PERCOLATION AND $O(1)$ LOOP MODEL

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Abstract. We present the "ultimate" proof of Cardy's formula for the critical percolation on the hexagonal lattice, showing the existence of the universal and conformally invariant scaling limit of crossing probabilities. The new approach is physically relevant, more conceptual, less technically demanding, and is amenable to generalizations. The original argument by the second author was never published, appearing only as an announcement without proof [23] and an unpublished preprint [24].

1. Introduction

Percolation was introduced by Broadbent and Hammersley [4] to model how a fluid spreads through a random medium. It is very easy to define: sites (or bonds) of a graph are declared open or closed independently (in Bernoulli percolation) with probabilities p and $1-p$ correspondingly, and connected open clusters are studied. Nevertheless, this percolation model exhibits a very rich and complicated behavior even on planar lattices, including a phase transition at some lattice-dependent value p_c.

In particular, the "crossing probability" (of the existence of an open cluster connecting two opposite sides of a fixed shape), as the mesh of the lattice tends to 0, tends to 0 when $p < p_c$ and tends to 1 when $p > p_c$ — a "sharp threshold phenomenon".

Meanwhile, for regular lattices, the Russo-Seymour-Welsh a priori estimates guarantee that for $p = p_c$ the "crossing probability" stays bounded away from 0 and 1, strongly suggesting the existence of a non-trivial "scaling limit".

In the seminal work [17] Langlands, Pouliot, and Saint-Aubin conducted a number of computer experiments suggesting that there is a universal (lattice-independent) scaling limit of the crossing probabilities at criticality which is furthermore conformally invariant, i.e. depends only on the conformal modulus of the quadrangular shape.

Almost immediately Cardy [5] derived (unrigorously) the exact formula for the limit as a hypergeometric function of the modulus, which Carleson observed to take a particularly nice form for an equilateral triangle with one more marked point on a side.

In 2000 the second author provided a rigorous proof of the Cardy's prediction for the critical percolation on the triangular lattice, which allowed to deduce many of its properties.

This proof has never appeared in a journal form not in the least because we felt it somehow artificial and having unexplained complications, albeit still elegant. The result was widely used to deduce various properties of percolation, such as the convergence of interfaces to SLE_6 and exact values of the critical exponents. It also stimulated an extremely fruitful: approach to study models by tools of discrete holomorphic or harmonic observables [18, 6, 8].

It took some time to arrive at what we think is "the proof from the Book", which we present in this article. On one hand, the new proof is more "ideologically fruitful": while it can be literally translated into the old one; the objects under consideration are classical disorder operators, rather than some curiosities of uncertain origins. The parafermionic nature of the observable and its relation to similar objects in the Ising and other models becomes clear, cf. [25, 7, 11]. On the other hand, the proof is much more straightforward. In particular, discrete holomorphicity becomes exact and there is no need to estimate error terms.

Moreover, the new description of the observable admits immediate generalizations allowing one to obtain several results (e.g. Schramm's formula [21] or formulae for the probabilities of the link patterns in the topological hexagon [9]) in the spirit of this article. We intend to show that in the subsequent papers [16, 15].

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Justification of Cardy's formula for graphs other than the hexagonal lattice remains an open problem and we have some hope that the new point of view could become useful there.

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1.1. **Percolation model.**

We will study critical site percolation on triangular lattice, or equivalently plaquette percolation on hexagonal lattice. Let $\mathbb{C}^\delta_{\hexagon}$ be a hexagonal lattice of mesh size δ on $\mathbb{C}$. A $\hexagon^\delta$-domain (hexagonal domain) is a bounded simply-connected domain glued from the faces of $\mathbb{C}^\delta_{\hexagon}$ and a $\hexagon$-domain is a domain that is $\hexagon^\delta$-domain for some δ. By $\mathcal{F}(\Omega)$ and $\mathcal{E}^{\text{half}}(\Omega)$ we denote the sets of faces and half-edges of a $\hexagon$-domain Ω respectively.

The percolation model on Ω is the uniform measure on the set of all $2^{\#\mathcal{F}(\Omega)}$ colorings of faces of Ω in two colors, say blue and yellow, we denote this measure by $\mathbb{P}^{\text{perc}}_\Omega$. For a given coloring

$$\sigma\colon \mathcal{F}(\Omega) \to \{\textbf{yellow}, \textbf{blue}\}$$

if there is a σ-blue path between two sets X and Y, we say that X and Y are *connected* and write $X \leftrightarrow Y$.

The scaling limits of probabilities to be connected in the percolation model are proven to exist and be conformally invariant. In this article we give a revised proof of the fundamental result in the area.

Theorem 1 (Smirnov'01, [23])**.** *If $\{(\Omega^\delta, A^\delta, B^\delta, C^\delta, D^\delta)\}_\delta$ approaches $(\Omega^\bullet, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ (in the sense of Definition 7) then*

$$\lim_{\delta\searrow 0} \mathbb{P}^{\text{perc}}_{\Omega^\delta}[\partial_{A^\delta B^\delta}\Omega^\delta \leftrightarrow \partial_{C^\delta D^\delta}\Omega^\delta] = \frac{\varphi(C^\bullet) - \varphi(D^\bullet)}{\varphi(C^\bullet) - \varphi(A^\bullet)}, \tag{1}$$

where φ is the conformal map from $\Omega^\bullet$ to an equilateral triangle, mapping $A^\bullet, B^\bullet, C^\bullet$ to vertices.

1.2. **Loop representation.**

For a collection of half-edges $\xi \subset \mathcal{E}^{\text{half}}(\Omega)$ we denote by $\partial\xi$ the set of vertices and mid-edges of Ω that are adjacent to an odd number of half-edges of ξ.

Let $U = \{u_1, \dots u_k\}$ be a set of k mid-edges of Ω, we call them *marked points*. We define

$$W_\Omega(u_1, \dots u_k) := W_\Omega(U) := \left\{\xi \subset \mathcal{E}^{\text{half}}(\Omega) : \partial\xi = U\right\}$$

and call elements of $W_\Omega(u_1, \dots u_k)$ *loop configurations with disorders at marked points.* Assume that k is even, then this set in non-empty.

Let ξ be such a loop configuration. The union of half-edges of ξ still will be denoted by ξ. The union of connectivity components of ξ containing at least one marked point we denote by $\mathcal{IP}(\xi)$ and call the *Interface Part* of ξ. Note that $\xi \setminus \mathcal{IP}(\xi)$ is a union of disjoint loops and $\mathcal{IP}(\xi)$ is a union of disjoint paths, matching marked points. This matching is called *Link Pattern* of ξ.

By $\mathbb{P}^{\text{loop}}_{\Omega,U}$ we denote the uniform measure on $W_\Omega(U)$. Note that $\mathbb{P}^{\text{loop}}_{\Omega,\emptyset}$ corresponds to the loop $O(1)$ model (or, equivalently, the Ising model at the infinite temperature). The matter of our interest is the law of the link pattern of the uniformly random loop configuration with disorders at marked points. Note that if ξ_1 and ξ_2 are loop configurations with the same disorders, then the symmetric difference $\xi_1 \oplus \xi_2$ is a union of loops. This implies that there are exactly $2^{\#\mathcal{F}(\Omega)}$ loop configurations with given disorders.

If z and w are two points on $\partial\Omega$ we denote by $\partial_{zw}\Omega$ the counterclockwise arc of $\partial\Omega$ from z to w. When $u_1, \dots, u_m$ are defined as points lying on the boundary of Ω we always mean that they go in the counterclockwise order and are indexed cyclically: $u_{n\pm m} := u_n$. For $j, j' \in \mathbb{Z}$ we use shorthands $\partial_{jz}\Omega := \partial_{u_j z}\Omega$, $\partial_{zj}\Omega := \partial_{z u_j}\Omega$, $\partial_{jj'}\Omega := \partial_{u_j u_{j'}}\Omega$. Additionally, if $m = 2l+1$ is odd then $\partial_j\Omega := \partial_{u_{j+l} u_{j-l}}\Omega$.

Lemma 2. *Let $u_1, \dots, u_4$ be four distinct mid-edges on $\partial\Omega$. Then there are two possible link patterns of a loop configuration ξ: either u_1 is linked to u_2 and u_3 to u_4 in $\mathcal{IP}(\xi)$ or u_1 is linked to u_4 and u_2*

to u_3. We denote the corresponding events by $[u_1 \leftrightsquigarrow u_2, u_3 \leftrightsquigarrow u_4]$ and $[u_1 \leftrightsquigarrow u_4, u_2 \leftrightsquigarrow u_3]$. Then

$$\mathbb{P}^{\mathrm{perc}}_{\Omega}[\partial_{u_1u_2}\Omega \leftrightarrow \partial_{u_3u_4}\Omega] = \mathbb{P}^{\mathrm{loop}}_{\Omega,\{u_1,u_2,u_3,u_4\}}[u_1 \leftrightsquigarrow u_4, u_2 \leftrightsquigarrow u_3].$$

Proof. For a coloring σ one can construct a loop configuration $\xi = \xi(\sigma)$ with disorders at $u_1, \dots u_4$ by the following rule: a half-edge e belongs to $\xi(\sigma)$ if and only if the colors on the left and on the right of e differ, (see Figure 1). Here we assume that the outer boundary is blue along $\partial_{12}\Omega$ and $\partial_{34}\Omega$ and is yellow along $\partial_{23}\Omega$ and $\partial_{41}\Omega$. This map is a bijection between colorings and loop configurations with disorders at $u_1, \dots u_4$, moreover $\partial_{u_1u_2}\Omega \leftrightarrow \partial_{u_3u_4}\Omega$ in σ if and only if $[u_1 \leftrightsquigarrow u_4, u_2 \leftrightsquigarrow u_3]$ in $\xi(\sigma)$. □

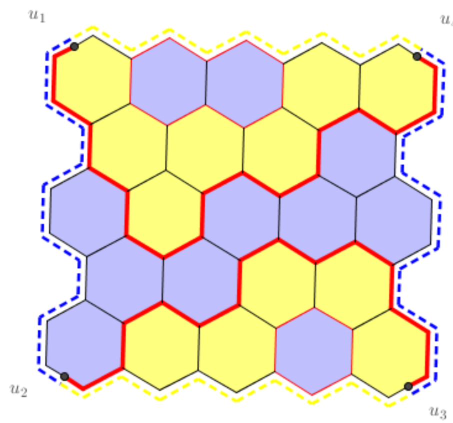

FIGURE 1. Here σ is drawn in blue and yellow and ξ in red; $\mathcal{IP}(\xi)$ is thick and outer boundaries are dashed.

1.3. Spinor percolation model.

Lemma 2 shows the correspondence between loop configurations with disorders on the boundary and colorings in two colors. One can naturally generalize this correspondence for the case when the disorders are allowed to lie inside the domain. Indeed, let $u_1, \dots u_k$ be mid-edges of Ω, and let $\rho\colon \tilde{\Omega}_{u_1,\dots u_k} \to \Omega$ be the double covering of Ω ramified at each u_j, so $\tilde{\Omega}_{u_1,\dots u_k}$ includes two copies of each face of Ω. A *spinor coloring* is a map

$$\sigma\colon \mathcal{F}(\tilde{\Omega}_{u_1,\dots u_k}) \to \{\mathbf{yellow}, \mathbf{blue}\}$$

such that two ρ-preimages of any face of Ω have different colors. Note that if each u_j lies on the boundary then $\tilde{\Omega}_{u_1,\dots u_k}$ has the same structure of faces and mid-edges as the disjoint union of two copies of Ω. If σ is a spinor coloring and $\tilde{\xi}(\sigma)$ is the set of half-edges such that σ-colors on the left and the right of it differ, then $\xi = \rho(\tilde{\xi}(\sigma))$ is a loop configuration with disorders at $u_1, \dots u_k$, the vice-versa is also true.

The spinor percolation model is the uniform measure on the set of all spinor colorings. There are several immediate advantages of working with it. In particular, the interfaces can be sampled by the standard revealment process (and those processes can be naturally coupled for models on the same domain with different disorders until the moment when the interface 'disconnects disorders').

2. DISCRETE HOLOMORPHICITY

Let u_1, u_2, u_3 be three distinct mid-edges lying in the counterclockwise order on $\partial\Omega$ and z be any mid-edge distinct from them. There are three possible link patterns for a loop configuration with disorders at u_1, u_2, u_3, z. If z is connected to u_j and u_{j-1} is connected to u_{j+1} by the edges of $\mathcal{IP}(\xi)$ we say that the event $[z \leftrightsquigarrow u_j] = [z \leftrightsquigarrow u_j, u_{j-1} \leftrightsquigarrow u_{j+1}]$ occurs.

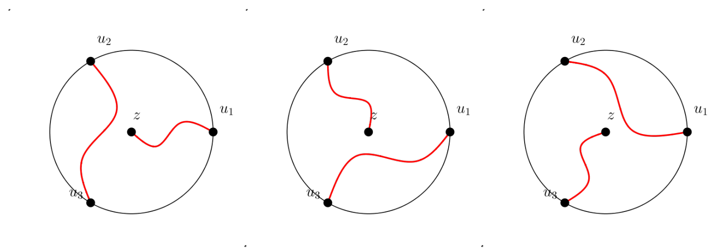

FIGURE 2. Link patterns $[z \leftrightsquigarrow u_1]$, $[z \leftrightsquigarrow u_2]$, $[z \leftrightsquigarrow u_3]$.

Definition 3. We set $\tau := \exp(2\pi i/3)$. Let u_1, u_2, u_3 be three distinct mid-edges lying in the counterclockwise order on $\partial\Omega$. By $\mathcal{E}^{\mathrm{mid}}_\circ(\Omega)$ we denote the set of all mid-edges of Ω *except for* u_1, u_2, u_3. Then the *observable* is a function $F = F_{\Omega,u_1,u_2,u_3} : \mathcal{E}^{\mathrm{mid}}_\circ(\Omega) \to \mathbb{C}$ given by the formula

$$F(z) := \mathbb{E}^{\mathrm{loop}}_{\Omega,u_1,u_2,u_3,z}[H(\xi)] = \sum_{j=1}^{3} \tau^j H_j(z), \tag{2}$$

where $H(\xi) = \sum_{j=1}^{3} \tau^j \mathbf{1}_{[z \leftrightsquigarrow u_j]}$ and $H_j(z) = \mathbb{P}^{\mathrm{loop}}_{\Omega,u_1,u_2,u_3,z}[z \leftrightsquigarrow u_j]$.

Lemma 4 (Discrete holomorphicity). *Let $z_1, z_2, z_3 \in \mathcal{E}^{\mathrm{mid}}_\circ(\Omega)$ be three mid-edges around a vertex v indexed in the counterclockwise order, then*

$$\sum_{k=1}^{3} \tau^k F(z_k) = 0. \tag{3}$$

Proof. We group loop configurations from $\cup_{z \in \{z_1,z_2,z_3\}} W(u_1, u_2, u_3, z)$ in triples such that any two loop configurations in the same triple differ by two half-edges adjacent to v (See Fig. 3). Each triple contributes zero to

$$\sum_{k=1}^{3} \tau^k \sum_{\xi \in W_\Omega(u_1,u_2,u_3,z_k)} H(\xi).$$

$\square$

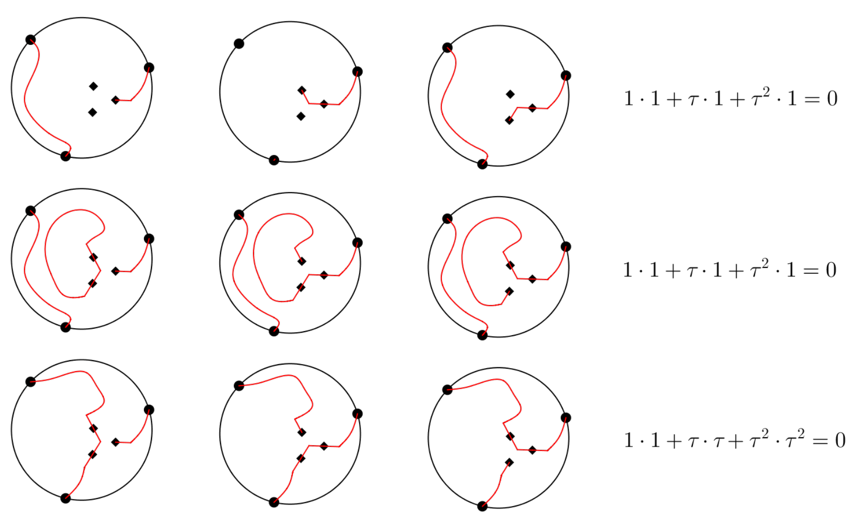

FIGURE 3. Graphical proof of Lemma 4. Mid-edges z_1, z_2, z_3 are marked with diamonds and u_1, u_2, u_3 with circles. Configurations are grouped horizontally.

Corollary 5. *Let γ be a dual contour, i.e. a sequence $(w_0, w_1, \dots w_n = w_0)$ of distinct faces where any two consecutive faces w_j and w_{j+1} share exactly one edge e_j. Then the discrete integral of F*

along γ defined by the formula

$$\int\limits_{\gamma}^{\#} F(z)\,d^{\#}z := \sum_{j=0}^{n-1} F(e_j)(w_{j+1}^{\circ} - w_j^{\circ})$$

(here w_j° stands for the center of w_j) vanishes.

Proof. For an elementary contour (i.e. that consists of three faces adjacent to the same vertex) the equality follows from (3). Since any contour can be decomposed into a union of elementary ones and the discrete integration is additive w.r.t contour, the corollary is also true for arbitrary contour. □

Remark 6. *The functions H_1, H_2, H_3 can also be defined on the vertices, though an interface can now arrive from three possible directions. Apparently, that would give exactly the same functions H_1, H_2, H_3 as were defined in [23] and F as was defined in [2] under name h. The Aizenman-Duplantier-Aharony recoloring [1] used in [23] corresponds to the last triple in the Figure 3.*

In terms of observable F Lemma 2 says that if mid-edges u_1, u_2, u_3 lie on the boundary of Ω and a mid-edge z lies on the boundary arc $\partial_j\Omega$ then

$$F(z) = \mathbb{P}_{\Omega}^{\mathrm{perc}}[\partial_{j+1,z}\Omega \leftrightarrow \partial_{j-1,j}\Omega] \cdot \tau^{j-1} + \mathbb{P}_{\Omega}^{\mathrm{perc}}[\partial_{j,j+1}\Omega \leftrightarrow \partial_{z,j-1}\Omega] \cdot \tau^{j+1} \subset [\tau^{j-1}, \tau^{j+1}]. \qquad (4)$$

3. Theorem 1 for the Jordan case

We denote by $\mathbb{T}$ the open domain bounded by the regular triangle with vertices $1, \tau, \tau^2$. For a simply-connected domain U with three chosen prime ends A, B, C we denote by $\varphi_{U;A,B,C}$ the conformal map from U to $\mathbb{T}$ that maps A, B, C to $\tau, \tau^2, \tau^3 = 1$ respectively.

Definition 7. Let $\Omega^\bullet \subset \mathbb{C}$ be a bounded simply-connected domain and $A^\bullet$, $B^\bullet$, $C^\bullet$, $D^\bullet$ be prime ends of $\Omega^\bullet$ lying in the counterclockwise order. Let $\{(\Omega^\delta, A^\delta, B^\delta, C^\delta, D^\delta)\}_\delta$ parametrized by $\delta \searrow 0$ be a sequence such that Ω^δ is a $\hexagon^\delta$-domain, $A^\delta, B^\delta, C^\delta, D^\delta$ are boundary mid-edges of Ω^δ. We say that the sequence $(\Omega^\delta, A^\delta, B^\delta, C^\delta, D^\delta)$ *approaches* $(\Omega^\bullet, A^\bullet, B^\bullet, C^\bullet, D^\bullet)$ if Assumption 1 or Assumption 2 holds, see below.

Assumption 1. *$\partial\Omega^\bullet$ is a Jordan curve, Ω^δ is the $\hexagon^\delta$-domain lying inside $\Omega^\bullet$ of the maximal area and $A^\delta, B^\delta, C^\delta, D^\delta$ are the boundary mid-edges of Ω^δ closest to $A^\bullet$, $B^\bullet$, $C^\bullet$, $D^\bullet$ respectively.*

Proof of Theorem 1 under Assumption 1. Let F_δ be defined by the formula (2) for $(\Omega, u_1, u_2, u_3) = (\Omega^\delta, A^\delta, B^\delta, C^\delta)$. We denote by f_δ the piecewise linear extension of F_δ defined as follows. First, define f_δ on centers, mid-edges and vertices of all the hexagons intersecting Ω by $f_\delta(u) := F_\delta(u^\delta)$, where u^δ the mid-edge of $\mathcal{E}_\circ^{\mathrm{mid}}(\Omega^\delta)$ closest to u (if there are several closest mid-edges we choose one arbitrary). Then extend f_δ linearly to each triangle spanned by adjacent vertex, mid-edge and center of a face.

Lemma 9 implies that the family $\{f_\delta\}_\delta$ is uniformly Hölder on any $K \Subset \Omega^\bullet$. Moreover, since $\Omega^\bullet$ is Jordan, it is locally connected: there exists $\zeta(\cdot) = o(1)$ near 0 such that any two points $x, y \in \Omega^\bullet$ can be joined inside $\Omega^\bullet$ by a curve of a diameter at most $\zeta(|x-y|)$. So from Lemma 9 we can derive that the family $\{f_\delta\}_\delta$ is equicontinuous on $\overline{\Omega^\bullet}$.

By Arzelà–Ascoli theorem there is a continuous function $f\colon \overline{\Omega^\bullet} \to \mathbb{C}$ and a sequence $\{\delta_n\}_n$ converging to 0 such that $f_{\delta_n} \rightrightarrows f$ on $\overline{\Omega^\bullet}$. Let $\gamma \Subset \Omega^\bullet$ be any rectangular contour and let γ_{δ_n} be a dual contour of the maximal area lying inside γ. Then

$$\int\limits_{\gamma} f(z)\,dz = \lim_{n\to\infty} \int\limits_{\gamma_{\delta_n}} f(z)\,dz = \lim_{n\to\infty} \int\limits_{\gamma_{\delta_n}}^{\#} F_{\delta_n}(z)\,d^{\#}z = 0,$$

so f is holomorphic by Morera's theorem.

From (4) we conclude that f maps $\partial_j\Omega^\bullet$ to $\partial_j\mathbb{T} = [\tau^{j+1}, \tau^{j-1}]$. The argument principle implies that $f = \varphi_{\Omega^\bullet, A^\bullet, B^\bullet, C^\bullet} =: \varphi$, so all subsequential limits of $\{f_\delta\}$ coincide. Again using (4) we find that

$$\lim_{\delta\searrow 0} \mathbb{P}_{\Omega^\delta}^{\mathrm{perc}}[\partial_{A^\delta B^\delta}\Omega^\delta \leftrightarrow \partial_{C^\delta D^\delta}\Omega^\delta] = \lim_{\delta\searrow 0} \frac{\varphi(C^\bullet) - F_\delta(D^\delta)}{\varphi(C^\bullet) - \varphi(A^\bullet)} = \frac{\varphi(C^\bullet) - \varphi(D^\bullet)}{\varphi(C^\bullet) - \varphi(A^\bullet)}.$$

□

4. A priori estimates

Our work requires only one non-trivial result on percolation: the famous Russo-Seymour-Welsh estimate. We state it in the following way:

Proposition 8 (RSW estimate)**.** *There exist $\eta > 0$ and $C_{\mathrm{RSW}} > 0$ such that for any $r < R$ and for any δ*

$$\mathbb{P}^{\mathrm{perc}}{}_{\mathbb{C}^\delta_{\hexagon}}[\partial B_r \leftrightarrow \partial B_R] < C_{\mathrm{RSW}}(r/R)^\eta.$$

In order to make the proofs work for domains with possibly complicated boundaries, we define a metric on the closure $\overline{U}$ of a Jordan domain U by formula

$$\rho_U(x, y) := \inf\{\operatorname{diam}\gamma : \gamma \subset \overline{U} \text{ is a curve from } x \text{ to } y\}$$

and formulate Lemma 9 in terms of this metric. To prove Theorem 1 for the case when $\Omega^\bullet$ is smooth one can use the Euclidian metric instead of it.

Lemma 9 (Hölder continuity)**.** *There exist $\eta, C > 0$ such that the following holds. Let Ω be a $\hexagon^\delta$-domain with three marked boundary mid-edges v_1, v_2, v_3. Assume that a set S is such that $\overline{\Omega} \setminus S$ has a path-connected component, containing two mid-edges $x, y \in \mathcal{E}^{\mathrm{mid}}_\circ(\Omega)$ and at most one marked mid-edge, then*

$$\forall j \quad |H_j(x) - H_j(y)| < C\left(\frac{\operatorname{diam} S}{R}\right)^\eta, \tag{5}$$

where $R = \max_k \rho_\Omega(S, \partial_k\Omega)$.

Proof. See Figure 4. We start by assuming that $R/100 > 100 \operatorname{diam} S > \delta$, otherwise Lemma follows by chosing large enough C. Without loss of generality $R = \rho_\Omega(S, \partial_3\Omega)$, so v_1, v_2 are outside of the path connected component of $\overline{\Omega} \setminus S$ that contains x, y. Let $\tilde{S}$ be the (10δ)-neighborhood of S with respect to ρ_Ω. We choose a $\hexagon$-path $[xy]$ such that no path joining $[xy]$ and $\{v_1, v_2\}$ is disjoint from $\tilde{S}$. Clearly, the LHS of (5) is bounded by $\mathbb{P}^{\mathrm{loop}}_{\Omega,v_1,v_2,v_3,x}[H(\xi) \neq H(\xi \oplus [xy])]$, and let us call configurations ξ such that the last event occurs *bad* and denote the set of bad configurations by W^{bad}.

If $\xi \in W^{\mathrm{bad}}$, then $[xy]$ should be connected to each of v_1, v_2, v_3 by edges of ξ; so $\tilde{S}$ is connected to v_1 and v_2 by edges of ξ. Let $\beta(\xi)$ be the minimal subset of ξ that connects $\tilde{S}$ to v_1 and v_2 (this is a union of two paths). Now note that there is a path in $\mathcal{E}^{\mathrm{half}}(\Omega) \setminus \beta(\xi)$ from x to u_3. For each $\xi \in W^{\mathrm{bad}}$ we choose such path $\alpha(\beta(\xi))$ in any way depending on $\beta(\xi)$ but not on ξ itself.

Note that the map $\xi \mapsto \xi \oplus \alpha(\beta(\xi))$ is injective on W^{bad}. Moreover, if $\xi \in W^{\mathrm{bad}}$, then $\xi \oplus \alpha(\beta(\xi)) \oplus \partial_3\Omega$ is a loop configuration without disorders that contains a loop touching $\tilde{S}$ and $\partial_3\Omega$. Then the corresponding coloring defined as in the proof of Lemma 2 (assuming that the outer boundary of Ω is yellow) contains a monochromatic path between $\tilde{S}$ and $\partial_3\Omega$. Since the diameter of any such path is at least $R/2$, we conclude by the RSW estimate. □

5. Theorem 1 for the general case

In this section we work under the following assumption, which is more general than Assumption 1.

Assumption 2 (convergence in the Carathéodory sense)**.** *$\Omega^\bullet$ is an arbitrary bounded simply-connected domain and the following properties hold:*

- *Any K such that $K \Subset \Omega^\bullet$ is contained in Ω^δ for δ small enough;*
- *$\varphi^{-1}_{\Omega^\delta;A^\delta,B^\delta,C^\delta} =: \varphi^{-1}_\delta$ converges to $\varphi^{-1}_{\Omega^\bullet;A^\bullet,B^\bullet,C^\bullet} =: \varphi^{-1}$ uniformly on any compact $K \Subset \mathbb{T}$;*
- *$\varphi_{\Omega^\delta;A^\delta,B^\delta,C^\delta}(D^\delta)$ converges to $\varphi_{\Omega^\bullet;A^\bullet,B^\bullet,C^\bullet}(D^\bullet)$;*
- *$\cup_\delta\Omega^\delta$ is bounded.*

Proof of Theorem 1 for the general case. As in the proof for the Jordan case, using Lemma 9 we define functions f_δ and find a sequence $\{\delta_n\}_n$ converging to 0 and a holomorphic function f on $\Omega^\bullet$ such that

$$f_{\delta_n} \rightrightarrows f \text{ on any } K \Subset \Omega^\bullet. \tag{6}$$

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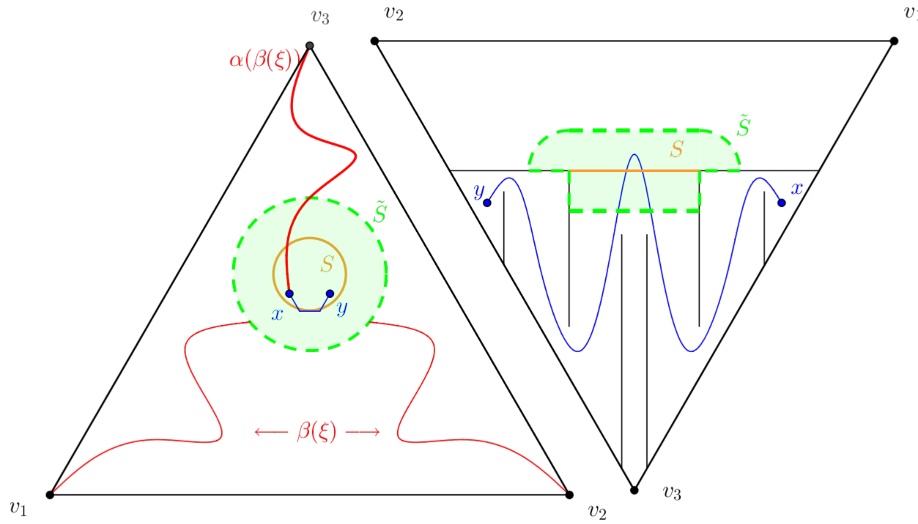

FIGURE 4. Two examples of a ⬡-path $[xy]$ (in blue), sets S (in orange) and $\tilde{S}$ (in green) and ⬡-paths $\alpha(\beta(\xi)), \beta(\xi)$ (in red, only on the left). In general, S either surrounds x and y as on the left, or cuts away the part of Ω containing them, as on the right. On the right we sketch how such cuts would look if $\Omega^\bullet$ is non-Jordan.

To analyze its boundary behavior in the prime end ('Carathéodory') compactification, we extend $\varphi_{\delta_n}^{-1}$ to $\overline{\mathbb{T}}$ by continuity and note that the sequence $f_{\delta_n} \circ \varphi_{\delta_n}^{-1} \colon \overline{\mathbb{T}} \to \mathbb{C}$ uniformly converges to $f \circ \varphi^{-1}$ on any compact subset of $\mathbb{T}$. Then we aim to show that this sequence is equicontinuous on $\overline{\mathbb{T}}$.

For $x, y \in \overline{\mathbb{T}}$ and any δ we consider the set of simple (possibly closed) curves $\gamma \subset \overline{\Omega^\delta}$ such that $\overline{\Omega^\delta} \setminus \gamma$ consist of exactly two path-connected components, one containing $\varphi_\delta^{-1}(x), \varphi_\delta^{-1}(y)$ and another one containing at least two of marked points $A^\delta, B^\delta, C^\delta$ and denote by $\tilde{\rho}_\delta(x,y)$ the infimum of lengths of those curves. By estimating the extremal length one can easily show that

$$\lim_{\epsilon \searrow 0} \sup_{\delta} \sup_{x,y \in \overline{\mathbb{T}}, |x-y|<\epsilon} \tilde{\rho}_\delta(x,y) = 0. \tag{7}$$

Now we note that the family $\{\Omega^\delta\}_\delta$ is non-degenerate in the following sense:

$$r^\bullet := \liminf_{\delta \searrow 0} \inf_{x \in \Omega^\delta} \max_k \rho_{\Omega^\delta}(x, \partial_k \Omega^\delta) > 0. \tag{8}$$

Indeed, assume the contrary, then for any δ there exists a path-connected $Y_\delta \subset \overline{\Omega^\delta}$ touching all three boundary arcs of Ω^δ such that $\liminf \operatorname{diam} Y_\delta = 0$ as $\delta \to 0$. Let O be the center of $\mathbb{T}$ and set $O_\delta := \varphi_\delta^{-1}(O)$. Since φ_δ^{-1} uniformly converges to φ^{-1} on some open neighborhood of O, the distance from O_δ to $\partial\Omega^\delta$ is bounded from below, so $\liminf \operatorname{dist}(O_\delta, Y_\delta) > 0$. At the same time the harmonic measures with the pole at O_δ of arcs $\partial_{A^\delta,B^\delta}\Omega^\delta$, $\partial_{B^\delta,C^\delta}\Omega^\delta$, $\partial_{C^\delta,A^\delta}\Omega^\delta$ equal to $1/3$. Since one of those arcs is separated from O_δ by Y_δ, (8) is proven by contradiction.

Now for $x, y \in \overline{\mathbb{T}}$ at a small distance, for any δ we can disconnect $\varphi_\delta^{-1}(x), \varphi_\delta^{-1}(y)$ from at least two marked points by a curve γ of small diameter by (7). Then we plug $S = \gamma$ in (5) and estimate the denominator in the RHS of (5) by the triangle inequality $\max_k \rho_{\Omega^\delta}(\gamma, \partial_k\Omega^\delta) \geq \inf_{x\in\Omega^\delta} \max_k \rho_{\Omega^\delta}(x, \partial_k \Omega^\delta) - \operatorname{diam}\gamma$ and (8). From that we conclude the sequence $f_{\delta_n} \circ \varphi_{\delta_n}^{-1}$ is equicontinuous on $\overline{\mathbb{T}}$. As in the proof for the Jordan case, it follows from (4) that $f \circ \varphi^{-1}$ maps $\partial_j \mathbb{T} = [\tau^{j+1}, \tau^{j-1}]$ to itself, so by the argument principle

$$f_\delta \circ \varphi_\delta^{-1} \rightrightarrows f \circ \varphi^{-1} = id \text{ on } \overline{\mathbb{T}}$$

which in turn implies that

$$\lim_{\delta \searrow 0} \mathbb{P}^{\text{perc}}_{\Omega^\delta}[\partial_{A^\delta B^\delta}\Omega^\delta \leftrightarrow \partial_{C^\delta D^\delta}\Omega^\delta] = \lim_{\delta \searrow 0} \frac{\varphi(C^\bullet) - (f_\delta \circ \varphi_\delta^{-1}) \circ \varphi_\delta(D^\delta)}{\varphi(C^\bullet) - \varphi(A^\bullet)} = \frac{\varphi(C^\bullet) - \varphi(D^\bullet)}{\varphi(C^\bullet) - \varphi(A^\bullet)}.$$

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