

# Families of rational maps and iterative root-finding algorithms

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## Abstract

In this paper we develop a rigidity theorem for algebraic families of rational maps and apply it to the study of iterative root-finding algorithms.

We answer a question of Smale's by showing there is no generally convergent algorithm for finding the roots of a polynomial of degree 4 or more. We settle the case of degree 3 by exhibiting a generally convergent algorithm for cubics; and we give a classification of all such algorithms.

In the context of conformal dynamics, our main result is the following: a stable algebraic family of rational maps is either *trivial* (all its members are conjugate by Möbius transformations), or *affine* (its members are obtained as quotients of iterated addition on a family of complex tori). Our classification of generally convergent algorithms follows easily from this result.

As another consequence of rigidity, we observe that the eigenvalues of a nonaffine rational map at its periodic points determine the map up to finitely many choices. This implies that bounded analytic functions nearly separate points on the moduli space of a rational map.

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### Introduction

Consider the map  $x \rightarrow nx$  on a complex torus  $T = \mathbf{C}/\Lambda$ . This map is equivariant with respect to the involution  $\iota: x \rightarrow -x$ ; so it descends to a rational map on the quotient space  $T/\iota = \mathbf{P}^1$ . The maps obtained in this way, first studied by Lattes [L], have many remarkable properties. We refer to them as *affine* rational maps.

If instead of a single torus  $T$  we have an algebraic family of tori, then the above construction produces an algebraic family of rational maps. Considered as a collection of dynamical systems on  $\mathbf{P}^1$ , this family is *stable* (free from bifurcations).

Our main result (Theorem 2.2) is a rigidity theorem for stable algebraic families of rational maps. The theorem states that any such family is either affine (it is obtained by the above construction) or trivial (all its members are conjugate by Möbius transformations). A consequence of interest in dynamical systems is the following: A nonaffine rational map is determined up to finitely many choices by its eigenvalues at its periodic points (Corollary 2.3).

Our study of algebraic families of rational maps was motivated by the following question of Smale's [Sm]: When does there exist a generally convergent purely iterative algorithm for finding the roots of a polynomial of degree  $d$ ? Such an algorithm associates to each polynomial  $p$  a rational function  $T_p(z)$  (depending rationally on the coefficients of  $p$ ), such that under iteration,  $T_p^n(z)$  tends to a root of  $p$  for most choices of  $p$  and  $z$ . It turns out that the maps  $T_p$  must form a stable algebraic family over the space of polynomials of degree  $d$ . Applying the result above, we conclude *there is no generally convergent algorithm for finding the roots of a polynomial of degree 4 or more* (Theorem 1.1). On the other hand, we exhibit an (apparently new) algorithm for finding the roots of cubics (Proposition 1.2). (It is well-known that Newton's method is generally convergent for quadratics but not cubics.)

Here is an outline of the proof of rigidity. Suppose we have a nontrivial stable algebraic family—parameterized, say, by a Riemann surface of finite type. Using the local theory of stable families of rational maps (as developed by Mañé, Sad and Sullivan [MSS]), we analyze the interaction between the periodic points of a rational map and the forward orbit of its critical points. This analysis, combined with the standard fact that a Riemann surface of finite type admits only finitely many nonconstant maps to the triply punctured sphere, allows us to conclude that *the critical points have finite forward orbits* (they eventually land

on periodic points). Then Thurston's theory  $[T_2]$  applies to give the dichotomy, rigid or affine. The main point is that we can pass from a general family to a critically finite family, and mappings of the latter type can be understood with finite-dimensional Teichmüller spaces.

Our exposition is organized into eight sections. In Sections 1 and 2 we present our results on iterative algorithms and rigidity of families respectively. The remaining sections are devoted to proofs. Section 3 contains proofs of the results concerning algorithms and *assumes* the rigidity theorem. Then in Section 4 we begin discussing the local and global theory of stable families of rational maps, drawing on material in [MSS]. It is here we obtain the reduction to the critically finite case. In Sections 5 and 6 we complete the proof of rigidity in two ways. The first approach uses only classical complex analysis and requires that at least one critical point is actually *periodic*; it is, however, sufficient for application to Smale's problem. The second proof is completely general, and is essentially the "easy half" of Thurston's theorem on the topology of rational maps. In Section 7 we use our rigidity result to show that the derivatives of a rational map at its periodic points determine the map up to finitely many choices.

We have included an appendix (§8) which discusses the monodromy of the Julia set's motion in a stable family. These considerations are related to material in Sections 4 and 6, but are not necessary for the main argument.

An excellent introduction to the dynamics of rational maps is the survey article of Blanchard  $[B_1]$ . We refer to this article for the standard results used implicitly or explicitly in the sequel.

This research owes much to the enthusiasm and insight of my advisor Dennis Sullivan. I wish to express my thanks to him, to Steve Smale for suggesting the problem, to John Hubbard for introducing me to Thurston's results, and to O. Gabber for useful conversations.

## 1. Iterative root-finding

In [Sm] Smale asks the question, when does there exist a generally convergent purely iterative algorithm for finding the roots of a polynomial of degree  $d$ ? It is well-known that Newton's method is generally convergent for quadratic polynomials, but not for cubics or polynomials of higher degree. Motivated by Smale's question, we prove below a rigidity theorem for algebraic families of rational maps, and use it to show that *there is no generally convergent purely iterative algorithm for finding the roots of a polynomial of degree 4 or more*. We settle the case of degree 3, perhaps surprisingly, by exhibiting a generally convergent algorithm for cubics.

*Definitions.* Let  $T: \mathbf{P}^1 \rightarrow \mathbf{P}^1$  be a rational mapping and let  $p(X)$  be a monic polynomial. Let  $T^n(z)$  denote the  $n^{\text{th}}$  iterate of  $T$ . We say  $T(z)$  is *convergent for  $p$*  if  $T^n(z)$  converges to a root of  $p$  for all  $z$  in a dense open subset of  $\mathbf{P}^1$ .

Let  $\text{Poly}_d (\simeq \mathbf{C}^d)$  and  $\text{Rat}_k (\simeq \mathbf{P}^{2k+1})$  denote the space of polynomials of degree  $d$  and the space of rational functions of degree  $k$  respectively. A *purely iterative algorithm* is a rational map

$$T: \text{Poly}_d \rightarrow \text{Rat}_k.$$

As a set theoretic mapping,  $T$  may be undefined on an algebraic subvariety of its domain. Let  $T_p(z)$  denote the rational function corresponding to  $p \in \text{Poly}_d$ . The algorithm  $T$  is *generally convergent* if there exists an open dense set of  $p$  in  $\text{Poly}_d$  such that  $T_p$  is convergent for  $p$ .

(Our definition of a “generally convergent algorithm” is more general than Smale’s. Thus our result on the non-existence of such an algorithm for degree  $\geq 4$  is stronger than his original conjecture.)

In the statement of the theorem below, the *centralizer*  $C(T)$  is the group of Möbius transformations commuting with  $T$ .

**THEOREM 1.1** (Classification of generally convergent algorithms).

I. *There is no generally convergent purely iterative algorithm for finding the roots of a polynomial of degree  $d \geq 4$ .*

II. *For  $d = 3$ , every generally convergent algorithm is obtained by specifying a rational map  $T(z)$  such that*

(i)  *$T$  is convergent for  $X^3 - 1$  and*

(ii) *its centralizer  $C(T)$  contains the group of Möbius transformations which permute the cube roots of unity. (This group is generated by  $z \rightarrow 1/z$  and  $z \rightarrow \zeta z$  where  $\zeta$  is a primitive cube root of 1.)*

*The algorithm is given (for  $p$  with distinct roots) by  $T_p = M_p T M_p^{-1}$ , where  $M_p$  is a Möbius transformation carrying the cube roots of unity to the roots of  $p$ . Furthermore any  $T$  with properties (i) and (ii) determines in this way a generally convergent algorithm for cubics.*

III. *For  $d = 2$ , every generally convergent algorithm is obtained by specifying a rational map  $T(z)$  and a rational function  $M: \text{Poly}_d \rightarrow \text{PSL}_2\mathbf{C}/C(T)$  such that*

(i)  *$T$  is convergent for  $X^2 - 1$ ,*

(ii) *its centralizer  $C(T)$  contains the element  $z \rightarrow -z$ , and*

(iii) *for  $p$  generic,  $M_p = M(p)$  carries the square roots of unity to the roots of  $p$ .*

The algorithm is given by  $T_p = M_p T M_p^{-1}$ . Furthermore any  $T$  and  $M_p$  satisfying (i)–(iii) determine a generally convergent algorithm for quadratics.

The proof of Theorem 1.1 (to be presented in §3) is straightforward once we have established that for a generally convergent algorithm, the maps  $T_p$  for varying  $p$  are all conjugate by Möbius transformations. For instance, Newton’s method for a quadratic with distinct roots is always conjugate to  $z \rightarrow z^2$ . The conjugacy moves 0 and  $\infty$  to the roots of the polynomial; since  $z^2$  is invariant under  $z \rightarrow 1/z$ , the conjugated map only depends upon symmetric functions of the roots. This principle turns up in the assertions about  $C(T)$  in parts II and III of the theorem above.

Among generally convergent algorithms we distinguish a special class which we call *superconvergent*. These algorithms have the property that the critical points of  $T_p$  are fixed by  $T_p$  and coincide with the roots of  $p$ . We see easily that a superconvergent algorithm is generally convergent, using the fact that the Julia set of  $T_p$  is nowhere dense and each stable region attracts a critical point.

In the proposition below, the *degree* of an algorithm refers to the degree of the map  $T_p$ .

PROPOSITION 1.2 (Examples of generally convergent algorithms).

I. Newton’s method is the unique degree 2 superconvergent algorithm for quadratics which fixes  $\infty$ .

II. There exists a unique degree 4 superconvergent algorithm for cubics. If the cubic polynomial  $p$  is given by

$$p(X) = X^3 + aX + b$$

then the algorithm is given by

$$T_p(X) = X - \frac{(X^3 + aX + b)(3aX^2 + 9bX - a^2)}{(3aX^4 + 18bX^3 - 6a^2X^2 - 6abX - 9b^2 - a^3)}.$$

(Note:  $T_p$  gives an algorithm for unnormalized cubics in the obvious way; we have restricted ourselves to  $p(X)$  with the  $X^2$  coefficient zero only to simplify the formula.)

Remarks. 1) Our algorithm for cubics is the same as Newton’s method applied to the rational function

$$q(X) = \frac{p(X)}{(3aX^2 + 9bx - a^2)}$$

(which of course has the same zeros as  $p$ ). The success of this algorithm can be

understood by checking that the *points of inflection* of  $q(X)$  coincide with the zeros of  $p(X)$ . Thus Newton's method certainly converges if we take one of the points of inflection of  $q$  as our initial guess. But it is known that convergence at the points of inflection implies convergence for an open dense set of initial guesses [Sm, Theorem 1 and discussion following].

2) The algorithms given above converge, not only for an open dense set of pairs  $(p, z)$ , but also for almost every  $(p, z)$  in the sense of measure. When  $p$  is quadratic with distinct roots, Newton's method converges for all initial guesses  $z$  avoiding a straight line passing between the roots. When  $p$  is a cubic with distinct roots, the method given in (II) above converges for all  $z$  outside of the Julia set of  $T_p$ . In this case, the Julia set has Hausdorff dimension strictly less than two; so in particular almost any initial guess  $z$  converges to a root. (The dimension assertion follows from a proposition in [Sul<sub>1</sub>], once we have observed that  $T_p$  is "expanding".)

3) If we allow our formula for  $T_p$  to include the operation of complex conjugation (so that the resulting function is real-algebraic but no longer complex-algebraic), then there exist generally convergent algorithms for polynomials of all degrees. This is a recent result of Shub and Smale [SS].

## 2. Rigidity of algebraic and analytic families of rational maps

In this section we present a rigidity theorem which will be used in the proof of Theorem 1.1. The proof of rigidity is postponed to Sections 4-6.

*Definitions.* An *algebraic family of rational maps* is a rational mapping of an irreducible (quasi-) projective variety  $V$  into the space  $\text{Rat}_k$  of rational functions of degree  $k$  for some  $k$ . A family is *stable* if there is a uniform bound on the period of the attracting cycles of the rational maps occurring in the family. (This notion of stability has many strong consequences; it essentially rules out bifurcation [MSS]). A family is *trivial* if all of its members are conjugate by Möbius transformations. A rational map is *critically finite* if the forward orbit of each critical point is finite.

LEMMA 2.1. *A stable algebraic family of rational maps is either trivial or all of its members are critically finite.*

Combining this result with recent work of Thurston [T<sub>2</sub>], we deduce:

THEOREM 2.2 (Rigidity of algebraic families). *A stable algebraic family of rational maps is either trivial or its members are affine.*

Here a family is *affine* if each member is a quotient of iterated addition on a complex torus (these are the Lattes examples). These maps will be discussed in detail in Section 6.

Theorem 1.1 follows easily from Theorem 2.2. A more classical proof (avoiding Teichmüller theory) can be given directly from Lemma 2.1 (see §5).

We will deduce a corollary of interest in the general study of rational dynamics: *Deformations of a nonaffine map are detected by changes in the eigenvalues at its periodic points.* More precisely: Let  $E_n(R)$  associate to  $R \in \text{Rat}_k$  the unordered set of values of  $(R^n)'(z)$  at the fixed points of  $R^n$ . Then  $E_n$  is a rational mapping from  $\text{Rat}_k$  to  $\mathbb{P}^{nk+1}$ . Clearly  $E_n$  is unchanged if we conjugate  $R$  by a Möbius transformation.

**COROLLARY 2.3** (Eigenvalues as moduli). *The conjugacy class of a nonaffine map  $R$  is determined up to finitely many choices by its eigenvalues  $\langle E_n(R) \rangle$ .*

Thus a nonaffine family in which  $E_n(R_\lambda)$  remains constant for all  $n$  must be trivial. This corollary allows us to extend our rigidity theorem to more general families.

**COROLLARY 2.4** (Rigidity of analytic families). *Let  $M$  be a complex manifold on which all bounded holomorphic functions are constant. Then any stable analytic family of rational functions parameterized by  $M$  is affine or trivial.*

This result is best possible in the sense that a manifold admitting a nonconstant bounded holomorphic function  $c(\lambda)$  also admits a nontrivial stable family of rational functions (e.g.  $R_\lambda = c(\lambda)z + z^2$ , if  $|c|$  is bounded by 1.)

Results concerning families of rational maps can often be translated into results concerning the geometry of the corresponding moduli space. Thus our rigidity result implies that the moduli space of a nonaffine rational map contains no algebraic curve (at least none which locally lifts to Teichmüller space.) We believe a stronger assertion is true.

*Conjecture.* On the moduli space of a nonaffine rational map, bounded analytic functions separate points.

The Teichmüller space  $T(R)$  and moduli space  $M(R)$  of a rational map are discussed in more detail in [Sul<sub>2</sub>].  $T(R)$  is a connected complex manifold whose points correspond to pairs  $\langle [S], [\phi] \rangle$ , where  $S$  is a rational map and  $\phi$  is a quasiconformal map conjugating  $R$  to  $S$ . The brackets indicate that only the conjugacy class of  $S$  and the homotopy class of  $\phi$  are remembered. The maps  $[S]$  form a stable family over  $T(R)$ .

$M(R)$  is the complex analytic space obtained as a quotient of  $T(R)$  by forgetting the map  $[\phi]$ . Its points consist of conformal conjugacy classes of maps  $S$  which are quasiconformally conjugate to  $R$ . Any  $\mathrm{PSL}_2\mathbb{C}$ -invariant holomorphic function on  $\mathrm{Rat}_k$  determines an analytic function on  $M(R)$ .

As evidence for the conjecture, note that symmetric functions of the reciprocals of the eigenvalues at the repelling periodic points of  $[S] \in M(R)$  determine bounded analytic functions on  $M(R)$ , and by Corollary 2.3 these functions separate points up to finitely many choices.

### 3. Proofs of the results concerning algorithms

In this section we prove the results in Section 1, *assuming* the theorems on rigidity of families described in Section 2. Subsequent sections are devoted to the proof of rigidity.

*Proof of Theorem 1.1. Part I.* The failure of purely iterative algorithms for polynomials of degree  $\geq 4$  results from the existence of the cross-ratio. We begin by observing that a general convergent algorithm must, by rigidity, be a trivial family of rational maps. Then we show that the sinks of  $T_p$  coincide with the roots of  $p$ . Thus the roots of any two (generic)  $p$  and  $q \in \mathrm{Poly}_d$  are related by a Möbius transformation, and this is clearly impossible for  $d \geq 4$  (since cross-ratios of the roots must be preserved).

To prove triviality, we first remark that  $T_p$  is a stable family. In fact, if  $T_p$  has an attracting cycle of order  $\geq 2$  for some  $p$ , then this cycle persists on an open neighborhood of  $p$  in  $\mathrm{Poly}_d$  and hence the algorithm is not generally convergent. Since the Julia set of an affine rational map is all of  $\mathbb{P}^1$ , it is obvious that the family is not affine. (This and other properties of affine maps are discussed in §6.) By rigidity (Theorem 2.2), the family is trivial (all members are conjugate).

We now argue that the sinks of  $T_p$  coincide with the roots of  $p$ . (Here  $x$  is a *sink* of  $T$  if there is an open set  $U$  such that  $T^n(u) \rightarrow x$  for all  $u \in U$ .) The idea is simple:  $T_p$  is generally convergent, so that the sinks of  $T_p$  are contained in the roots of  $p$ . But  $T$  only depends upon symmetric functions of the roots, so if one root is a sink so are the rest.

To make this precise, we first replace the family  $T_p$  by the family  $T_r$ , where  $r$  ranges in  $\mathbb{C}^d$  and  $T_r = T_p$  for  $p$  the polynomial with roots  $(r_1, \dots, r_d)$ . In other words  $T_r$  denotes the pull-back of  $T_p$  via the map  $\mathbb{C}^d \rightarrow \mathrm{Poly}_d$  by symmetric functions. Then  $T_r$  is a well-defined rational map for all  $r$  outside of a proper algebraic subvariety  $W$  of  $\mathbb{C}^d$ , and  $T_r = T_{\sigma r}$  for any permutation  $\sigma$  of the coordinates. For simplicity we adjoin to  $W$  the discriminant locus (the set of  $r$  whose coordinates are not all distinct).

Now for each  $r$  not in  $W$ , associate to  $T_r$  the nonempty set  $I_r$ , consisting of those  $i$  such that  $T_r$  has a sink at  $r_i$ . We claim this set is locally constant: Indeed, the sinks of  $T_r$  are a continuously varying subset of the coordinates of  $r$ , and these coordinates are distinct for  $r$  outside of  $W$ . But  $\mathbb{C}^d - W$  is connected; so  $I_r$  is globally constant. Hence there exists an  $i$  such that for all  $r$ ,  $T_r$  has a sink at  $r_i$ . On the other hand,  $T_r = T_{\sigma r}$  for any permutation  $\sigma$ ; thus  $T_r$  has a sink at all the coordinates of  $r$ .

Since  $T_r$  is a trivial family, the sink locations of any two members are related by a Möbius transformation. This is clearly impossible if  $d$  is greater than 4; so part I of the theorem is proved.

*Part II.* We now pass to the analysis of the case  $d = 3$ . By triviality,  $T_p$  and  $T_q$  are conjugate for all  $p$  and  $q$  outside a proper subvariety  $V$  of  $\text{Poly}_d$  (containing the discriminant locus); since the conjugacy must take the roots of  $p$  to the roots of  $q$ , it is uniquely determined up to six choices (the group of permutations of the roots). Along any path  $\gamma$  in  $\text{Poly}_d - V$ , the conjugacy of  $T_{\gamma(t)}$  to  $T_{\gamma(0)}$  can be chosen to vary continuously with  $t$ ; by choosing  $\gamma$  to be a loop such that the roots of  $\gamma(t)$  move with prescribed monodromy, we deduce that  $C(T_{\gamma(0)})$  contains the full group of Möbius transformations permuting the roots of  $\gamma(0)$ .

Conjugating  $T_p$  for a generic  $p$  so that its sinks become the cube roots of unity, we obtain a rational function  $T(z)$  satisfying conditions (i) and (ii) and determining  $T_p$  in the manner stated in part II.

Conversely, any such  $T$  determines a generally convergent algorithm. All that has to be checked is that  $T_p$  is well-defined, but this follows from the fact that its centralizer contains the full group of permutations of the cube roots of unity. Put differently, this centralizer condition guarantees that  $T_p$  only depends upon symmetric functions of the roots of  $p$ .

*Part III.* The analysis for degree 2 is almost the same except for the additional complication that with only two sinks the conjugacy has an additional degree of freedom. We omit the details.

*Proof of Proposition 1.2.* Part I is easy and is left to the reader. Part II follows from Theorem 1.1 and the observation that

$$T(X) = \frac{X^4 + 2X}{2X^3 + 1}$$

is the unique degree 4 rational map with superattracting fixed points at the cube roots of unity and symmetric under the group of Möbius transformations permuting those points.

The explicit formula for  $T_p$ , given in the statement of Proposition 2, is in principle straightforward to calculate, but a brute-force approach proves difficult. We describe how the calculation was made tractable. Notice that the fixed points of  $T$ , other than the cube roots of unity, are exactly 0 and  $\infty$ . Writing  $M_p$  explicitly, we compute  $M_p(0) + M_p(\infty)$  and  $M_p(0)M_p(\infty)$  in terms of symmetric functions of the roots of  $p$ , and hence determine a quadratic polynomial  $Q(X)$  such that the fixed points of  $T_p$  are exactly the roots of  $P(X)Q(X)$ . Then  $T_p$  takes the form

$$T_p(X) = X - \frac{P(X)Q(X)}{R(X)}$$

where  $R$  is a yet to be determined degree 4 polynomial. Using the fact that  $(T_p)'$  vanishes at the roots of  $P$ , we easily determine that  $R = SP + (PQ)'$  for some unknown  $S(X) = (cX + d)$ . The value of  $c$  is forced simply by the requirement that  $T$  is a degree 4 map, and  $d$  is determined from the fact that  $(T_p)'$  in fact vanishes to second order at each of the roots of  $p$ .  $\square$

#### 4. Stability, holomorphic motions and critical finiteness

In this section we develop some results on stable families which we use to prove Lemma 2.1 (stable algebraic families are either trivial or critically finite). The idea of the proof is simple: A critical point which is not pre-periodic provides an infinite set of dynamically labelled points (its forward orbit); we study how these points move as a function of the underlying parameter. Using the fact that a Riemann surface of finite type admits only finitely many nonconstant maps to the triply punctured sphere, we see that the tail of the forward orbit must not be moving at all, and from this we conclude that the entire family is rigid.

To carry out this argument in the case of a critical point in the Julia set, we need to understand the interplay between stability, holomorphic motions, and the  $\lambda$ -lemma, as developed in [MSS]. Here we review and extend some of those results; the reader is referred to [MSS] for more details.

*Definitions.* A *labelled holomorphic motion* of a set  $A$  (in  $\mathbf{P}^1$ ) over a complex manifold  $M$  with distinguished point  $m$  is a map  $f: (M, m) \times A \rightarrow \mathbf{P}^1$  such that:

- (i) For any fixed  $a \in A$ ,  $f(\lambda, a)$  is a holomorphic function of  $\lambda$ ;
- (ii) For any fixed  $\lambda \in M$ ,  $f(\lambda, a)$  is an injective function of  $a$ ; and
- (iii)  $f(m, a) = a$  for all  $a \in A$ .

We think of  $M$  as providing the time parameter for the motion; the motion of any point is a holomorphic function of time, and the points move without collision.

**PROPOSITION 4.1** (the  $\lambda$ -lemma). *A labelled holomorphic motion  $f: (\Delta, 0) \times A \rightarrow \mathbf{P}^1$  extends by continuity to a holomorphic motion of  $\bar{A}$  (the closure of  $A$ ).*

*Proof.* See [MSS]. The idea is that, whenever we normalize so that three points of  $A$  stay at 0, 1 and  $\infty$  for all time, the motion of any other point is given by a map of the disk into the triply punctured sphere, and the modulus of continuity is controlled by the Schwarz lemma.  $\square$

**PROPOSITION 4.2.** *For any fixed  $\lambda$ ,  $f(\lambda, \cdot): A \rightarrow \mathbf{P}^1$  can be extended to a quasiconformal map of the Riemann sphere to itself, with a dilatation  $K$  depending only upon  $\lambda$ .*

*Proof.* This follows from the extended  $\lambda$ -lemma of Sullivan and Thurston [ST]. It can also be proved by the following argument, due to Bers and Royden [BR]: Restricted to any finite subset of  $A$ ,  $f$  provides a map of  $M$  into the Teichmüller space of configurations of points on the Riemann sphere; such a map is contracting with respect to the Kobayashi metrics on the domain and range, and the Kobayashi metric agrees with the Teichmüller metric [Roy]; hence the restriction of  $f$  to any finite subset of  $A$  admits a qc extension with dilatation depending only upon  $\lambda$ ; and normalized qc mappings with bounded dilatation form a normal family, so that we can pass to a limit and obtain an extension over all of  $A$ .  $\square$

The next proposition shows that when the parameter space is a Riemann surface of finite type (finite genus with finitely many punctures), most labelled holomorphic motions are trivial.

**PROPOSITION 4.3.** *Let  $f: (S, s) \times A \rightarrow \mathbf{P}^1$  be a labelled holomorphic motion of an infinite set  $A$  over a Riemann surface  $S$  of finite type. Then there exists a holomorphic map  $M: S \rightarrow \text{PSL}_2\mathbf{C}$  such that  $f(\lambda, a) = M_\lambda(a)$ . In other words, after renormalization by a Möbius transformation, the motion becomes trivial.*

*Proof.* By composing  $f$  with a holomorphically varying Möbius transformation, we can assume that three points of  $A$  stay at 0, 1 and  $\infty$  for all  $\lambda$ ; to prove the theorem, it suffices to show that the remaining points do not move either. But each of the remaining points determines a different holomorphic map  $f(\cdot, a): S \rightarrow \mathbf{P}^1 - \{0, 1, \infty\}$ , and there are only finitely many such maps which are nonconstant; hence an infinite subset  $A_0$  of  $A$  must remain fixed as  $\lambda$  varies. Then any point not in  $A_0$  determines a map of  $S$  into the complement of an

infinite set, and such a map must also be constant. So once we have normalized so that three points are fixed, the motion becomes trivial, as claimed.

Since the argument uses the finiteness of the set of nonconstant maps of  $S \rightarrow \mathbf{P}^1 - \{0, 1, \infty\}$ , we briefly sketch a proof of this standard fact. First, a Riemann surface  $S$  of finite type is always obtained from some compact Riemann surface  $S'$  by deleting a finite set of points  $E$ . A holomorphic map of  $S$  to the triply punctured sphere determines a meromorphic function  $f$  on  $S'$ , with the values 0, 1, and  $\infty$  assumed only on the finite set  $E$ . It is easy to see that a nonconstant  $f$  is completely determined once we know where it assumes the values 0, 1 and  $\infty$  and with what *multiplicities*; but the multiplicities are bounded by Euler characteristic considerations; in fact

$$-\chi(S) \geq -\deg(f) \cdot \chi(\mathbf{P}^1 - \{0, 1, \infty\}) = \deg(f).$$

Thus  $f$  is completely determined by data which can take on only finitely many values, and hence the number of nonconstant maps of  $S$  to the triply punctured sphere is finite.  $\square$

The proof shows that up to renormalization, *there are only a finite number of nontrivial labelled holomorphic motions parameterized by a given Riemann surface of finite type.*

We will see below that the Julia set in a stable family moves by an *unlabelled* holomorphic motion. One might hope to prove rigidity by showing that this motion can be labelled by passing to a finite cover. Then the above proposition would hold the Julia set fixed, and triviality of the family would follow easily. Unfortunately there exist otherwise well-behaved stable families on the punctured disk whose Julia sets cannot be globally labelled on any finite cover (see the appendix). Nevertheless our proof of Lemma 2.1 will be closely related to the above proposition.

To give a general (unlabelled) holomorphic motion over a complex manifold  $M$ , we specify holomorphic motions  $f_i: (W_i, \omega_i) \times A_i \rightarrow \mathbf{P}^1$ , defined on a covering of  $M$  by open sets  $\langle W_i \rangle$ , such that on the overlaps the motion is the same up to a relabelling of the set: There exist bijective maps  $g_{ij}: A_i \rightarrow A_j$  with

$$f_i(\lambda, a) = f_j(\lambda, g_{ij}(a))$$

for all  $\lambda$  in the intersection of  $W_i$  and  $W_j$ . The  $g_{ij}$  are required to satisfy the cocycle condition  $g_{ij} \circ g_{jk} = g_{ik}$ .

One of the principal properties of a stable family of rational maps is that the corresponding Julia sets move by a canonical holomorphic motion which is also a conjugacy. This motion is first defined on the repelling periodic points, and then extended to the entire Julia set by the  $\lambda$ -lemma. Here a periodic point of a rational map is called *attracting*, *indifferent* or *repelling* according to whether

the derivative of the first return map is in absolute value strictly less than one, equal to one or greater than one. (The attracting points include the *superattracting* periodic points—for which the derivative is zero.) The attracting and repelling points together form the *hyperbolic* periodic points.

In a family of rational maps, a periodic point is *persistently non-hyperbolic* if it remains non-hyperbolic as the parameter varies. A more precise definition is given in [MSS].

**PROPOSITION 4.4.** *Let  $R_\lambda$  be a family of rational functions parameterized by a complex manifold  $M$ . Then the following conditions are equivalent:*

(i) *The period of the longest attracting cycle of  $R_\lambda$  is bounded uniformly in  $\lambda$ .*

(ii) *For each  $\lambda$ , the periodic points of  $R_\lambda$  are either hyperbolic or persistently non-hyperbolic.*

(iii) *As  $\lambda$  varies, the Julia set of  $R_\lambda$  moves by a holomorphic motion.*

*Proof.* See [MSS]. □

A family of rational maps parameterized by a complex manifold is called *stable* (J-stable, in the terminology of [MSS]) if the equivalent conditions (i)–(iii) above are satisfied. (Condition (i) is the definition of “stable” given previously in §2.) Another characterization of stability is given in Corollary 4.7 below.

To prove our rigidity theorem we need to elaborate point (iii) above. Specifically, we observe that the motion of the Julia set is compatible with the dynamics and with the labelling of a critical point.

**PROPOSITION 4.5.** *Let  $R_\lambda$  be a stable family of rational maps parameterized by  $(\Delta, 0)$ . Then there is a unique labelled holomorphic motion of the Julia set  $J_\lambda$  of  $R_\lambda$  such that*

$$R_\lambda(f(\lambda, z)) = f(\lambda, R_0(z))$$

*for all  $z \in J_0$ . If  $c_\lambda$  is a holomorphically labelled critical point of  $R_\lambda$ , and  $c_0$  is in the Julia set of  $R_0$ , then  $c_\lambda$  is in the Julia set of  $J_\lambda$  for all  $\lambda$  and  $f(\lambda, c_0) = c_\lambda$ . (In other words,  $f$  carries critical points to critical points.)*

*Proof.* The existence of the holomorphically varying conjugacy  $f(\lambda, \cdot)$  is proved exactly as in [MSS]. Observe that the repelling periodic points move holomorphically and never collide because they stay repelling by stability. The motion of the repelling points provides a conjugacy on a dense subset of the Julia set and extends to a holomorphic motion of the closure by the  $\lambda$ -lemma. It is unique because its values can be specified in only one way on the repelling periodic points.

Now suppose the holomorphically labelled critical point  $c_\lambda$  lies in the Julia set when  $\lambda = 0$ . Notice that  $c_\lambda$  moves injectively with the repelling periodic points—they never collide because a repelling cycle cannot include a critical point. By the  $\lambda$ -lemma, this motion also has a continuous extension to the Julia set. But the new motion agrees with  $f$  on a dense set; so  $f$  and  $c_\lambda$  determine the same motion of  $c_0$ , as claimed.  $\square$

**COROLLARY 4.6.** *Let  $R_\lambda$  be a stable family of rational maps, and let  $c_\lambda$  be a holomorphically labelled critical point of  $R_\lambda$ . Suppose for some  $\lambda_0$  the forward orbit of  $c_{\lambda_0}$  lands on a repelling periodic point of order  $N$ . Then this behavior persists for all  $\lambda$ .*

*Proof.* A critical point which lands on a repelling point must itself be in the Julia set. By Proposition 4.5, the holomorphic motion of the Julia set in a stable family is a conjugacy which is compatible with the labelling of a critical point. Hence the combinatorics of the forward orbit of the critical point is locally preserved.  $\square$

Recall that a point  $z$  lies outside of the Julia set of  $R(z)$  if and only if the iterated mappings  $\langle R^n \rangle$  form a normal family at  $z$ . The following corollary, while not used in the sequel, is of interest because it gives an analogous definition of the stable points of an analytic family.

**COROLLARY 4.7.** *Let  $R_\lambda$  be an analytic family of rational functions. Suppose the critical points of  $R_\lambda$  are labelled by holomorphic function  $c_1(\lambda)$ ,  $c_2(\lambda)$ , etc. Then  $R_\lambda$  is stable if and only if the functions*

$$c_i(\lambda), R_\lambda(c_i(\lambda)), R_\lambda^2(c_i(\lambda)), \dots$$

*(with parameterization of the forward orbit of the  $i^{\text{th}}$  critical point) form a normal family for each  $i$ .*

*Proof.* Suppose  $R_\lambda$  is stable. We claim that, by stability, there is a periodic cycle (of period  $\geq 3$ ) which remains repelling for all  $\lambda$  and which avoids the forward orbit of the critical points. Indeed, in a stable family any repelling cycle remains repelling for all  $\lambda$ ; so we need only justify our assertion about the critical points. But any critical point whose forward orbit meets a repelling cycle for some *fixed*  $\lambda$  must be involved with the same cycle for *all*  $\lambda$  (by Corollary 4.6); since there are an infinite number of cycles and only finitely many critical points, we may choose a cycle which consistently avoids the post-critical set.

We can locally normalize so that three points of the cycle are constantly at 0, 1 and  $\infty$ . Then the sequence of functions listed above are all maps to the triply punctured sphere, and hence they form a normal family.

On the other hand, suppose the sequence of functions is normal. For each  $i$ , let  $f_i(\lambda)$  denote a function which is the limit of a subsequence of the forward orbit of the  $i^{\text{th}}$  critical point. Now suppose for some  $\lambda_0$ ,  $R_{\lambda_0}$  has an attracting cycle of order  $n$ . Then this cycle attracts some critical point, and hence there exists an  $i$  such that

$$R_{\lambda_0}^n(f_i(\lambda_0)) = f_i(\lambda_0).$$

Equality persists on a neighborhood of  $\lambda_0$  and hence holds for all  $\lambda$ . Therefore the  $i^{\text{th}}$  critical point is never attracted to a cycle of period greater than  $n$ . We easily deduce an upper bound on the length of the longest attracting cycle, and therefore the family is stable.  $\square$

*Proof of Lemma 2.1* (a stable algebraic family is trivial or critically finite). Suppose we have a stable algebraic family of rational functions  $R_\lambda$  parameterized by a (quasiprojective) variety  $S$ , which we imagine imbedded in some  $\mathbf{P}^N$ . By intersecting the parameter space with a hyperplane of the appropriate codimension, we easily reduce the proof of the lemma to the case where  $S$  is a Riemann surface of finite type (genus  $g$ , possibly with punctures).

As in the proof of Corollary 4.7 above, stability implies there is a periodic cycle which remains repelling for all  $\lambda$  and which avoids the forward orbit of the critical points.

Replace  $S$  by a covering surface on which the points of this cycle and all of the critical points are labelled as functions of  $\lambda$ . This covering will be branched over the points of  $S$  where the critical points collide; since the family is algebraic, these branch points form a finite set and the covering space is again a Riemann surface of finite type. (We do not have to worry about points in the repelling cycle colliding, since the associated eigenvalue is a constant of absolute value  $> 1$  and hence the roots of the corresponding factor of  $R^n(z) - z$  are always distinct.)

Now normalize the family (conjugate by a Möbius transformation varying holomorphically in  $\lambda$ ) so that three points in the labelled repelling cycle are constantly at  $0, 1$  and  $\infty$ . Let  $c_\lambda$  denote a critical point as a function of  $\lambda$ . We will show that either  $c_\lambda$  is pre-periodic or the family is trivial.

Since the cycle does not meet the forward orbit of the critical points, the functions  $c_\lambda, R_\lambda(c_\lambda), R_\lambda^2(c_\lambda), \dots$ , define maps of  $S \rightarrow \mathbf{P}^1 - \{0, 1, \infty\}$ . But there are only finitely many such maps which are nonconstant. Therefore either  $c_\lambda$  is pre-periodic or its forward orbit is eventually constant in  $\lambda$  and infinite. Assume we are in the latter case. Then we claim all periodic cycles of sufficiently high order are constant in  $\lambda$ . Indeed, any such cycle is parameterized on a finite covering  $S'$  of  $S$ , and avoids the forward orbit of  $c_\lambda$ ; hence it defines a map of  $S'$

into  $\mathbf{P}^1$  – (an infinite set), which must be constant. So now we have all repelling cycles of sufficiently high order constant. There are infinitely many such cycles, and the critical points avoid them; so each critical point defines a map of  $S \rightarrow \mathbf{P}^1$  – (an infinite set), which must be constant, similarly for all the pre-images of the critical values (once labelled by passing to a covering). Then for each  $\lambda$ ,  $R_\lambda$  defines a covering map from  $\mathbf{P}^1$  – (the pre-images of the critical values, a set which is constant in  $\lambda$ ) to  $\mathbf{P}^1$  – (the critical values themselves, also constant); there are only finitely many such coverings, and since  $S$  is connected, the family  $R_\lambda$  is constant, i.e. trivial.  $\square$

### 5. Rigidity with a periodic critical point

In this section we give a proof of Theorem 2.2 for families of rational maps with at least one attracting region. This proof is of interest because it shows that our main result on iterative algorithms (Theorem 1.1) can be derived entirely from classical function theory. In the next section we will discuss Thurston's Teichmüller space approach, which will be used to prove Theorem 2.2 in the general case.

**PROPOSITION 5.1.** *Let  $R_\lambda$  be a stable algebraic family of rational maps with at least one attracting cycle. Then the family is trivial.*

*Remark.* A generally convergent algorithm must have at least one attracting cycle; so the above proposition is sufficient to prove Theorem 1.1.

The proof of 5.1 rests on our ability to “linearize” a neighborhood of a superattracting periodic point—that is, introduce local coordinates in which the map takes the form  $z \rightarrow z^n$ . We begin by discussing the situation for a single rational map  $R(z)$ . Assume that 0 is a superattracting fixed point, and the power series for  $R(z)$  at 0 is

$$R(z) = z^n + [\text{higher order terms}].$$

Furthermore, assume that  $\infty$  is not attracted to zero. We define a negative harmonic function  $\psi(z)$  on the basin of attraction of 0, as follows:

$$\psi(z) = \lim_{k \rightarrow \infty} n^{-k} \log |R^k(z)|$$

(the convergence is easily checked). Then

$$\psi(R(z)) = n\psi(z).$$

Let  $C(0)$  denote the critical points of  $R$  which are attracted to zero, but do not eventually land on zero.

LEMMA 5.2. *The basin of attraction of zero contains a ball about zero of radius  $r$ , where*

$$r = \frac{1}{4} \min \{ \exp \psi(c) : c \in C(0) \}.$$

*Proof.* It is well-known (see for example [B<sub>1</sub>], Thm 3.4) that there is a conformal map  $\phi: U \rightarrow \Delta$  carrying a neighborhood  $U$  of zero into the unit disk and conjugating  $R$  to  $z \rightarrow z^n$ ; in other words,

$$\phi(R(z)) = [\phi(z)]^n.$$

Since the power series for  $R(z)$  begins with  $z^n$ ,  $|\phi'(0)| = 1$  and one checks that

$$\log |\phi(z)| = \psi(z) \quad \text{for } z \in U.$$

Now  $\phi$  can be extended to any connected neighborhood  $U$  of 0 such that  $R(U)$  is contained in  $U$  and such that  $U$  contains no critical points other than 0. In particular, we may take  $U$  to be a connected component of

$$\{ z : \psi(z) < \psi(c) \quad \text{for all } c \in C(0) \}.$$

Then  $\phi$  carries  $U$  univalently onto a disk of radius  $4r$ , where  $r$  is given in the statement of the lemma. Apply the Koebe theorem to  $\phi^{-1}$ , and the lemma follows. (We need  $\infty$  to lie outside of  $U$  to apply the Koebe theorem, and this is why we have normalized so that  $\infty$  is not attracted to zero.) □

*Proof of Proposition 5.1.* Let  $R_\lambda$  be a family of rational maps parameterized by  $\lambda \in S$ . As usual, it is enough to carry out the proof in the case where  $S$  is a Riemann surface of finite type, and we may assume that the critical points of  $R_\lambda$  are labelled by meromorphic functions  $c_1(\lambda)$ ,  $c_2(\lambda)$ , etc.

By Lemma 2.1, we need only analyze the critically finite case. It is easy to see that a critically finite map with an attracting cycle must have a periodic critical point. (Any attracting point attracts a critical point—see [B<sub>1</sub>], Thm 5.8.) Assume for simplicity that the critical point is actually a fixed point.

By renormalizing the family we may assume that the fixed critical point is constantly at zero and that  $\infty$  is not attracted to zero (for example, take  $\infty$  to be a repelling fixed point). Then the power series for  $R_\lambda$  at  $z = 0$  is given by

$$R_\lambda(z) = a(\lambda)z^n + [\text{higher order terms}]$$

where  $a(\lambda)$  is a meromorphic function on  $S$ .

Replacing  $S$  by a finite covering surface, we may assume there exists a meromorphic function  $b(\lambda)$  such that  $b(\lambda)^{n-1} = a(\lambda)$ . Then after conjugation of  $R_\lambda$  by  $z \rightarrow b(\lambda)z$ , the functions  $R_\lambda$  all assume the normalized form required for Lemma 5.2 above.

Now assume  $c_1(\lambda), \dots, c_m(\lambda)$  parameterize the critical points which are attracted to zero. For each  $i$ , the function

$$\psi_i(\lambda) = \lim_{k \rightarrow \infty} n^{-k} \log |R_\lambda(c_i(\lambda))|$$

is a well-defined negative harmonic function of  $\lambda$ . But on a Riemann surface of finite type any such function is constant. Hence

$$\frac{1}{4} \exp \min \{ \psi_i(\lambda) : i = 1, \dots, m \} = r$$

is a positive constant independent of  $\lambda$ . By Lemma 5.2, the basin of attraction of zero contains a ball of radius  $r$  for all  $\lambda$ .

Thus the Julia set of  $R_\lambda$  always stays a definite distance away from zero.

Now symmetric functions of the reciprocals of the locations of the repelling periodic points of a given order define bounded holomorphic functions of  $\lambda$ , which must be constant. Therefore the Julia set is fixed. But the critical points not in the Julia set must remain outside of it, so they too are fixed. It is now easy to check that the critical values and their pre-images must be constant in  $\lambda$  and hence the family is trivial.  $\square$

## 6. Periodic critical points: Thurston's theorem

To prove rigidity of stable algebraic families in general we need a uniqueness result on critically finite rational maps. We must also locate the affine examples—which give nontrivial stable families of rational maps—and show these are the only cases which must be avoided.

The required result is one component of a recent theorem of Thurston, which gives a necessary and sufficient condition for a critically finite branched covering of the sphere to be realizable as a rational map. With the exception of the affine examples, this realization is unique. To keep the exposition self-contained, the uniqueness assertion will be proved here and used to complete the proof of our rigidity theorem. A careful and complete proof of Thurston's theorem is given in a recent manuscript of Douady and Hubbard [DH], and Thurston's own exposition [T<sub>2</sub>] is forthcoming.

*Definitions.* Let  $R(z)$  be a rational map. The *critical set*  $C$  is the set of critical points of  $R$ . The *post-critical set*  $P$  is given by

$$P = \bigcup_{n \geq 1} R^n(C).$$

Note that the critical set is not necessarily included in the post-critical set.  $R$  is *critically finite* if  $P$  is a finite set. We associate to each  $p \in P$  the integer  $n(p)$  defined as the least common multiple of the branching order of  $R^k$  over  $p$ , for  $k = 1, 2, 3, \dots$  and for all pre-images of  $p$ . This number is finite unless  $p$  is in

the forward orbit of a periodic critical point. Then to  $R$  we associate the complex orbifold  $Q(R)$  obtained from  $\mathbf{P}^1$  by making each  $p \in P$  a cone point of order  $n(p)$ ; i.e. a neighborhood of  $p$  is modelled on  $\Delta/(z \rightarrow \zeta z)$  where  $\zeta$  is a primitive  $n(p)^{\text{th}}$  root of unity. If  $n(p) = \infty$  the associated orbifold has a puncture at  $p$ . The signature of the orbifold associated to  $R$  is the collection of numbers  $\{n(p): p \in P\}$ . Its Euler characteristic is given by

$$\chi(Q) = 2 - \sum_{p \in P} \left( 1 - \frac{1}{n(p)} \right).$$

The map  $R$  is hyperbolic if  $\chi(Q(R)) < 0$ .

We say that  $R$  is affine if it is obtained as the quotient of iterated addition on a complex torus. These are among the Lattes examples [Lat]; they are constructed as follows. Let  $E = \mathbf{C}/\Lambda$  be a compact Riemann surface of genus 1 and let  $\alpha > 1$  be an integer. Then multiplication by  $\alpha$  induces a map of  $E \rightarrow E$  which is a group homomorphism, and hence is equivariant with respect to the involution  $\iota: E \rightarrow E$  by  $\iota(x) = -x$ . Now  $E/\iota = \mathbf{P}^1$  (the map is given by the Weierstrass  $p$ -function); so  $\alpha$  descends to a rational map  $R_{\alpha, \Lambda}: \mathbf{P}^1 \rightarrow \mathbf{P}^1$ . In this case the Euler characteristic of the associated orbifold is zero.

This map is interesting because we can keep  $\alpha$  fixed and vary  $\Lambda$  to obtain a nontrivial stable family. Notice that  $\alpha: E \rightarrow E$  has a dense set of periodic points, all repelling (since  $\alpha > 1$ ). We easily deduce that the same is true of  $R_{\alpha, \Lambda}$ . Hence the Julia set of an affine rational map is all of  $\mathbf{P}^1$ .

The construction described above can be carried out slightly more generally: We can use any  $\alpha$  carrying  $\Lambda$  into itself ( $\alpha$  does not have to be an integer if  $\Lambda$  has complex multiplication), and we can take a quotient by any nontrivial group of automorphisms of  $E$  (this is more general only when  $\Lambda = \mathbf{Z}[i]$  or  $\mathbf{Z}[\rho]$ ). These more general constructions do not admit deformations, because they depend upon special properties of the lattice  $\Lambda$ . Thus we have used the word affine to single out the nonrigid examples.

*Example.* The functions

$$R_\lambda(z) = \frac{z^4 + 2z^2 + 2\lambda z + 1}{4z^3 - 4z - \lambda}, \quad \lambda \in \mathbf{P}^1,$$

determine a nontrivial stable algebraic family of affine rational maps parameterized by the Riemann sphere. This family is obtained from the addition law

$$p(2u) = -2p(u) + \frac{1}{4} \left( \frac{p''(u)}{p'(u)} \right)^2$$

for the family of lattices whose Weierstrass  $p$ -functions satisfy

$$(p')^2 = 4p^3 - 4p - \lambda$$

(the addition law is derived in [Lang]). Using the relation between  $p'$  and  $p$ , we express the right hand side of the addition formula in terms of  $p(u)$  alone. Letting  $z$  play the role of  $p(u)$ , we obtain the rational function  $R_\lambda$  given above. The family is nontrivial because the corresponding elliptic curves are not all conformally equivalent.

Two critically finite rational maps  $R_0$  and  $R_1$  (with post-critical sets  $P_0$  and  $P_1$ ) are *combinatorially equivalent* if there is a pair of homeomorphisms

$$\phi, \psi: (\mathbf{P}^1, P_0) \rightarrow (\mathbf{P}^1, P_1)$$

such that  $\phi \circ R_0 = R_1 \circ \psi$ , and such that  $\phi$  and  $\psi$  are in the same isotopy class relative to  $(P_0, P_1)$ . The topology of the orbifold  $Q(R)$  depends only upon the combinatorial equivalence class of  $R$ .

**PROPOSITION 6.1.** *Let  $R_0$  and  $R_1$  be combinatorially equivalent rational maps. Suppose their associated orbifold has signature  $(2, 2, 2, 2)$ . Then either*

- (i)  $R_0$  is conformally conjugate to  $R_1$  and each is a quotient of an endomorphism of a torus with complex multiplication, or
- (ii)  $R_0$  is quasiconformally conjugate to  $R_1$  and both are affine.

*Proof.* The universal cover of a  $(2, 2, 2, 2)$  orbifold (say that for  $R_0$ ) can be obtained as follows: Construct the unique elliptic curve  $E = \mathbf{C}/\Lambda$  such that the Weierstrass function is ramified over  $P_0$  (up to a Möbius transformation). Then the composed map  $\mathbf{C} \rightarrow E \rightarrow \mathbf{P}^1$  gives the universal covering. By [T<sub>2</sub>, 13.3], the map  $R_0$  is a holomorphic self-map of its underlying orbifold, so that it lifts to a conformal automorphism of  $\mathbf{C}$ , which must be given by  $z \rightarrow \alpha z$  for some  $\alpha$  (once normalized so that 0 is a fixed point).

Let  $\{1, \tau\}$  be a basis for  $\Lambda$ . Since  $\alpha$  carries  $\Lambda$  into itself, there is a  $2 \times 2$  integer matrix such that

$$\begin{bmatrix} \alpha \\ \alpha\tau \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ \tau \end{bmatrix}.$$

Up to conjugation by elements of  $SL_2\mathbf{Z}$ , this matrix is an invariant of the combinatorial equivalence class of  $R_0$ . If the matrix is not diagonal, it determines a nontrivial quadratic equation for  $\tau$ , and hence *both  $\tau$  and  $\alpha$  can be recovered from the combinatorial type of  $R_0$* . Then clearly  $R_0$  and  $R_1$  are conformally conjugate.

On the other hand, if the matrix is diagonal then  $\alpha$  is an integer and  $R_0$  and  $R_1$  are affine rational maps (of the same degree,  $\alpha^2$ ). Thus  $R_i = R_{\alpha, \Lambda_i}$  for

possibly distinct lattices  $\Lambda_0$  and  $\Lambda_1$ . There is an  $\mathbf{R}$ -linear map carrying  $\Lambda_0$  to  $\Lambda_1$ , and this map descends from  $\mathbf{C}$  to give a quasiconformal conjugacy between  $R_0$  and  $R_1$ .  $\square$

**THEOREM 6.2.** *Let  $R_0$  and  $R_1$  be two critically finite, combinatorially equivalent rational maps. Then either both are affine or  $R_0$  and  $R_1$  are conformally conjugate.*

*Remark.* In other words, a nonaffine critically finite rational map is uniquely determined by its combinatorial type.

*Proof.* We want to apply Teichmüller theory to homeomorphisms of the complement of the post-critical set; so we begin by dispensing with the cases where the post-critical set consists of fewer than four points.

Notice that  $R_0$  is a covering map from the complement of  $R_0^{-1}(P_0)$  to the complement of  $P_0$ , and combinatorial equivalence implies that  $R_0$  and  $R_1$  are equivalent coverings.

If  $P_0$  consists of two points, one easily checks that  $R_0$  is conjugate to  $z \rightarrow z^n$  for some  $n$  which depends only upon the combinatorial type of  $R_0$ .

If  $P_0$  consists of three points, the conformal structure of its complement is unique and hence the conformal structure of the complement of  $R_0^{-1}(P_0)$  is uniquely determined by combinatorial data. Now the pre-image of  $P_0$  contains  $P_0$  and there is a unique Möbius transformation identifying the corresponding points; thus  $R_0$  is completely determined by its combinatorial type.

We now assume the post-critical set contains at least four points. Let  $\phi, \psi$  be homeomorphisms providing a conjugation up to isotopy as in the definition of combinatorial equivalence. By Teichmüller theory we may replace  $\phi$  by the unique quasiconformal map of minimal dilatation in its isotopy class. Similarly we may replace  $\psi$  by the map  $R_1^{-1} \circ \phi \circ R_0$ , where the inverse branch of  $R_1$  is chosen so that the resulting map is in the same isotopy class as the original  $\psi$ . This new  $\psi$  is isotopic to  $\phi$  and is quasiconformal with the same dilatation as  $\phi$  because the maps  $R_0$  and  $R_1$  are conformal; by uniqueness of the Teichmüller mapping,  $\phi = \psi$ , and  $\phi$  now provides an exact conjugation between  $R_0$  and  $R_1$ .

Let  $\Phi_0, \Phi_1$  denote quadratic differentials on  $(\mathbf{P}^1, P_0)$  and  $(\mathbf{P}^1, P_1)$  corresponding to the extremal map  $\phi$ . We claim that  $|R_0^* \Phi_0| = \deg(R_0) |\Phi_0|$ . This follows simply from the fact that  $R_0^* \Phi_0$  is again a quadratic differential corresponding to  $R_1^{-1} \circ \phi \circ R_0 = \phi$ .

If  $\Phi_0 = 0$ , the map  $\phi$  is conformal, and  $R_0$  and  $R_1$  are conjugate by a Möbius transformation. So assume  $\Phi_0 \neq 0$ ; we will show that  $R_0$  (and hence  $R_1$ ) is affine.

The differential  $\Phi_0$  is holomorphic outside of  $P_0$  and has at worst simple poles on  $P_0$ . How are the singularities (zeros and poles) of a quadratic differential altered by pulling back? If we pull back by a locally invertible mapping, the zero or pole is preserved; if we pull back through a critical point which looks like  $z \rightarrow z^n$ , a simple pole becomes a zero of order  $(n - 2)$ , a regular point becomes a zero of order  $(2n - 2)$ , and a zero of order  $k$  becomes a zero of order  $(nk + 2n - 2)$ . A zero is *never destroyed* by pulling back. Therefore  $\Phi_0$  has *no zeros*; for a zero at a point  $z$  implies zeros along the entire inverse orbit of  $z$  under  $R_0$ , which is an infinite set. Also, there must be a pole at every critical value, and every critical point must be simple; otherwise pulling back through a critical point would create a zero. Since the only way to obtain a pole is by pulling back a pole by an invertible mapping, we find that *there are poles all along the post-critical set  $P$  and there are no critical points in the post-critical set*. Finally a quadratic differential on the sphere has degree  $-4$ , so the number of points in  $P$  is exactly four.

Thus  $R_0$  is associated to the  $(2, 2, 2, 2)$  orbifold. The preceding proposition applies to complete the proof.  $\square$

*Proof of Theorem 2.2* (a stable algebraic family is affine or trivial). By Lemma 2.1, it suffices to show that a family of nonaffine critically finite maps is trivial. It is easy to see that any two maps in such a family are combinatorially equivalent. By Thurston's theorem, such a map is determined up to conformal conjugacy by its combinatorial type. Thus any two maps in the family are conjugate by a Möbius transformation, as claimed.  $\square$

## 7. Eigenvalues as moduli

In this section we use the fact that rational functions with prescribed eigenvalues form an algebraic variety to prove our assertions about eigenvalues as moduli, and obtain a rigidity statement in the analytic category.

*Proof of Corollary 2.3.* Given a rational function  $R \in \text{Rat}_k$ , consider the algebraic variety  $V$  consisting of those functions with the same eigenvalues as  $R$ . This variety is stable family of rational maps since there occur no attracting cycles of order longer than the longest attracting cycle of  $R$ . If  $R$  is nonaffine, then by Theorem 2.2 the family is trivial and the component of  $V$  containing  $R$  consists exactly of those maps which are conjugate to  $R$  by Möbius transformations. There are only finitely many components so the conjugacy class of  $R$  is determined by the data  $\langle E_n(R) \rangle$  up to finitely many choices.  $\square$

*Proof of Corollary 2.4.* Let  $R_\lambda$  be an analytic family of rational maps parameterized by a complex manifold  $M$  which admits no nonconstant bounded

holomorphic functions. Assume  $R_\lambda$  is stable. Then as  $\lambda$  varies, any given periodic cycle is either always repelling or always attracting. Hence the associated eigenvalue, or its reciprocal, defines a bounded holomorphic function on  $M$ , and therefore the eigenvalue is constant. Now apply Corollary 2.3.  $\square$

*Questions and Remarks.* Noetherian properties imply there are an  $N$  and an  $M$  such that  $E_1(R), \dots, E_N(R)$  determine  $R$  up to at most  $M$  choices. What is  $N$  as a function of the degree of  $R$ ? And is  $R$  determined uniquely?

Notice that the “spectrum” of a rational map—its set of eigenvalues—is not necessarily discrete. For example, if we take  $R(z) = \lambda z + z^2$ , where  $\lambda = \exp(2\pi i\theta)$  and  $\theta$  is very Liouville (well approximated by rationals), then  $R$  will have a set of periodic points accumulating at zero whose eigenvalues accumulate at one. Indeed,

$$R^n(z) - z = z^{2^n} + \dots + (\lambda^n - 1)z$$

so that there exists a periodic point  $r_n$  with

$$|r_n| \leq |\lambda^n - 1|^{2^{-n}}.$$

Given any sequence  $\varepsilon_n > 0$  there is a dense  $G_\delta$  of  $\lambda$  in  $S^1$  such that  $|\lambda^n - 1| < \varepsilon_n$  for infinitely many  $n$ , and by choosing  $\varepsilon_n$  appropriately we can assure that the corresponding values of  $(R^n)'(r_n)$  accumulate at one. (This argument is essentially due to Cremer [C].) Is discreteness of the spectrum a generic property?

The situation is closely analogous to that for Kleinian groups obtained as limits of quasifuchsian groups, where a Liouville property of the ending lamination gives rise to a hyperbolic 3-manifold carrying arbitrarily short closed geodesics. (The corresponding group then contains loxodromic elements whose derivatives at their fixed points tend to one in absolute value.)

### 8. Appendix: Monodromy and the quasiconformal centralizer

Let  $R_\lambda$  be a stable family of rational maps parameterized by a Riemann surface  $S$ , as discussed in Section 4. Then the associated Julia sets  $J_\lambda$  move by an unlabelled holomorphic motion which is also a conjugacy. The fundamental group of  $S$  gives rise to an obstruction to production of a global labelling.

For simplicity, we will assume that the family is not only stable but *qc-stable* (*C-stable* in the terminology of [MSS]). This is equivalent to the condition that the critical points and all of their forward images move by a holomorphic motion. One easily checks that the forward orbit of any particular critical point then determines a normal family of functions of  $\lambda$ ; so (by Cor 4.7) *qc-stability* implies stability.

In [MSS] and [Sul<sub>2</sub>], it is shown that *locally* the canonical holomorphic motion of the Julia set in a *qc*-stable family *extends* (noncanonically) to a holomorphic motion of all of  $\mathbf{P}^1$  *which is still a conjugacy*. In particular, all members of a (connected) *qc*-stable family are quasiconformal conjugate.

In addition, the monodromy of the Julia set's motion around a loop on  $S$  based at  $s$  can be extended to a quasiconformal self-conjugacy of the map  $R_s$ . So in a *qc*-stable family, the obstruction to globally labelling the Julia set is a map

$$\pi_1(S, s) \rightarrow C_{qc}(R_s)|J$$

where

$$C_{qc}(R) = \{ \text{quasiconformal maps } \phi: \mathbf{P}^1 \rightarrow \mathbf{P}^1: R\phi = \phi R \}$$

denotes the quasiconformal centralizer of a rational map  $R$ , and  $C_{qc}(R)|J$  denotes the restriction of these quasiconformal maps to the Julia set of  $R$ .

*Problem.* When is  $C_{qc}(R)|J$  a finite group?

If  $R$  is affine,  $J = \mathbf{P}^1$  and (as observed by M. Herman) the above group is  $\text{PSL}_2\mathbf{Z}$ . (All  $\phi$  lift to automorphisms of the torus, but the involution on the torus descends to the identity.) Hyperbolic maps are better behaved.

**PROPOSITION 8.1.** *If  $R$  is critically finite and hyperbolic, its conformal and quasiconformal centralizers agree when restricted to the Julia set. In particular,  $C_{qc}(R)|J$  is a finite group.*

*Proof.* Let  $\phi$  be a quasiconformal map which centralizes  $R$ . In Theorem 6.2 we proved that  $R$  is uniquely determined by its combinatorial type. The argument shows there is a Möbius transformation  $M$  in the conformal centralizer of  $R$  and in the same isotopy class as  $\phi$ , relative to the post-critical set  $P$ . ( $M$  is just the map of minimal dilatation in the same isotopy class as  $\phi$ .) We claim that  $M$  and  $\phi$  agree on the Julia set of  $R$ . To see this, let  $\phi_t$  be an explicit isotopy connecting  $\phi = \phi_0$  and  $M = \phi_1$ . This is an isotopy relative to  $P$ ; so  $M$  and  $\phi$  agree on  $P$ .

Construct a new isotopy by replacing each  $\phi_t$  with  $\psi_t = R^{-1} \circ \phi_t \circ R$ , where the branch of  $R^{-1}$  has been chosen so that  $\psi_0 = \phi_0 = \phi$  (then also  $\psi_1 = \phi_1 = M$ ). Now *this* is an isotopy relative to  $R^{-1}(P)$ , and hence  $\phi$  and  $M$  agree on this larger set. Replacing  $\psi_t$  with  $R^{-1} \circ \psi_t \circ R$  and continuing inductively, we deduce that  $\phi$  and  $M$  agree on the whole backwards orbit of  $P$ . But the backwards orbit of any point accumulates on the Julia set; therefore  $\phi|J = M|J$  as claimed.  $\square$

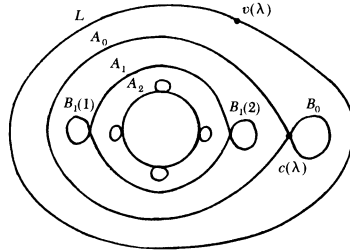


FIGURE A.1. DYNAMIC FOLIATIONS FOR  $z^3 + \lambda z^2$ ,  $\lambda \gg 0$ .

However, even if we ignore affine rational maps, there are still many qc-stable families for which the Julia set cannot be globally labelled on any finite covering of  $S$ . For example, consider the family of polynomials

$$P_\lambda(z) = z^3 + \lambda z^2.$$

For each  $\lambda$ , the map  $P_\lambda$  has four critical points: two at infinity, one at zero, and one at  $-2\lambda/3$ . The critical points at zero and infinity are fixed, while for  $\lambda$  sufficiently large, the remaining critical point is attracted to infinity.

Let  $\lambda$  vary through the region  $S$  consisting of those finite  $\lambda$  of absolute value  $\geq 10$  (such  $\lambda$  are certainly “sufficiently large”). Then  $S$  (which does not include  $\infty$ ) is isomorphic to the punctured disk, and  $P_\lambda$  forms a qc-stable family over  $S$ . Let  $s$  be any basepoint in  $S$ .

**PROPOSITION 8.2.** *The monodromy of the Julia set of  $P_s$ , as  $\lambda$  follows a loop based at  $s$  and passes once around the puncture of  $S$ , is of infinite order.*

*Remark.* Thus  $C_{qc}(P_s)|J$  contains an element of infinite order, and the Julia sets of  $P_\lambda$  cannot be globally labelled on any finite covering of  $S$ .

*Proof.* We begin by describing a picture of the dynamics of  $P_\lambda$  (Fig. A.1). A more detailed discussion of the dynamics of cubic polynomials is given by Blanchard in [B<sub>2</sub>].

When  $\lambda \in S$ , the critical point  $c(\lambda) = -2\lambda/3$  is attracted to infinity. The basin of attraction of infinity is foliated by dynamically defined leaves; the leaf through  $x$  consists of the closure of the set of  $y$  such that for some  $n$ ,  $P_\lambda^n(x) = P_\lambda^n(y)$ . Near infinity there is a coordinate chart on which  $P_\lambda$  becomes  $z \rightarrow z^3$ , and in this chart the leaves are actual circles centered at zero.

Topologically, circular shape persists until we reach the critical point  $c(\lambda)$ . The leaf through the critical point has the shape of a figure 8, with  $c(\lambda)$  at the crossing. One loop of the figure eight encloses the critical point at zero; call this

loop  $A_0$  and the other loop  $B_0$ . (The two loops are attached at  $c(\lambda)$ .) Under  $P_\lambda$ ,  $A_0$  maps (by degree two) and  $B_0$  (by degree one) to the smooth leaf  $L$  through the critical value  $v(\lambda) = R_\lambda(c(\lambda)) = 4\lambda^3/27$ .

Within  $A_0$  there is a loop  $A_1$ , enclosing zero and mapping by degree 2 to  $A_0$ , and *two* loops  $B_1(1)$  and  $B_1(2)$ , attached to  $A_1$  and mapping to  $B_0$ . Within  $A_1$  there is a similarly defined loop  $A_2$  with *four* loops  $B_2(1), \dots, B_2(4)$  attached, and in general we have a nested sequence of loops  $A_0, A_1, \dots, A_n$ , with  $2^n$  loops  $B_n(i)$  attached to  $A_n$ . The loop  $A_n$  and the loops at level  $n + 1$  bound a topological annulus which we denote by  $R_n$ .  $P_\lambda$  carries  $R_{n+1}$  to  $R_n$  by a degree-two covering map.

How does this configuration move as  $\lambda$  follows a simple loop around the puncture of  $S^2$ ? Clearly  $c(\lambda)$ , being proportional to  $\lambda$ , moves once around the origin and returns to its original position; and the monodromy map is the identity on  $(A_0, B_0)$ . Since the monodromy is a conjugacy, it is also the identity on the leaf  $L$  through the critical value  $v(\lambda)$ ; but since  $v(\lambda)$  is proportional to  $\lambda^3$ , it has moved *three* times around the origin. Thus on the annulus bounded by  $L$  and  $(A_0, B_0)$ , the monodromy is a Dehn twist of  $720^\circ$ .

This annulus is double-covered by the annulus  $R_0$  between  $A_0$  and  $(A_1, B_1(1), B_1(2))$ . Being a conjugacy, the monodromy on  $R_0$  is a  $360^\circ$  Dehn twist, and again the boundary of the annulus is fixed. But on  $R_1$  the twist is only  $180^\circ$ , so its inner boundary is not fixed; the loops  $B_2(i)$  undergo an order-two permutation. Similarly the twist through  $R_2$  is  $90^\circ$ , giving rise to a degree four permutation of the loops  $B_3(i)$ . At the  $n^{\text{th}}$  level, the  $2^n$  loops attached to  $A_n$  are not brought back to their original positions until the monodromy map has been iterated  $2^{n-1}$  times. But each loop  $B_n(i)$  encloses a portion of the Julia set, so that the monodromy on the Julia has infinite order.  $\square$

We have included this example to show that a certain *program* for the proof of rigidity (which would begin by passing to a finite cover on which the Julia set is labelled, as discussed in §4) cannot succeed in general.

A better picture of the quasiconformal centralizer would be useful in understanding *local* degenerations in qc-stable families. Our rigidity theorem guarantees the existence of bifurcations in purely iterative algorithms for the solution of polynomials of degree 4 or more; we expect that this behavior does not really depend upon the *global* properties of the algorithm that we have used, and is in fact a local phenomenon concentrated near the discriminant locus (where the roots collide). It is only common sense to suspect that an algorithm becomes confused as the roots are intertwined.

**Added in proof:** Quasiconformal automorphisms of rational maps are studied further in [Mc1], leading to a local obstruction to general convergence, phrased in the language of braids [Mc2].

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