0$, the map $f^k : A_{n+k} \rightarrow A_n$ has bounded nonlinearity, since the intervals $(A_0, A_1, A_2, \dots)$ are disjoint; that is, we have

$$\eta(A_{n+k}, f^k) = O\left(\sum_0^\infty |A_i|\right) = O(|I_p|) = O(1).$$

Consider an interval $J \subset I_p$ disjoint from R_p . Then there is a unique n such that $J \subset A_n$ or $J \subset B_n$; by symmetry we may assume we are in the former case. Let $J_n = J$ and let $J_k = f^{n-k}(J_n) \subset A_k$, $k = 0, 1, \dots, n$. Since f^k sends the pair of intervals (J_k, A_k) to (J_0, A_0) with bounded nonlinearity, equation (5.3) implies that

$$|J_k|/|A_k| \asymp |J_0|/|A_0|$$

for all k between 0 and n . It follows that the map $f^n : J_n \rightarrow J_0$ has nonlinearity controlled by $|J_0|$, since

$$\eta(J_n, f^n) = O\left(\sum_0^n |J_k|\right) = O\left(|J_0/A_0| \cdot \sum_0^\infty |A_k|\right) = O(|J_0|). \quad (5.5)$$

Similarly reasoning leads to the same conclusion when $|J \cap R_p| < 3$. In this case, we can assume that $J \subset A_n \cup A_{n+1} \cup A_{n+2}$ for some n ; then $J_0 = f^n(J)$ satisfies

$$J_0 \cap [a_3, b_3] = \emptyset. \quad (5.6)$$

Now let $J \subset S^1$ be an arbitrary interval. Let $J_n = J$ and $J_k = f^{n-k}(J_n)$ for $0 \leq k \leq n$. Suppose that (i) $f^n|_{J_n}$ is injective and (ii) none of the intervals J_k is controlled by the thin part of f . Then each time J_k lands in I_p , equation (5.5) shows that the total nonlinearity accumulated during its sojourn in I_p is bounded by its length at the moment it exits. In view of equation (5.6), it follows that

$$\eta(J, f^n) = O\left(\sum \{|J_k| : J_k \cap [a_3, b_3] = \emptyset\}\right).$$

On the other hand, f is *uniformly* expanding for the metric ρ when restricted to the compact set $S^1 - (a_3, b_3)$. Thus the sum above is uniformly bounded by a geometric series, just as in the case $P(f) = \emptyset$. As before, this shows $f^n|_J$ is k -quasisymmetric, and hence f is geometrically finite.

This argument can be easily modified to handle any finite number of parabolic periodic cycles.

Finally we treat the case where $B(f)$ is nonempty. Consider any interval $J \subset S^1$ and integer $n \geq 0$ such that $f^n|_J$ is injective. As usual let $J = J_n$ and $J_k = f^{n-k}(J_n)$ for $0 \leq k \leq n$. Suppose that for $k < n$, J_k is not controlled by the thin part of f and J_k does not meet $B(f)$. Then by the preceding arguments, $\eta(J, f^n) = O(1)$, and $f^n : J_n \rightarrow J_0$ is uniformly quasisymmetric.

It may happen, however, that J_n itself meets $B(f)$. We then need to show that under further iteration of f , either J_n blows up to definite size with bounded distortion, or it encounters the thin part of f . It is here that conditions (1) and (2) on f come in to play. We may assume that $|J_n|$ is small, and thus it contains a unique break point $b \in B(f)$. By (1) the forward orbit of b lands on a hyperbolic or parabolic point p . In the parabolic case, this means the forward orbit of J_n encounters the thin part of f , while in the hyperbolic case, condition (2) on balanced derivatives insures that under further iteration, J_n blows up to definite size with bounded distortion.

Thus f is geometrically finite. ■

1. Condition (1) in the statement of Theorem 5.1 can be compared to the definition of a postcritically finite rational map, with $B(f)$ playing the role of critical points.
2. In condition (4), $P(f)$ can be replaced by a set $Q \subset S^1$ such that $f(Q) \subset Q$ and $f^n(Q) = P(f)$ for some n .
3. The requirement that f is piecewise C^∞ can be relaxed to piecewise C^2 , and even to $C^{1+\epsilon}$ when f has no parabolic points. However, the argument requires smoothness close to C^2 when there are parabolic points of exponent $e \gg 1$.
4. See [SS], [McS, Theorem 2.9] and the discussion of nonlinearity in [Sul, §2] for related arguments (without parabolics).

6 Construction of Minkowski pairs

In this section we prove Theorem 1.5 on the universality of [?].

Minkowski pairs. Fix $d \geq 2$. We say $(f, g) \in P_d S \times P_d A$ is a *Minkowski pair* if f and g are geometrically finite and $g = ?f?^{-1}$.

Following §4, we say (f, g) has type (d, n) if f has type (d, n) ; that is, if $\deg(f) = d$, $P(f) \subset \text{Fix}(f)$ and $|P(f)| = n$. The map g always has type $(d, 0)$.

Theorem 6.1 *For any nonempty set $Q \subset \text{Fix}(\chi_d)$, there exists a Minkowski pair (f, g) and a marking ϕ for f such that $\phi(P(f)) = Q$.*

Corollary 6.2 *There exists a Minkowski pair of type (d, n) for every possible type with $n, d > 0$.*

Templates. Let $(B_1, \dots, B_n)$ and $(A_1, \dots, A_{dn})$ be two cyclically ordered tilings of S^1 by Farey intervals. Extend the indexing to $\mathbb{Z}$, so that $B_{i+n} = B_i$. There is then a unique $f \in P_d S$ such that $B(f) \subset \bigcup \partial A_i$ and $f(A_i) = B_i$. The map $f|_{A_i}$ agrees with the unique element of S sending A_i to B_i .

The tilings (A_i) and (B_i) form a *Farey template of type (d, n)* if:

1. Every interval A_i is a proper subset of some B_j ;
2. The level k of A_i is strictly less than the level of $B_i = f(A_i)$; and
3. Every point in $\bigcup \partial B_i$ is fixed by f .

Clearly $B(f)$ and $f(B(f))$ are contained in $\bigcup \partial A_i$ and $\bigcup \partial B_i$ respectively.

A dyadic template for a map $g \in P_d A$ is defined similarly, and $?(x)$ gives a bijection between the two types of templates. In the dyadic case, the level condition just means $f'(x) > 1$ where defined.

Lemma 6.3 *The map f associated to a Farey template is geometrically finite, with $P(f) = \bigcup_1^n \partial B_i \subset \text{Fix}(f)$.*

Proof. Since endpoints of the Farey intervals (B_i) are rational fixed points of f , their union $Q = \bigcup_1^n \partial B_i$ is contained in $P(f)$. Moreover the image of $B(f)$ lies in Q . To complete the proof, it suffices to show that f expands the *hyperbolic metric* ρ on $S^1 - Q$ (and thus $Q = P(f)$).

In general, on the interior of an interval $I = [a, b]$, the hyperbolic metric is given by

$$\rho_I = \frac{|a - b| |dx|}{(x - a)(x - b)}.$$

It has two key properties: any proper inclusion $I \subset J$ is a hyperbolic contraction, and any Möbius transformation $g : I \rightarrow J$ is a hyperbolic isometry.

By definition, the hyperbolic metric ρ on $S^1 - Q$ is given by ρ_{B_i} on the interior of each interval B_i . To see it is expanded, observe that $f : A_i \rightarrow B_i$ can be factored as the composition of the *inverse* of the (contracting) inclusion $A_i \subset B_j$ with the (isometric) Möbius transformation $f : A_i \rightarrow B_i$. ■

Lemma 6.4 *The map g associated to a dyadic template is geometrically finite if and only if*

$$\frac{|B_i|}{|A_j|} = \frac{|B_{i+1}|}{|A_{j+1}|} \tag{6.1}$$

whenever all 4 intervals above meet at a single point p .

Proof. By condition (2) in definition of a template, we have $|B_i| > |A_i|$ for all i . Thus g expands the metric $\rho = |dx|$, and $P(g)$ is empty. Every periodic point $p \in B(g)$ is actually a fixed point, and the condition above says that the right and left derivatives of g agree at p . ■

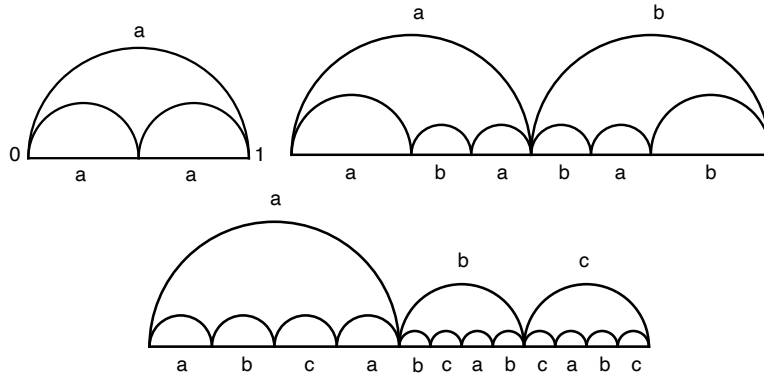


Figure 10. Dyadic templates of types $(2, 1)$, $(3, 2)$ and $(4, 3)$.

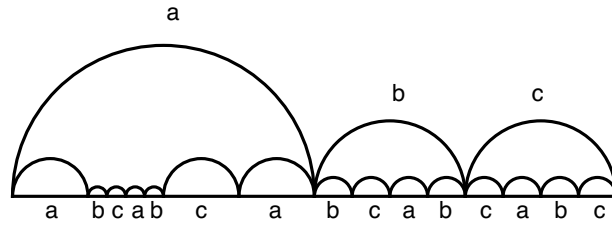


Figure 11. A template of type $(5, 3)$.

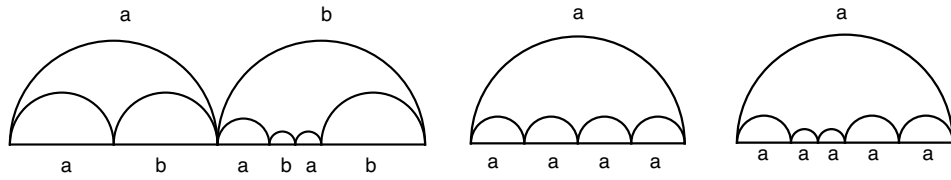


Figure 12. Realizing Minkowski pairs of type $(d, 1)$, for $d = 3, 4$ and 5 .

Parabolic points. Note that every geometrically finite covering map $f \in PS_d$ has a parabolic point. Indeed, there must be at least one break point b , which lies in $\mathbb{Q}/\mathbb{Z}$; its forward orbit must be finite, and lie entirely in $\mathbb{Q}$; and any periodic rational point must be parabolic.

Proof of Theorem 6.1. To begin, suppose $n = |Q| = d - 1$. Write $n = 2^k + m$ with $0 \leq m < 2^k$. Let $(B_i)_1^n$ be the unique dyadic tiling of $[0, 1]$ obtained by starting with 2^k tiles of equal length, and then cutting the last m of them in half.

Next, further subdivide each B_i into d dyadic subintervals, to obtain a finer tiling $(A_i)_1^{dn}$. One can again arrange that intervals of only two lengths occur in this tiling, and that $|A_i| < |B_j|$ for all i, j .

In addition, one can arrange that the balancing condition in Lemma 6.4 holds. This is especially easy when $d \geq 4$: for each $p \in \bigcup \partial B_i$, first choose the intervals A_j and A_{j+1} adjacent to p so equation (6.1) holds, and then fill in the rest of the intervals afterwards.

Finally, identifying the endpoints of $[0, 1]$, we can regard (A_i) and (B_i) as a template on S^1 . Its associated dyadic map g is geometrically finite by Lemma 6.4, as is the map $f = g^{-1}g$ by Lemma 6.3. By construction, $P(f) = \text{Fix}(f)$ and hence (f, g) is a Minkowski pair of type $(d, d - 1)$.

Explicit examples for $d = 2, 3$ and 4 are shown in Figure 10. The diameters of the large circles labeled $a, b, c, \dots$ are the intervals (B_i) ; the smaller circles below them indicate the finer subdivision (A_i) . The labels a, b, c below each A_i designate the interval B_i that it maps to.

By Corollary 4.5, this construction handles the case where $n = |Q| = d - 1$. To handle the case $n = d - 2$, start with a template of type $(d - 1, d - 2)$, and then refine (A_i) by dividing A_2 into $(d - 1)$ dyadic subintervals. The result is a new template $(A'_i), (B_i)$ of type $(d, d - 2)$, with a new, non-parabolic fixed point introduced in the interval A_2 . Provided A_2 does not meet ∂B_1 , the balancing condition (6.1) continues to hold, and we obtain a Minkowski pair (f, g) of type $(d, d - 2)$.

As an example, Figure 11 shows how to modify the template of type $(4, 3)$ in Figure 10 to obtain the required template of type $(5, 3)$.

By repeating this construction, and varying the location of the newly inserted fixed points, one can produce from a template of degree $(d, d - 1)$ a Minkowski pair of type $(e, d - 1)$ realizing any given $Q \subset \text{Fix}(\chi_e)$ with $|Q| = d - 1$.

The only caveat is that the construction cannot begin if A_2 meets ∂B_1 . This only happens when $(d, d - 1) = (2, 1)$; equivalently, when $n = |Q| = 1$. But the case $(e, 1)$ is easily handled directly for all $e \geq 2$; see Figure 12.

Note that the case $e = 3$ is not quite based on a template (since $x = 1/2$ is not fixed); to see it produces a Minkowski pair (f, g) , check that f expands $\rho = |dx|$ and use Theorem 5.1 $\blacksquare$

A prime example. The template of type $(2, 1)$ shown in Figure 10 gives the simplest and most important example of a Minkowski pair: the pair (f, χ_2) discussed in §2, where f is given by equation (2.4).

Proof of Theorem 1.5. Let ϕ be a topological conjugacy between a pair of geometrically finite maps f and g of degree d . Suppose f has at least one parabolic cycle, but g has none. Replacing f and g with suitable iterates, we can assume that $d > 0$ and that $P(f) \subset \text{Fix}(f)$.

By Theorem 6.1, there exists a Minkowski pair (f_1, g_1) and a topological conjugacy α from f to f_1 such that $\alpha(P(f)) = P(f_1)$. Then $\beta = ? \circ \alpha \circ \phi^{-1}$ is a conjugacy between g and g_1 making the diagram

$$\begin{array}{ccc} f \circ S^1 & \xrightarrow{\phi} & S^1 \circ g \\ \alpha \downarrow & & \downarrow \beta \\ f_1 \circ S^1 & \xrightarrow{?} & S^1 \circ g_1 \end{array}$$

commute. By Theorem 1.4 we have $\alpha, \beta \in \text{QS}$, and hence ϕ and $?$ determine the same double coset in the moduli space $\mathcal{M} = \text{QS} \setminus \text{Homeo}^+(S^1) / \text{QS}$. $\blacksquare$

7 Welding and Julia sets of polynomials

In this section we prove Theorem 1.1, stating that $[W(?)] = [J(z^d + c)]$ when the latter Julia set contains a parabolic fixed point. We also prove Theorem 1.8 on $\text{Aut}(?)$.

We begin by showing:

Theorem 7.1 *The map $?(x)$ is a strong welding homeomorphism.*

Theorem 7.2 *The Jordan curve $W(?)$ is quasiconformally equivalent to the cauliflower Julia set $J(z^2 + 1/4)$.*

The generalization to $J(z^d + c)$ is given at the end of this section.

Corollary 7.3 *The curve $W(?)$ bounds a John domain.*

Welding. Let Δ_0 and Δ_∞ denote the two components of $\widehat{\mathbb{C}} - S^1$, with $0 \in \Delta_0$ and $\infty \in \Delta_\infty$. A homeomorphism $\omega : S^1 \rightarrow S^1$ is a *welding map* if it arises from an oriented Jordan curve $J \subset \widehat{\mathbb{C}}$ by forming the difference

$$\omega = f_\infty^{-1} \circ f_0$$

between the Riemann mappings

$$f_0 : \Delta_0 \rightarrow \Omega_0 \quad \text{and} \quad f_\infty : \Delta_\infty \rightarrow \Omega_\infty$$

to the two components of $\widehat{\mathbb{C}} - J$ (indexed using the orientation). One can regard J as the common boundary of two round disks that have been welded together by ω to produce the Riemann sphere $\widehat{\mathbb{C}}$.

Since the Riemann mappings can be normalized in different ways, the set of all welding maps arising from J forms a double coset

$$\Omega(J) = \text{Möb} \cdot \omega \cdot \text{Möb} \subset \text{Homeo}^+(S^1).$$

More generally we have:

$$\text{QS} \cdot \Omega(J) \cdot \text{QS} = \bigcup_{\phi \in \text{QC}(\widehat{\mathbb{C}})} \Omega(\phi(J)); \quad (7.1)$$

compare [Pf]. Recalling from §1 that

$$\mathcal{J} = \{\text{Jordan curves } J \subset \mathbb{C}\} / \text{QC}(\mathbb{C}),$$

and noting each $J \subset \mathbb{C}$ has a canonical orientation, we obtain from equation (7.1) a natural map

$$\mathcal{J} \rightarrow \mathcal{M} = \text{QS} \setminus \text{Homeo}^+(S^1) / \text{QS}.$$

Strong welding. A closed set $E \subset \widehat{\mathbb{C}}$ is *conformally removable* if any homeomorphism of $\widehat{\mathbb{C}}$ that is holomorphic on $\widehat{\mathbb{C}} - E$ is holomorphic on $\widehat{\mathbb{C}}$.

We say ω is a *strong* welding homeomorphism if it arises from a conformally removable Jordan curve. In this case we let $J = W(\omega)$ denote the Jordan curve giving rise to ω ; it is unique up to a Möbius transformation.

The set of strong welding maps is a union of QS double cosets. It is easy to check:

If ω is a strong welding homeomorphism, then there is a unique point $[J] \in \mathcal{J}$ mapping to $[\omega] \in \mathcal{M}$.

A useful criterion for strong welding is provided by [Jones]:

If J bounds a John domain, then every $\omega \in \Omega(J)$ is a strong welding map.

We remark that a Jordan curve bounds a John domain iff it satisfies a 1-sided bounded turning condition [GHM, Theorem 4.4].

Normalized welding. Let $\mathcal{J}^*$ denote the set of Jordan curves J in $\widehat{\mathbb{C}}$ separating 0 from ∞ . In this case we can make the Riemann mappings f_0 and f_∞ *unique* by requiring that they fix 0 and ∞ respectively, and have positive derivatives there. Then J determines a canonical welding map

$$\omega_J = f_\infty^{-1} \circ f_0 : S^1 \rightarrow S^1. \quad (7.2)$$

Here we identify $S^1 = \mathbb{R}/\mathbb{Z}$ with $S^1 \subset \mathbb{C}$ via the change of coordinates $z = \exp(2\pi i x)$. Note that $\omega_J = \omega_{tJ}$ for all $t > 0$.

Proof of Theorems 7.1 and 7.2. Let $P(z) = z^2 + 1/4$ and let $J = J(P)$. The map P has a parabolic fixed point at $z = 1/2$, whose basin of attraction Ω is a topological disk bounded by J . The region Ω is a John domain by [CJK, Theorem 6.1]. In particular, J is a Jordan curve and $\omega = \omega_J$ is a strong welding homeomorphism.

Let $F_2(z) = (3z^2 + 1)/(z^2 + 3)$ be the parabolic Blaschke product appearing in equation (5.1). The map $P|_\Omega$ is holomorphically conjugate to $F_2|_\Delta$, while on $\widehat{\mathbb{C}} - \overline{\Omega}$, $P(z)$ is conjugate to z^2 . Since these two dynamical systems must agree along J , we have

$$\omega f_2 \omega^{-1} = \chi_2,$$

where $f_2 = F_2|_{S^1}$. Note that f_2 is geometrically finite of type $(2, 1)$.

Now consider the basic Minkowski pair (f, χ_2) of type $(2, 1)$ discussed in §2, with $f(x)$ defined by equation (2.4). On the one hand, we have

$$?f?^{-1} = \chi_2$$

by the definition of a Minkowski pair (and as shown in §2); on the other hand, f_2 and f are quasisymmetrically conjugate by Corollary 4.5; say

$$\phi f_2 \phi^{-1} = f.$$

Since a topological conjugacy between covering maps is unique in degree 2, it follows that

$$\omega = ? \circ \phi. \quad (7.3)$$

Consequently $[?] = [\omega]$ in $\mathcal{M}$ and $[W(?)] = [W(\omega)] = [J(z^2 + 1/4)]$ in $\mathcal{J}$. In particular $?(x)$, like $\omega(x)$, is a strong welding map. ■

Proof of Theorem 1.1. Using the Blaschke product $F_d(z)$ from equation 5.1), and [CJK] again, the same argument shows that $[J(z^d + c)] = [W(?)]$ in $\mathcal{J}$ whenever $z^d + c$ has a parabolic fixed point. ■

Proof of Theorem 1.8 and Corollary 1.9. We wish to show that $\text{Aut}(?)$ preserves $\mathbb{Q}/\mathbb{Z} \subset S^1$, or equivalently that $\text{Aut}(J(z^2 + 1/2))$ preserves cusps.

Proof 1. Given $x \in S^1$ let

$$\delta(x) = \limsup_{t \rightarrow 0} \frac{?(x+t) - ?(x)}{?(x+2t) - ?(x)}.$$

We claim $\delta(x) = 0 \iff x \in \mathbb{Q}/\mathbb{Z}$. Indeed, if x is rational then all derivatives of $?$ vanish at x and hence $\delta(x) = 0$. On the other hand, if x is irrational then x belongs to infinitely many Farey intervals $[a, b]$ with $\text{mid}_S[a, b]$ approximately centered in $[a, b]$, which readily implies $\delta(x) > 0$.

For any (quasisymmetric) $\phi \in \text{Aut}(?)$ we have $\delta(x) \asymp \delta(\phi(x))$; hence ϕ preserves $\mathbb{Q}/\mathbb{Z}$.

Proof 2. Let $P(z) = z^2 + 1/4$ and $J = J(P)$. For each $z \in J$ let $K(z)$ denote the set of all possible limits, in the Hausdorff topology on closed subsets of $\mathbb{C}$, of the blowups $(1/r)(J - z)$ as $r \rightarrow 0$. Each $F \in K(z)$ is a closed set containing $z = 0$.

At the parabolic fixed point p of $P(z)$, $K(p)$ consists of a single point, namely the ray $[0, \infty) \subset \mathbb{C}$. Since $P|_J$ is locally injective, we also have $K(P(z)) = P'(z) \cdot K(z)$ for all $z \in J$. It follows that $K(z)$ consists of a single ray based at the origin whenever z is a cusp (an element of the inverse orbit of p).

On the other hand, if z is not a cusp, then have $d(P^n(z), \mathbb{R}) \asymp 1$ for infinitely many n . Since the postcritical set of P is contained in the real axis, for each such n we obtain — pulling back by a univalent branch of P^{-n} — an arc $J_n \subset J$ through z which is, up to scaling, a mildly distorted copy of an arc of J through $P^n(z)$ of definite size. Moreover $\text{diam}(J_n) \rightarrow 0$. Thus at least one limit $F \in K(z)$ contains an arc through the origin.

Now suppose $\phi \in \text{Aut}(J)$. By compactness of quasiconformal maps, for any $z \in J$ and $F \in K(z)$, there is a set $F' \in K(\phi(z))$ related to F by a homeomorphism of $\mathbb{C}$ fixing the origin. Since the sets appearing in $K(z)$ at cusps and non-cusps are topologically distinct, the map ϕ must send cusps to cusps. ■

8 Dynamics from hypocycloids

In this section we prove Theorem 1.2, stating that the Julia sets J_n associated to hypocycloids, as well as their quotients $J_{n,k}$, satisfy

$$[J_n] = [J_{n,k}] = [W(?)] \text{ in } \mathcal{J}.$$

To do so we pass from the study of polynomials to iterated relations $R(x, y)$.

Relations. A dynamical *relation* (or *correspondence*) R on the Riemann sphere is specified by an algebraic curve

$$R \subset \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$$

defined in coordinates by a polynomial $R(x, y) = 0$. We assume that $R(x, y)$ is *separable* (a product of distinct irreducible polynomials), and R has no vertical or horizontal components. The *inverse* relation is defined by $R^{-1}(x, y) = R(y, x)$; and R is *symmetric* if $R = R^{-1}$.

The *bidegree* (d_1, d_2) of R gives the degree of $R(x, y)$ in y and x respectively. In $H_2(\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}, \mathbb{Z})$ we have

$$[R] = d_1 \cdot [\widehat{\mathbb{C}} \times \{0\}] + d_2 \cdot [\{0\} \times \widehat{\mathbb{C}}].$$

For example, the graph of a polynomial, $y = P(x)$, has degree $(1, \deg P)$.

A relation gives a set-valued map defined by

$$R(x) = \{y : R(x, y) = 0\},$$

and more generally by $R(A) = \bigcup \{R(x) : x \in A\}$. For generic x and y , we have $|R(x)| = d_2$ and $|R^{-1}(y)| = d_1$.

For our purposes below, we define the *Julia set* $J(R)$ to be the smallest non-elementary, R -invariant subset of $\widehat{\mathbb{C}}$. This means $|J| \geq 3$ and $R(J) = R^{-1}(J) = J$.

Hypocycloids. Consider, for $n \geq 3$, the rational map of degree n defined by:

$$\Phi_n(z) = z \left(1 + \frac{z^{-n}}{n-1} \right).$$

This map has simple critical points at the n th roots of unity, and a critical point of multiplicity $n-2$ at $z = 0$. It defines a univalent map

$$\Phi_n : (\widehat{\mathbb{C}} - \overline{\Delta}) \rightarrow (\widehat{\mathbb{C}} - P_n),$$

where ∂P_n is a hypocycloid with n cusps. (The curve ∂P_n gives the trace of a point on a circle of radius r as it rolls along inside a circle of radius nr .) The cusps of P_n lie at the critical values $\Phi_n(\zeta_n^i)$, $i = 1, 2, \dots, n$.

Dynamics. We let J_n denote the Julia set of the symmetric, degree (n, n) relation defined by

$$R_n(x, y) = \{(x, y) : \Phi_n(z) = x \text{ and } \Phi_n(1/z) = y \text{ for some } z \in \widehat{\mathbb{C}}\}; \quad (8.1)$$

equivalently, by $R_n(x) = \Phi_n(1/\Phi_n^{-1}(x))$. The locus J_n is invariant under $z \mapsto \zeta_n z$, and for any $k|n$ the quotient

$$J_{n,k} = J_n/(\mathbb{Z}/k) = \{z^k : z \in J_n\}$$

is the Julia set of the relation $R_{n,k}$ associated, as in equation (8.1), to the rational map

$$\Phi_{n,k}(z) = z \left(1 + \frac{z^{-n/k}}{n-1} \right)^k.$$

This map is characterized by $\Phi_n(z^{1/k})^k = \Phi_{n,k}(z)$.

Main properties. For each $n \geq 3$:

1. The Julia set J_n is a Jordan curve, bounding a John domain Ω_n in $\mathbb{C}$.
2. Let $Q_n \subset \overline{\Delta}$ be the ideal polygon with vertices the n th roots of unity. The normalized Riemann mapping

$$f_0 : \Delta \rightarrow \Omega_n$$

sends Q_n to the hypocycloid $P_n \subset \overline{\Omega}_n$.

3. The relation $R_n(x, y)|_{\Omega_n}$ corresponds to the relation $\prod (x - \rho_e(\bar{y}))$ on Δ , where e ranges over the n edges of Q_n and ρ_e denotes reflection through e .
4. Let $m = n - 1$. Under the outer Riemann mapping

$$f_\infty : (\widehat{\mathbb{C}} - \overline{\Delta}) \rightarrow (\widehat{\mathbb{C}} - \overline{\Omega}),$$

the relation $R_n(x, y)$ on its target corresponds to the relation $(x^m - y)(y^m - x)$ on its domain.

See [LLMM] for details. In the proof below, the most essential point is that $J = J_n$ is a *Jordan curve*; it then has a well-defined welding map ω_J , which can be compared to $?(x)$.

Proof of Theorem 1.2. Define $f, g : S^1 \rightarrow S^1$ by

$$f(z) = \rho_e(\bar{z}),$$

where e is the edge of Q_n separating $\bar{z}$ from 0, and by $g(z) = z^{n-1}$. The maps f and g are geometrically finite, with $P(f)$ equal to the n th roots of unity and $P(g) = \emptyset$. In view of the properties above of (J_n, R_n) above, the welding map ω_J gives a topological conjugacy between f and g . Thus $[\omega_J] = [?]$ in $\mathcal{M}$ by Theorem 1.5, and hence $[J] = [W(?)]$ in $\mathcal{J}$. Similar reasoning shows, more generally, $[J_{n,k}] = [W(?)]$ whenever $k|n$. ■

Remarks.

1. A topological disk $U \subset \widehat{\mathbb{C}}$ is a *quadrature domain* if the Riemann mapping $f : \Delta \rightarrow U$ extends to a rational function on $\widehat{\mathbb{C}}$. The region $\widehat{\mathbb{C}} - P_n$ is a quadrature domain, since $\Phi_n(z)$ is rational; so is the image of the outside of the unit disk under $\Phi_{n,k}(z)$.
2. Note the close similarity between the Farey tiling of Δ^* in Figure 7 and the corresponding tiling of $\Omega_{3,3}$ in Figure 3.
3. The antiholomorphic relation $R_n(x, \bar{y}) = 0$ has the same Julia set as R_n ; it gives a mating between reflections through the sides Q_n and the map $z \mapsto \bar{z}^n$.

9 $\mathrm{SL}_2(\mathbb{Z})$ and the deltoid

In this section we study the unique extension of $?[0, 1]$ to a map

$$\widehat{?} : \widehat{\mathbb{R}} \rightarrow S^1(3) = \mathbb{R}/3\mathbb{Z}$$

satisfying

$$\widehat{?}(R(x)) = 1 + \widehat{?}(x) \bmod 3, \tag{9.1}$$

where $R(x) = 1/(1-x)$. Note that R has order 3 and $[0, 1]$ is a fundamental domain for its action on $\widehat{\mathbb{R}}$, so $\widehat{?}(x)$ is well-defined.

Let $\zeta = \zeta_3$, and define $e : S^1(3) \rightarrow \widehat{\mathbb{R}}$ by

$$e(t) = \frac{\zeta(\zeta^t - 1)}{\zeta^2 - \zeta^t}.$$

Note that e sends $(0, 1, 2)$ to $(0, 1, \infty)$. We will prove:

Theorem 9.1 *The welding map ω for the Julia set J_3 associated to the deltoid is given by*

$$\omega(t) = \widehat{?}(e(3t))/3.$$

Here $J = J_3 \in \mathcal{J}^*$ and $\omega = \omega_J$ is normalized as in equation (7.2).

Theorem 9.2 *The map $\widehat{?}$ sends $\mathrm{PSL}_2(\mathbb{Z})$ acting on $\widehat{\mathbb{R}}$ into the group $PA(3)$ of dyadic piecewise linear maps on $S^1(3)$.*

(The breakpoints of maps in $PA(3)$ are required to lie in $\mathbb{Z}[1/2]/(3\mathbb{Z})$).

Note that the factorization of ω in Theorem 9.1 corresponds to the composition:

$$S^1 \cong S^1(3) \xrightarrow{e} \widehat{\mathbb{R}} \xrightarrow{\widehat{?}} S^1(3) \cong S^1.$$

To see how these two results are related, recall that J_3 arises by mating the reflection group for an ideal triangle with the doubling map $z \mapsto \bar{z}^2$. Since the former group is commensurable to $\mathrm{SL}_2(\mathbb{Z})$, its welding map — which is given, up to a change of coordinates, by $\widehat{?}$ — sends the *global* dynamics of $\mathrm{SL}_2(\mathbb{Z})$ into dyadic dynamics on $S^1(3)$.

In the next section we will use $\widehat{?}$ to study $\mathrm{Aut}(?)$.

Characterization of ω . Using the change of coordinates $z = \exp(2\pi ix)$, we can regard $\omega : S^1 \rightarrow S^1$ as a function on the unit circle in $\mathbb{C}$. It is characterized by the properties:

- (i) $\omega(1) = 1$, and
- (ii) $\omega(\rho(z)) = \overline{\omega(z)}^2 = \omega(z)^{-2}$.

Here $\rho(z)$ is defined along the arc $[\zeta^i, \zeta^{i+1}] \subset S^1$ by reflection through the hyperbolic geodesic $E(\zeta^i, \zeta^{i+1}) \subset \widehat{\Delta}$. In view of the dihedral symmetry of the ideal triangle $T(1, \zeta, \zeta^2)$, we also have:

- (iii) $\omega(\zeta z) = \zeta \omega(z)$, and
- (iv) $\omega(\bar{z}) = \overline{\omega(z)}$.

In fact, if we assume (i), (iii) and (iv) hold, then (ii) follows from

$$(ii') \quad \omega(\rho(z)) = \omega(z)^2 \text{ for all } z \text{ in the arc } [1, \zeta_6].$$

Proof of Theorem 9.1. Consider the Möbius transformations

$$F(x) = 1 - x \quad \text{and} \quad T(x) = x/(1 - x)$$

acting on $\widehat{\mathbb{R}}$. Note that

$$\widehat{?}(F(x)) = 1 - \widehat{?}(x) = F(\widehat{?}(x)), \quad (9.2)$$

since $?(1 - x) = 1 - ?(x)$ on $?[0, 1]$; and that $FRT(x) = x/(2x - 1)$ is reflection through $E(0, 1)$.

Since ω is characterized by (i)–(iv) above, to prove Theorem 9.1 it suffices to check that the corresponding properties hold for $\widehat{?}(x)$; namely:

- (i) $\widehat{?}(0) = 0$,
- (ii') $\widehat{?}(FRT(x)) = -2\widehat{?}(x)$ for $x \in [0, 1/2]$,
- (iii) $\widehat{?}(R(z)) = 1 + \widehat{?}(z)$, and
- (iv) $\widehat{?}(F(x)) = F(\widehat{?}(x))$.

But (i) and (iii) are immediate from the definition of $\widehat{?}$, and we have just verified (iv). As for (ii'): by equation (2.5) we have

$$?(T(x)) = 2?(x) = 2\widehat{?}(x) \text{ for } x \in [0, 1/2];$$

hence equations (9.1) and (9.2) imply that

$$\widehat{?}(FRT(x)) = 1 - (1 + ?(T(x))) = -?(T(x)) = -2\widehat{?}(x)$$

as required. ■

Proof of Theorem 9.2. Let $I = [0, 1] \subset \widehat{\mathbb{R}}$. Given $f \in \text{PSL}_2(\mathbb{Z})$, subdivide $\widehat{\mathbb{R}}$ into finitely many intervals (J_k) such that each J_k is contained in I , $R(I)$ or $R^2(I)$, and the same is true for $f(J_k)$. To complete the proof, it suffices to show that $\widehat{?}$ conjugates $f|_{J_k}$ into $PA(3)$ for each k . By (9.1), this is true when $f = R$, so we can reduce to the case where both J_k and $f(J_k)$ lie in I . But $\widehat{?} = ?$ on I , so the desired conclusion follows from Theorem 2.1. ■

Examples. Under conjugation by $\widehat{?}$, the map $f(x) = x + 1$ in $\text{PSL}_2(\mathbb{Z})$ is sent to the unique g in $PA(3)$, with two linear branches, that sends $(0, 2, 3)$ to $(1, 2, 4) \bmod 3$. Similarly, for $f(x) = x/(1 + x)$, we obtain the map sending $(0, 2, 3)$ to $(0, 1, 3)$.

Comparison between ω and $?$. One might guess that $3\omega(x) = ?(3x)$, but this is false. The two maps nearly agree, because $|e(t) - t| < 0.2$ on $[0, 1]$; but they are fundamentally distinct, because the projective structures on $S^1 = \mathbb{R}/\mathbb{Z} \subset \mathbb{C}$ inherited from $\mathbb{R}$ and $\mathbb{C}$ are different.

10 Automorphisms and $\mathrm{PGL}_2^+(\mathbb{Q})$

In this section we prove Theorem 1.7. Let $\mathrm{PGL}_2^+(\mathbb{Q})$ denote the group of Möbius transformations $g : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$ of the form

$$g(x) = \frac{ax + b}{cx + d}, \text{ with } a, b, c, d \in \mathbb{Q} \text{ and } ad - bc > 0.$$

Our main result is:

Theorem 10.1 *The group $\mathrm{PGL}_2^+(\mathbb{Q})$ lies in $\mathrm{Aut}(\widehat{?})$.*

This means the map

$$\phi = \widehat{?} \circ g \circ \widehat{?}^{-1} : S^1(3) \rightarrow S^1(3)$$

is quasisymmetric for every $g \in \mathrm{PGL}_2^+(\mathbb{Q})$. Since $\widehat{?}|[0, 1] = ?$, we obtain:

Corollary 10.2 *For any $g \in \mathrm{PGL}_2^+(\mathbb{Q})$ and interval $I \subset [0, 1]$ such that $g(I)$ also lies in $[0, 1]$, the map $?(g|I)?^{-1}$ is quasisymmetric.*

Proof of Theorem 1.7. We have $\mathrm{Aut}(\widehat{?}) \cong \mathrm{Aut}(\omega) \cong \mathrm{Aut}(?)$ by Theorems 9.1 and 1.2, so $\mathrm{PGL}_2^+(\mathbb{Q})$ embeds in $\mathrm{Aut}(?)$ by Theorem 10.1. ■

Rational points and congruence groups. By Theorem 1.8, $\mathrm{Aut}(\widehat{?})$ stabilizes the set $\mathbb{Q} \cup \{\infty\} \subset \widehat{\mathbb{R}}$. Theorem 10.1 shows that, as a partial converse, any Möbius transformation stabilizing $\mathbb{Q} \cup \{\infty\}$ lies in $\mathrm{Aut}(\widehat{?})$.

For the proof we will use the fact that $\mathrm{PGL}_2^+(\mathbb{Q})$ is the commensurator of $\mathrm{PSL}_2(\mathbb{Z})$ in $\mathrm{PSL}_2(\mathbb{R})$. More precisely, for any $g \in \mathrm{PGL}_2^+(\mathbb{Q})$ there is a congruence group

$$\Gamma = \Gamma(n) = \{A \in \mathrm{PSL}_2(\mathbb{Z}) : A = I \bmod n\}$$

such that both

$$\Gamma \text{ and } \Gamma' = g\Gamma g^{-1}$$

are subgroups of finite index in $\mathrm{PSL}_2(\mathbb{Z})$. Conjugating by $\widehat{?}$, we obtain a pair of subgroups $L, L' \subset PA(3)$ that are topologically conjugate by

$$\phi = \widehat{?} \circ g \circ \widehat{?}^{-1}.$$

We will show that L and L' inherit a version of geometric finiteness from $\mathrm{SL}_2(\mathbb{Z})$, and use this to prove ϕ is quasisymmetric.

Fuchsian groups. To avoid going back and forth between $\overline{\mathbb{H}}$ and $\overline{\Delta}$, we will measure diameters, distances and norms of derivatives using the spherical metric

$$\rho = \frac{2|dz|}{1 + |z|^2}$$

on $\overline{\mathbb{H}}$.

Lemma 10.3 *Given a lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$, for every $x \in \widehat{\mathbb{R}}$ there exists a $g \in \Gamma$ such that either*

- (i) $|g'(x)|_\rho > 1$, or
- (ii) g is parabolic and $g(x) = x$.

Proof. Let $\pi : \mathbb{H} \rightarrow X = \mathbb{H}/\Gamma$ be the quotient map. Choose a compact set $K \subset \mathbb{H}$ such that $X - \pi(K)$ consists of horoball neighborhoods of cusps. Let $\gamma(s) \rightarrow x$ be the hyperbolic geodesic ray starting at $\gamma(0) = i$ and heading towards x . If $\pi \circ \gamma(s)$ tends to a cusp of X then we are in case (ii); otherwise, $\pi \circ \gamma(s)$ returns infinitely often to $\pi(K)$. In the latter case we can find $s_n \rightarrow \infty$ and $g_n \in \Gamma$ such that $g_n \cdot \gamma(s_n) \in K$. Let H_n be the horocycle resting on x and passing through $\gamma(s_n)$; then in the spherical metric, $\mathrm{diam}(H_n) \rightarrow 0$ while $\mathrm{diam}(g_n(H_n)) \asymp 1$. It follows that $|g'_n(x)|_\rho \rightarrow \infty$, since this quantity controls the expansion of H_n ; thus we are in case (i). ■

Making parabolics repelling. Let $U \subset \widehat{\mathbb{R}}$ be a small interval containing the fixed point x of a parabolic element $g \in \mathrm{PSL}_2(\mathbb{Z})$. Topologically, g is repelling on one side of x and attracting on the other. By replacing g with g^{-1} on one component of $U - \{x\}$, we obtain a piecewise Möbius, topologically repelling transformation $\bar{g} : U \rightarrow \widehat{\mathbb{R}}$.

Lemma 10.4 *The map $h = \widehat{?} \circ \bar{g} \circ \widehat{?}^{-1}$ is linear on a neighborhood of $y = \widehat{?}(x)$, with $h'(y) > 1$.*

Proof. There exists an integer $k \neq 0$ such that g translates the fan of Farey edges incident to x by k steps to the right. Thus $\widehat{?}g\widehat{?}^{-1}$ has a breakpoint at y ; its derivative is 2^{-k} from the right, and 2^k from the left. But then h itself is linear near y , with $h'(y) = 2^{|k|}$. ■

Proof of Theorem 10.1. Given $g \in \mathrm{PGL}_2^+(\mathbb{Q})$ let Γ, Γ' and ϕ be defined as above. Then by Lemma 10.3 and compactness, we can find finite collection (U_i, g_i) of open intervals in $\widehat{\mathbb{R}}$ and $g_i \in \Gamma$ such that for each i , either $|g'_i(x)|_\rho > 1$ on U_i , or g_i has a parabolic fixed point in U_i . In the latter case we define $\bar{g}_i$ as above so it is topologically expanding on U_i . Then the whole dynamical system (g_i, U_i) is topologically expanding, in the sense that any small interval $I \subset \widehat{\mathbb{R}}$ can be blown up to definite size by a finite composition of maps of the form $g_i|_{U_i}$ or $\bar{g}_i|_{U_i}$.

Conjugating by ϕ , we obtain a dynamical system (h_i, V_i) acting on $S^1(3)$. Here we set $h_i = \widehat{g}_i \widehat{g}^{-1}$ when g_i is parabolic. Conjugating (h_i, V_i) by ϕ , we obtain a similar piecewise linear dynamical system (h'_i, V'_i) .

By Theorem 9.2 we have $h_i, h'_i \in PA(3)$, since $\Gamma \cup \Gamma' \subset \mathrm{PSL}_2(\mathbb{Z})$. Using Lemma 10.4, we can assume the original U_i were chosen so that $h_i|_{V_i}$ and $h'_i|_{V'_i}$ are linear maps whenever g_i is parabolic.

When g_i is not parabolic, $h_i|_{V_i}$ need not be linear. Its break points, however, can only occur at dyadic rationals, and the dyadic rationals are the images, under $\widehat{g}$, of the cusps of Γ . Thus by adding more parabolic elements to the list (g_i, U_i) , and suitably shrinking or subdividing the domains of the original maps, these break points can be removed. Then $h_i|_{V_i}$ and $h'_i|_{V'_i}$ are simply *linear maps* for all i .

By topological expansion, any pair of adjacent intervals A, B in $S^1(3)$ of equal length can be blown up to definite size by a composition f of maps in the family $(h_i|_{V_i})$. Since all these maps are linear, f has no distortion; we have:

$$\frac{|f(A)|}{|f(B)|} = \frac{|A|}{|B|}.$$

A similar statement holds for (h'_i, V'_i) . Since ϕ distorts the lengths of intervals of definite size by only a bounded amount, it follows that $|\phi(A)| \asymp |\phi(B)|$, and hence ϕ is quasisymmetric. ■

Remark: On the continued fractions of x and nx . Let $g(x) = nx$, $n > 1$. Since $g \in \mathrm{PGL}_2^+(\mathbb{Q})$, the map $\phi(x) = \widehat{g} \widehat{g}^{-1}$ is quasisymmetric, and hence Hölder continuous, say of exponent α ; and for $x \in [0, 1/n]$ we have $\phi(\widehat{g}x) = \widehat{g}(nx)$.

This Hölder continuity describes how accurately the continued fraction expansion of nx can be predicted from that of x . More precisely, once the continued fractions have been converted to the binary numbers (see the end of §2), it means that the first $m + O(1)$ digits of $\widehat{g}(x)$ determine the first αm digits of $\widehat{g}(nx)$.

11 Welding is not closed under composition

In this section we prove Theorem 1.10, showing that an explicit composition of welding maps is not a welding.

Power laws. For convenience we will make a change of coordinates and study homeomorphisms not of S^1 , but of $\widehat{\mathbb{R}}$. Define $s : \widehat{\mathbb{R}} \rightarrow \widehat{\mathbb{R}}$ by

$$s(x) = x \text{ for } x < 0, \text{ and } s(x) = x^2 \text{ for } x \geq 0.$$

We begin with a new proof of:

Lemma 11.1 (Oikawa) *The map s is not a welding homeomorphism.*

Proof. Suppose s is a welding homeomorphism; then $\overline{\mathbb{H}} \cup_s (-\overline{\mathbb{H}}) \cong \widehat{\mathbb{C}}$, and hence

$$X = H \cup_s (-H) \cong \mathbb{C}^*,$$

where $H = \overline{\mathbb{H}} - \{0, \infty\}$. Since $s(x) = x$ for $x < 0$, we can also construct X by first gluing H to $-H$ using the identity map on the negative real axis, and then gluing by $s(x) = x^2$ on $(0, \infty)$.

Equivalently — taking the logarithms, with $\log(1) = 2\pi i$, — we find that X is isomorphic to S/g , where S is the strip

$$S = \{z : \operatorname{Im}(z) \in [2\pi, 4\pi]\}$$

and $g(z) = 2z$ glues the edges of S together. But it is easy to see that S/g is an annulus of finite modulus, not $\mathbb{C}^*$; in fact, S is fundamental domain for $\langle g \rangle$ acting on $\mathbb{H}$. In view of this contradiction, s itself is not a welding homeomorphism. ■

More generally, if $\omega(x) = -|x|^\alpha$ for $x < 0$ and $|x|^\beta$ for $x \geq 0$, then $\omega(x)$ is a welding homeomorphism $\iff \alpha = \beta$; see [Oi, Example 1]. This statement can also be proved as above, using two strips.

Local welding. The germ of a homeomorphism defined near $p \in \widehat{\mathbb{R}}$ is a *local welding map* if it can be extended to a global welding map on $\widehat{\mathbb{R}}$.

Corollary 11.2 *The germ of $s(x)$ at $x = 0$ is not a local welding map.*

Proof. By a standard argument, a piecewise quasymmetric map such as $s(x)$ gives a global welding $\iff$ it gives a local welding at each of its break points; and its behavior at the breakpoints 0 and ∞ is the same. ■

Lemma 11.3 *Suppose a homeomorphism $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is quasymmetric away from $x = 0$, and satisfies $\psi(\lambda x) = \lambda\psi(x)$ for some $\lambda > 1$. Then ψ is globally quasymmetric.*

Proof. Let $g(z) = \lambda z$. The map ψ covers a quasymmetric map $\bar{\psi}$ between two circles on the quotient torus $X = \mathbb{C}^*/\langle g \rangle$. By extending $\bar{\psi}$ to a quasiconformal automorphism of X , lifting back to $\mathbb{C}^*$, and completing across 0, we obtain a quasiconformal extension of ψ from $\widehat{\mathbb{R}}$ to $\widehat{\mathbb{C}}$. This extension property characterizes quasymmetric maps [LV, Ch. 6]. ■

Proof of Theorem 1.10. Recall that $h : S^1 \rightarrow S^1$ is a piecewise linear homeomorphism such that, when $|x|$ is small, $h(x) = x$ for $x < 0$ and $2x$ for $x \geq 0$. We will show that $?h?^{-1}$ is not a local welding map at $x = 0$.

Let $f(x) = x/(1-x)$ and recall from equation (2.5) that $?f?^{-1}(x) = 2x$ on $[0, 1/2]$. Let $g(x) = 2x$; then $\phi = \widehat{?g?}^{-1}$ is quasymmetric, by Theorem 10.1. Moreover, the identity

$$f(g(x)) = g(f^2(x))$$

on $\widehat{\mathbb{R}}$ gives, under conjugation by $\widehat{?}$, the key identity

$$2\phi(x) = \phi(4x)$$

for positive x near 0 in $S^1(3)$, which in turn implies:

$$4s(\phi(x)) = s(2\phi(x)) = s(\phi(4x)).$$

Now define $\psi(x)$ near $x = 0$ by $\psi(x) = s(\phi(x))$ for $x > 0$ and by $\psi(x) = x$ for $x \leq 0$. Then locally $\psi(4x) = 4\psi(x)$. Since $s(x)$ and $\phi(x)$ are quasymmetric for $x > 0$, $\psi(x)$ is quasymmetric on a *neighborhood* of 0 by Lemma 11.3. But $s(x)$ is not a local welding map at $x = 0$, so neither is the map

$$s^{-1} \circ \psi(x) = \begin{cases} \phi(x) = ?(2?^{-1}(x)) & \text{for } x \geq 0, \text{ and} \\ x & \text{for } x < 0. \end{cases}$$

This formula agrees with $?h?^{-1}$ near $x = 0$, so $?h?^{-1}$ fails to be a local welding as well. ■

Remarks. This example shows, more precisely, that the composition of two *strong* welding maps, $? \circ h$ and $?^{-1}$, need not be a welding.

As can be seen in Figure 13, it is geometrically plausible that $?h?^{-1}$ is not a welding, since it is linear for $x < 0$ and behaves like $\sqrt{x}$ for $x > 0$.

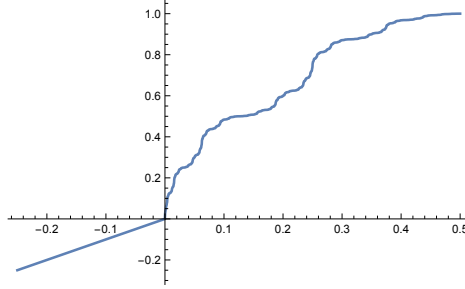


Figure 13. The function $h^{-1}(x)$ is not a welding.

A Appendix: The modular group and quadratic polynomials

The first matings between Fuchsian groups and polynomials were described by Bullett and Penrose in the mid-1990s [BP].

For suitable values of c , their construction combines $\mathrm{SL}_2(\mathbb{Z})$ and $P_c(z) = z^2 + c$ into a single complex dynamical system, given by an algebraic relation $S_c(x, y) = 0$ of degree $(2, 2)$ satisfying $S_c(x, y) = S_c(-y, -x)$. Here we will consider the special cases:

$$\begin{aligned} S_0(x, y) &= 18x^2y^2 - 21x^2y + 7x^2 + 21xy^2 - 16xy + 3x + 7y^2 - 3y \quad \text{and} \\ S_{1/4}(x, y) &= 8x^2y^2 - 9x^2y + 3x^2 + 9xy^2 - 6xy + x + 3y^2 - y. \end{aligned}$$

(These arise from $a = 5$ and 7 in [BP].)

The Julia set $J(S_c)$ is invariant under $z \mapsto -z$, but it is never a Jordan curve. This can be rectified by making the change of variables

$$w = 1/(p_c^2 - z^2), \tag{A.1}$$

where $p_0 = 1/3$ and $p_{1/4} = 1/5$. We then obtain a new *symmetric* relation $R_c(x, y)$ of degree $(4, 4)$.

Dynamics. Suppose $c = 0$ or $c = 1/4$. From the description of S_c as a holomorphic mating (see e.g. [BLS, §2.1]), it follows that the Julia set $J(R_c)$ is a Jordan curve, cutting $\widehat{\mathbb{C}}$ into disks Ω_∞ and Ω_0 containing ∞ and 0 respectively. The relation $R_c|_{\Omega_\infty}$ is analytically conjugate to the relation

$$z \mapsto \{P_c(z), -z\} \cup P_c^{-1}(z)$$

on the interior of the filled Julia set of P_c ; while $R_c|_{\Omega_0}$ is conjugate to the relation

$$z \mapsto \{T^{\pm 1}(z), U^{\pm 1}(z)\}$$

acting on the open disk

$$D = \mathbb{H}/\langle S \rangle.$$

Here

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Note that the involution S normalizes $\langle T, U \rangle$, and the change of variables (A.1) moves the critical of P_c to ∞ .

The corresponding Julia sets, along with the images of the Farey tiling of D , are shown in Figure 4 (which aims to clarify [BP, Figs. 8 and 11]).

Theorem A.1 *The Julia set $J(R_0)$ is quasiconformally equivalent to $W(?)$, while $J(R_{1/4})$ is a quasicircle.*

Proof. This follows from Theorems 1.4 and 1.5, using a suitable Markov partition of $J(R_c)$ to convert the action of $\text{SL}_2(\mathbb{Z})$ into a covering map. For $c = 0$, the parabolic dynamics of $\text{SL}_2(\mathbb{Z})$ is mated with the hyperbolic dynamics of $P_0(z) = z^2$, so $J(R_0)$ is equivalent to $W(?)$; while for $c = 1/4$, the parabolic fixed point of T is mated to the parabolic fixed point of $z^2 + 1/4$, so the result is a quasicircle. ■

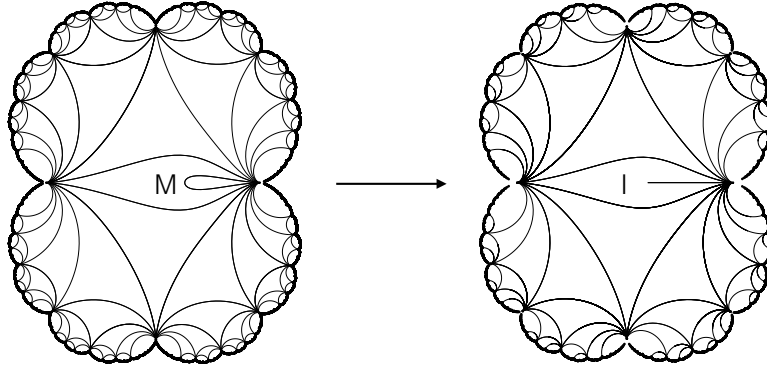


Figure 14. The Julia sets $J_{3,3}$ and $J(R_0)$ are analytically isomorphic outside one point.

Remarks. There is a close relationship between R_0 and the deltoid. In fact, $J(R_0)$ is the image of $J_{3,3}$ under a conformal map that collapses a monogon

M to an interval I as shown in Figure 14. In view of this relation, the fact that $J(R_0)$ is a Jordan curve follows from the same assertion for $J_{3,3}$.

It can be shown that $J(R_{1/4})$ makes an angle of 120° at the two points where it meets the real axis, either by analyzing the welding map or by studying the parabolic dynamics of $R_{1/4}$ in the spirit of the Leau–Fatou flower theorem [Mil, §7].

See [LLM, Theorem 1.9] and [BLLM, §5] for more details and other perspectives on $J(R_0)$ and $J(R_{1/4})$.

References

- [Bi] C. Bishop. Conformal welding and Koebe’s theorem. *Annals of Math.* **166** (2007), 613–656.
- [BL] S. Bullett and L. Lomonaco. Mating quadratic maps with the modular group III: The modular Mandelbrot set. *Adv. Math.* **458** (2024), Paper No. 109956.
- [BLLM] S. Bullett, L. Lomonaco, M. Lyubich, and S. Mukherjee. Mating parabolic rational maps with Hecke groups. *Preprint*, 2024.
- [BLS] S. Bullett, L. Lomonaco, and C. Siqueira. Correspondences in complex dynamics. In *New trends in one-dimensional dynamics*, pages 51–75. Springer, 2019.
- [BP] S. Bullett and C. Penrose. Mating quadratic maps with the modular group. *Invent. math.* **115** (1994), 483–511.
- [CFP] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson’s groups. *Enseig. Math.* **42** (1996), 215–256.
- [CJK] L. Carleson, P. W. Jones, and J.-C. Yoccoz. Julia and John. *Bol. Soc. Brasil. Mat.* **25** (1994), 1–30.
- [Con] J. H. Conway. *On Numbers and Games*. Academic Press, 1976.
- [CFS] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai. *Ergodic Theory*. Springer-Verlag, 1982.
- [GHM] F. W. Gehring, K. Hag, and O. Martio. Quasihyperbolic geodesics in John domains. *Math. Scand.* **65** (1989), 75–92.

- [Gh] E. Ghys. Sur l’invariance topologique de la classe de Godbillon-Vey. *Ann. scient. Éc. Norm. Sup.* **37** (1987), 59–76.
- [ILRS] O. Ivrii, P. Lin, S. Rohde, and E. Syga. Shapes of infinite conformally balanced trees. *Preprint*, 2023.
- [Jones] P. Jones. On removable sets for Sobolev spaces in the plane. In *Essays on Fourier analysis in honor of Elias M. Stein*, pages 250–267. Princeton University Press, 1995.
- [KS] Y. Komori and T. Sugawa. Bers embedding of the Teichmüller space of a once-punctured torus. *Conform. Geom. Dyn.* **8** (2004), 115–142.
- [LLMM] S.-Y. Lee, M. Lyubich, N. G. Makarov, and S. Mukherjee. Dynamics of Schwarz reflections: The mating phenomena. *Ann. scient. Éc. Norm. Sup.* **56** (2023), 1825–1881.
- [LM] S.-Y. Lee and N. Makarov. Topology of quadrature domains. *J. Amer. Math. Soc.* **29** (2016), 333–369.
- [LV] O. Lehto and K. J. Virtanen. *Quasiconformal Mappings in the Plane*. Springer-Verlag, 1973.
- [LLM] Y. Luo, M. Lyubich, and S. Mukerjee. A general dynamical theory of Schwarz reflections, B-involutions, and algebraic correspondences. *Preprint*, 2024.
- [LN] Y. Luo and D. Ntalampekos. Piecewise quasiconformal dynamical systems of the unit circle. *Preprint*, 2024.
- [LM] M. Lyubich and S. Merenkov. Quasisymmetries of the basilica and the Thompson group. *Geom. Funct. Anal.* **28** (2018), 727–754.
- [Mar] N. F. G. Martin. On finite Blaschke products whose restrictions to the unit circle are exact endomorphisms. *Bull. London Math. Soc.* **15** (1983), 343–348.
- [Mc1] C. McMullen. A compactification of the space of expanding maps on the circle. *Geom. Funct. Anal.* **18** (2009), 2101–2119.
- [Mc2] C. McMullen. Ribbon $\mathbb{R}$ -trees and holomorphic dynamics on the unit disk. *J. Topol.* **2** (2009), 23–76.

- [Mc3] C. McMullen. Dynamics on the unit disk: short geodesics and simple cycles. *Comm. Math. Helv.* **85** (2010), 723–749.
- [McS] C. McMullen and D. Sullivan. Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system. *Adv. Math.* **135** (1998), 351–395.
- [Mil] J. Milnor. *Dynamics in One Complex Variable*, volume 160 of *Annals of Math. Studies*. Princeton University Press, 2006.
- [Min] H. Minkowski. Zur Geometrie der Zahlen. In *Verhandlungen des Dritten Internationalen Mathematiker-Kongresses*, pages 164–173. B. G. Teubner, 1904.
- [MM] M. Mj and S. Mukerjee. Matings, holomorphic correspondences, and a Bers slice. *Preprint, 2023*.
- [MSW] D. Mumford. *Indra’s Pearls*. Cambridge University Press, 2002.
- [Oi] K. Oikawa. Welding of polygons and the type of Riemann surfaces. *Kodai Math. Sem. Rep.* **13** (1961), 37–52.
- [Pf] A. Pfluger. Über die Konstruktion Riemannscher Flächen durch Verheftung. *J. Indian Math. Soc.* **24** (1960), 401–412.
- [Rod] A. Rodriguez. Every circle homeomorphism is the composition of two weldings. *Preprint, 2025*.
- [Sa] R. Salem. On some singular monotonic functions which are strictly increasing. *Trans. Amer. Math. Soc.* **53** (1943), 427–439.
- [SWW] D. Saric, Y. Wang, and C. Wolfram. Circle homeomorphisms with square summable diamond shears. *Int. Math. Res. Not. IMRN*, 2024 pages 12219–12268.
- [Sh] M. Shub. Endomorphisms of compact differentiable manifolds. *Amer. J. Math.* **91** (1969), 175–199.
- [SS] M. Shub and D. Sullivan. Expanding endomorphisms of the circle revisited. *Ergodic Theory Dynam. Systems* **5** (1985), 285–289.
- [Sul] D. Sullivan. Conformal dynamical systems. In *Geometric Dynamics*, volume 1007 of *Lecture Notes in Math.*, pages 725–752. Springer, 1983.

- [Th] W. P. Thurston. Earthquakes in two-dimensional hyperbolic geometry. In D. B. Epstein, editor, *Low-dimensional Topology and Kleinian Groups*, pages 91–112. Cambridge Univ. Press, 1987.
- [Vai] J. Vainio. Conditions for the Possibility of Conformal Sewing. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* **53** (1985).

MATHEMATICS DEPARTMENT, HARVARD UNIVERSITY, 1 OXFORD ST,
CAMBRIDGE, MA 02138-2901