FRACTAL ASPECTS OF THE ITERATION OF $z \rightarrow \lambda z(1 - z)$ FOR COMPLEX λ AND z

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INTRODUCTION

Given a mapping $z \rightarrow f(z, \lambda)$, where f is a rational function of both z and λ, *consider the iterated maps* $z_n = f(f(\ldots f(z_0)\ldots))$ *of the starting point* z_0. To achieve a global understanding of these iterates' behavior, it is necessary to allow λ and z_0, hence z_n also, to be complex variables. Contrarily, the extensive recent studies of the mapping $z \rightarrow \lambda z(1 - z)$, for example, those found in Reference 1 and in the present volume, are largely restricted to λ real $\in [1, 4]$ and z real $\in [0, 1]$. Hence, they are powerful but local and incomplete. The global study for unrestricted complex λ and z throws fresh light upon the results of these restricted studies, and reveals important new facts. In this light, an immediate change of emphasis from the restricted studies to even more general mappings $\{x \rightarrow f(x, y, \lambda);\ y \rightarrow g(x, y, \lambda)\}$ appears to be premature.

The present paper stresses the role played in the unrestricted study of rational mappings by diverse fractal sets, including λ-fractals (sets in the λ plane), and z-fractals (sets in the z plane). Some are fractal curves (of topological dimension 1), and others are fractal "dusts" (of topological dimension 0). The z-fractals are of special interest, since they can be interpreted as the fractal attractors of appropriately defined (generalized) discrete dynamical systems, based upon inverse mappings. This role is foreshadowed in the work of P. Fatou[2,3] and of G. Julia[4] (and even that of H. Poincaré, in the related context of Kleinian groups), but the topic was never pursued. Indeed, an explicit and systematic concern with fractals only came with my book,[5] in which, for the first time, the notion itself was defined and given a name: A fractal set is one for which the fractal (Hausdorff-Besicovitch) dimension strictly exceeds the topological dimension. This paper's illustrations are fresh (and better than ever) examples of what this definition implies intuitively. The text is a summarized excerpt from Reference 6. A related excerpt concerning the fractal attractors of Kleinian groups is Reference 7. The final section comments on "strange" attractors.

THE λ-FRACTAL Q

We denote by Q the set of values of λ with the property that the initial points z_0 for which l.u.b. $|z_n| < \infty$ include a closed domain (that is, a set having interior points). It is well known that it suffices that l.u.b. $|z_n| < \infty$ hold when the initial point is the "critical" point $z_0 = 0.5$. The portion of Q for which $\text{Re}(\lambda) > 1$ is illustrated in FIGURE

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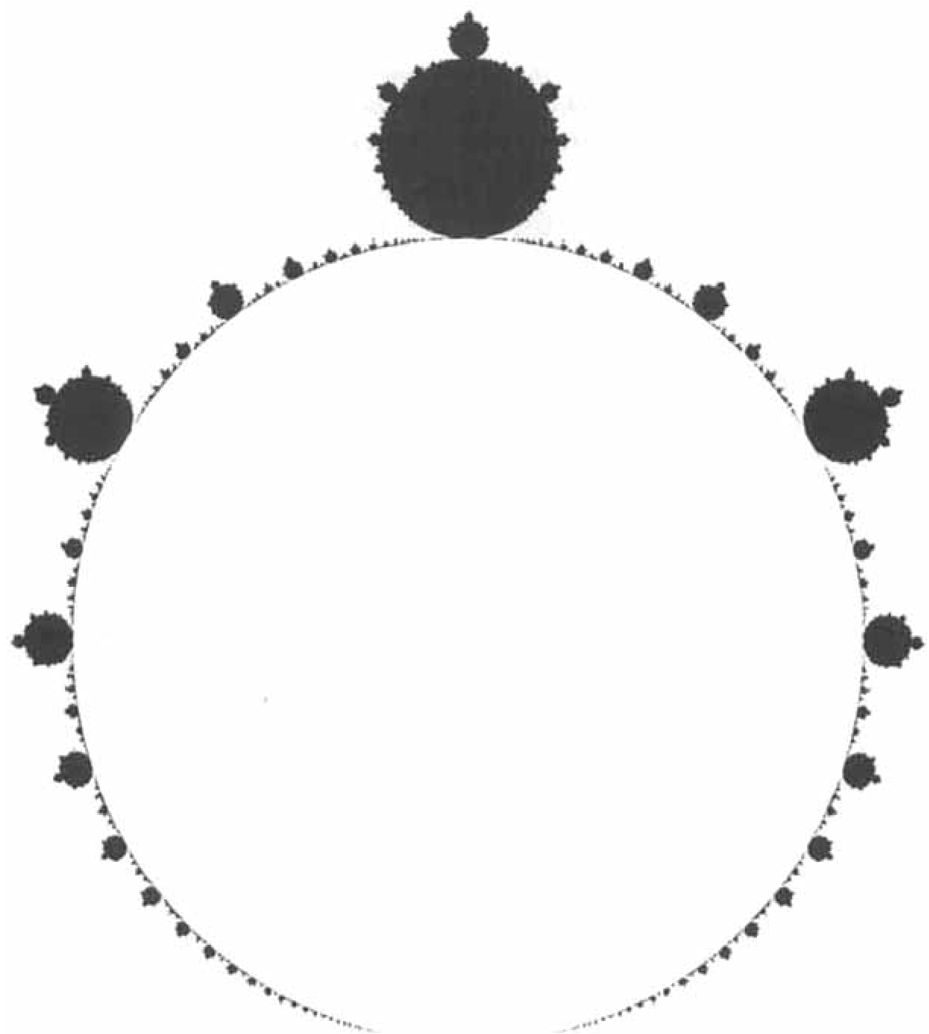

FIGURE 1. Complex plane map of the λ-domain Q. The real axis of the λ-plane points up from $\lambda = 1$. The center of the circle is $\lambda = 2$ and the tip of the whole is $\lambda = 4$.

1, the remainder of Q being symmetric to this figure with respect to the line $\mathrm{Re}(\lambda) = 1$.

A striking fact, which I think is new, becomes apparent here: FIGURE 1 is made of several disconnected portions, as follows.

The Domain of Confluence $\mathcal{L}$, and Its Fractal Boundary

The most visible feature of FIGURE 1 is the large connected domain $\mathcal{L}$ surrounding $\lambda = 2$. This $\mathcal{L}$ splits into a sequence of subdomains one can introduce in successive stages.

The first stage subdomain, $\mathcal{L}_0$, is constituted by the point $\lambda = 1$ plus the open disc $|\lambda - 2| < 1$ (left blank on FIGURE 1 to clarify the remainder's structure). Iff $\lambda \in \mathcal{L}_0$, there is a finite stable fixed point. (Proof: When $\mathrm{Re}\lambda > 1$, the stable limit point, if it exists, is $1 - 1/\lambda$; the condition $|f'(1 - 1/\lambda)| < 1$ boils down to $|\lambda - 2| < 1$.)

The remaining, and truly interesting, portion of $\mathcal{L}$ is shown in black on FIGURE 1. The fact that this black area is "small" means that the mapping $z \to \lambda z(1 - z)$ is mostly not bizarre. However, many interesting and bizarre behaviors (some of them unknown so far, and others thought to be associated with much more complex transformations) are obtained here in small but nonvanishing domains of λ.

Each of the second stage subdomains of $\mathcal{L}$ is indexed by one or several rational numbers α/β. The subdomain $\mathcal{L}(\alpha/\beta)$ is open, except that we include in it the limit point where it attaches "sprout"-like to $\mathcal{L}_0$; this is the point $\lambda - 2 = -e^{-i\theta} = -\exp[-2\pi i(\alpha/\beta)]$. When $\lambda \in \mathcal{L}(\alpha/\beta)$, the sequence z_n has a stable limit cycle of period β. This cycle can be obtained through a single β-fold bifurcation by a continuous change of λ that starts with any stable fixed point, for example, with the stable fixed point $z_0 = 0.5$ corresponding to $\lambda_0 = 2$.

Each of the third stage subdomains of $\mathcal{L}$ is indexed by two rational numbers: $\mathcal{L}(\alpha_1/\beta_1, \alpha_2/\beta_2)$; it is open, save for the point where it attaches, again sprout-like, to $\mathcal{L}(\alpha_1/\beta_1)$. When $\lambda \in \mathcal{L}(\alpha_1/\beta_1, \alpha_2/\beta_2)$, the sequence z_n has a stable limit cycle of period $\beta_1\beta_2$ resulting from two successive bifurcations, respectively β_1-fold and β_2-fold, which started with a stable fixed point in $\mathcal{L}_0$.

Further series of subdomains are similarly indexed by increasingly many rational numbers $\alpha_1\beta_1 \ldots \alpha_g/\beta_g$. $\mathcal{L}$ combines all the values of λ that lead either to stable limit points of z_n or to stable limit cycles that can be reduced to stable limit points by the inverse of the bifurcation process. I propose for this process the term confluence, which is why I call $\mathcal{L}$ the domain of confluence.

The domains $\mathcal{L}(\alpha/\beta)$ etc. are nearly disc shaped, but not precisely so. More generally, the boundary of each sprout is nearly a reduced scale version of the whole boundary of $\mathcal{L}$. Recalling the classic construction of the "snowflake curve,"[5] one can have little doubt but that the boundary of $\mathcal{L}$ is a fractal curve.

The Transformed Domain $\mathcal{M}$

Using the often invoked transformed variable $w = (2z - 1)\lambda/2$ re-expresses the mapping $z \to \lambda z(1 - z)$ into $w \to \mu - w^2$, where $\mu = (\lambda^2/4) - (\lambda/2)$. This leads to the replacement of the λ-set $\mathcal{L}$ by a μ-set $\mathcal{M}$. The counterpart to the discs $|\lambda - 2| \leq 1$ ($\equiv \mathcal{L}_0$) and $|\lambda| \leq 1$ (the symmetric of $\mathcal{L}_0$ with respect to $\mathrm{Re}(\lambda) = 1$) is a shape $\mathcal{M}_0$ bounded by the fourth order curve of equation $\mu = e^{2i\phi}/4 - e^{i\phi}/2$. The sets $\mathcal{M}$ and $\mathcal{M}_0$ will be needed momentarily. Hence, the scholars' familiar hesitation between the notations involving λ or μ is not resolved here: the shape $\mathcal{L}_0$ is far simpler than $\mathcal{M}_0$, but $\mathcal{M}$ is more useful than $\mathcal{L}$.

The Domains of Nonconfluent, or K-Confluent, Stable Cycles

In addition to $\mathcal{L}$, the domain Q is made of many smaller subdomains. Indeed, I discovered that at least some of what are, apparently, specks of dirt or ink on FIGURE 1

are indeed real: more detailed maps reveal a well-defined "island" whose shape is like that of $\mathcal{M}$, except for a nonlinear one-to-one deformation. Each island is, in turn, accompanied by subislands, doubtless ad infinitum.

When λ lies in an island's deformed counterpart to $\mathcal{L}_0$, z_n has a stable limit cycle of period $\omega > 1$. When λ lies in an island's deformed counterpart to $\mathcal{L}(\alpha_1/\beta_1, \ldots \alpha_g/\beta_g)$, z_n has a limit stable cycle of period $\omega\beta_1, \ldots \beta_g$. One would like again to be able to reduce these cycles, through successive confluences provoked by continuous changes in λ, to the fixed point $\lambda_0 = 2$. But this is impossible. None of these fixed cycles is confluent to a fixed point.

Some islands of $\mathcal{L}$ that intersect the real axis create intervals that have been previously recognized and extensively studied. It was clear that a cycle with λ in such an interval is not confluent to $\lambda = 2$ through real values of λ. We see that it is not confluent through complex λs, either.

The Radial Patterns in the Distribution of the Domains of Nonconfluence

The islands that intersect the real axis can be called "subordinate" to the value of $\lambda = 3.569$, which is known to mark the right-most point of $\mathcal{L}_0$ and corresponds to an infinite sequence of successive 2-bifurcations. More generally, I observe that every island is subordinate to a λ corresponding to an infinite sequence of successive bifurcations. The subordination is spectacular (on a detailed λ-map) when the first of these bifurcations is of high order, that is, when $\theta_1/2\pi = \alpha_1/\beta_1$ with a high value of β_1. But the subordination is already apparent in FIGURE 1 for the outermost point of the sprout linked to $\mathcal{L}_0$ at $\theta_1/2\pi = 1/3$. Moreover, the islands are arrayed along β_1 directions radiating from an "offshore point." In particular, if λ corresponds to several successive bifurcations, the other β_i do not affect the number of radii. For details, see Reference 6.

THE z-FRACTAL $\mathcal{F}(\lambda)$

We proceed now to a family of z-plane fractals associated with $z \rightarrow \lambda z(1 - z)$.

Definition

First, recall that $z = \infty$ is a stable fixed point for all λ. (Proof: In terms of $u = 1/z$, the mapping is $u \rightarrow g(u) = u^2/\lambda(u - 1)$; we see that $g'(0) = 0 < 1$.) For each λ, the z-fractal $\mathcal{F}(\lambda)$ is defined as the (closed) set of points z_0 such that l.u.b. $|z_n| \neq \infty$, that is, as the set of points whose iterates fail to converge to ∞. This set, $\mathcal{F}(\lambda)$, is never empty: it includes $z_0 = 0$, which is an unstable fixed point, all of whose iterates also satisfy $z_n = 0$, plus all the finite preimages of z_0 and their limit points. Furthermore, $\mathcal{F}(\lambda)$ is always bounded: it is easy to see that it is contained—with room to spare!—in the circle $|z - 0.5| = 2.5$. The boundary of $\mathcal{F}(\lambda)$ is to be denoted by $\mathcal{F}^*(\lambda)$.

FIGURE 2 shows an example involving a 7-fold bifurcation.

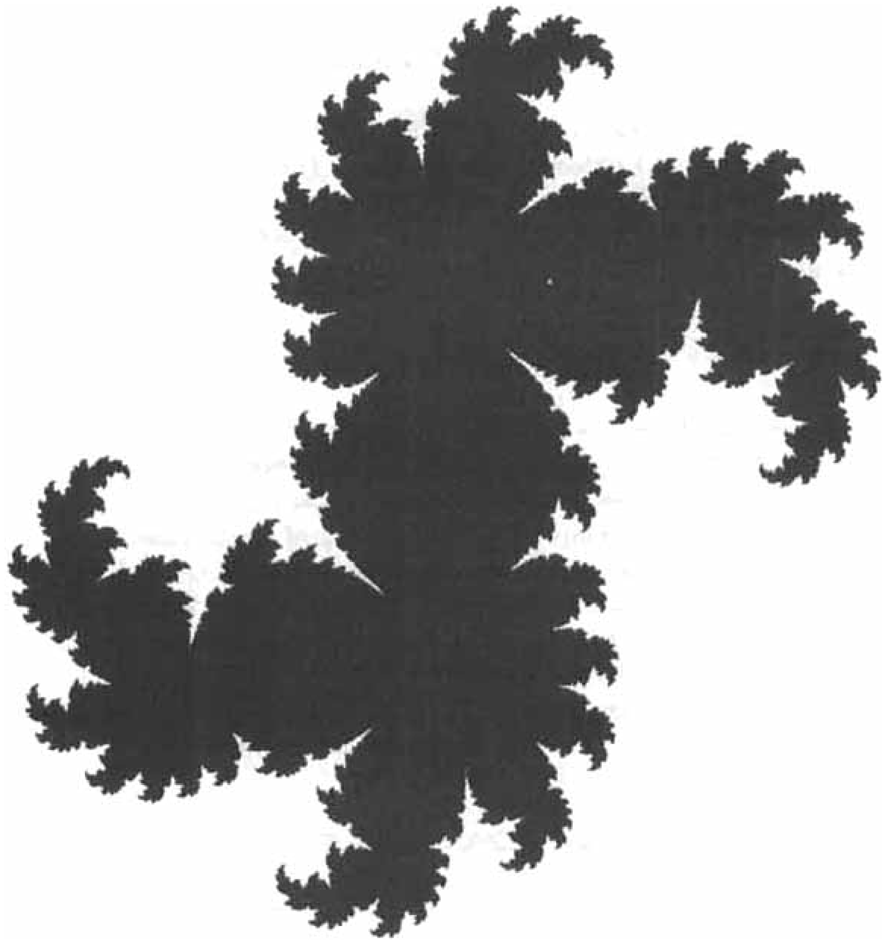

FIGURE 2. Map of $\mathcal{J}(\lambda)$ for λ near a 7-fold bifurcation.

Exceptional Values of λ, for Which $\mathcal{J}(\lambda)$ is a Standard Shape

The only such values of λ are $\lambda = 4$, $\lambda = 2$, and $\lambda = \infty$.

For $\lambda = \infty$, $\mathcal{J}^*(\infty)$, hence $\mathcal{J}(\infty)$, reduces to the points 0 and 1. Obviously, $z_n = \infty$ except conceivably for $z_0 \neq 0$ and $z_0 = 1$: these values yield an indeterminate expression $z_1 = 0 \cdot \infty$. The expression is made determinate by noting that the inverse transform leaves these points invariant. The relevance of the inverse transform will be made clear below.

For $\lambda = 4$, $\mathcal{J}^*(4)$, hence $\mathcal{J}(4)$, reduces the segment $[0, 1]$. Indeed, introducing the new variable $w = -(2z - 1)$ changes $z \to z(1 - z)$ into $w \to 2w^2 - 1$, and the further new variable $u = \cos^{-1}w$ yields $u \to 2u$, hence $u_n = 2^n u$. When $\mathrm{Im}(u_0) \neq 0$, $|\mathrm{Im}(u_n)| \to \infty$ and $|z_n| \to \infty$. Hence, the u coordinate representation of $\mathcal{J}(4)$ is the real axis, implying that $w \in [-1,1]$ and $z \in [0, 1]$.

For $\lambda = 2$, the same variable w changes $z \to 2z(1 - z)$ into $w \to w^2$, meaning that $\mathcal{J}^*(2)$ is the circle $|w| = 1$, i.e., $|z - 0.5| = 0.5$, and $\mathcal{J}(2)$ is the closed disc bounded by this circle. Clearly, $w_n \to 0$, hence $z_n \to 0.5$ iff $z_0 \in \mathcal{J}(2) - \mathcal{J}^*(2)$, and $w_n \to \infty$, hence

$z_n \to \infty$ iff (C denoting the complex plane) $z_0 \in \mathbf{C} - \mathcal{J}(2)$. When $z_0 \in \mathcal{J}^*(2)$, so that $z_0 = \exp(2\pi i\phi)$, z_n is ergodic on $\mathcal{J}^*(2)$ iff $\phi/2\pi$ is irrational; if $\phi/2\pi = \alpha/\beta$, z_n follows an unstable cycle of period β.

The preceding examples show that $\mathcal{J}(\lambda)$ can be of topological dimension 0 (isolated points or dusts), 1 (curves), or 2 (domains). For all other values of λ, $\mathcal{J}(\lambda)$ is a nonstandard set, namely a fractal, but examples of every topological dimension continue to be encountered. Due to lack of space, only a few can be described here (see Reference 6).

The Shape of $\mathcal{J}(\lambda)$ When $\lambda \in \mathcal{L}$

As λ moves away from $\lambda = 2$ a little, the circle $\mathcal{J}^*(2)$ "crumples" locally, then bigger folds gradually appear. As long as $\lambda \in \mathcal{L}_0$, the topology of $\mathcal{J}^*(\lambda)$ remains that of a circle. As λ reaches a point of β-fold bifurcation, the topology of $\mathcal{J}^*(\lambda)$ changes: it becomes "pinched" at an infinity of points, to each of which converge β points of $\mathcal{J}^*(\lambda)$. For example, as λ follows the real axis to the right and $\lambda \to 3$, $\mathcal{J}(\lambda)$ converges to the characteristic shape shown in FIGURE 3. (I call it the *San Marco shape,* in honor of the Basilica in Venice plus its reflection in a flooded Piazza and an infinite extrapolation.) When a β_1-fold bifurcation is followed by a β_2-fold bifurcation, the β_1-fold and the β_2-fold pinches generally occur at different points. In any event, $\mathcal{J}(\lambda)$ remains connected as long as λ lies in the domain of confluence $\mathcal{L}$.

The Shape of $\mathcal{J}(\lambda)$ When $\lambda \in \mathrm{Q} - \mathcal{L}$

I discovered that a totally different shape of $\mathcal{J}(\lambda)$ prevails when λ lies in a domain of nonconfluence.

1. The interior of $\mathcal{J}(\lambda)$ ceases to be connected.
2. Components of the interior of $\mathcal{J}(\lambda)$ have a common shape, except for deformations induced by the mapping $z \to \lambda z(1 - z)$ itself.
3. This common shape is close to that of the (connected) interior of $\mathcal{J}(\lambda^*)$, where λ^* is the point that is mapped upon λ when the domain of confluence $\mathcal{L}$ is mapped nonlinearly on the island under consideration.
4. There is strong evidence that $\mathcal{J}(\lambda)$ itself is connected because, in addition to the components of its interior and their boundaries, it includes a web of fractal filaments.

The Shape of $\mathcal{J}(\lambda)$ When λ is an Irrational Boundary Point of $\mathcal{L}$

An irrational boundary point of $\mathcal{L}$ is defined as a boundary point of $\mathcal{L}$ other than the rational points where a "sprout" is attached to either $\mathcal{L}_0$ or another sprout. Letting λ tend from an interior point of $\mathcal{L}$ to an irrational boundary point provokes either an infinite sequence of finite-fold bifucations, or an immediate ∞-fold bifurcation (as when $\lambda \to \exp(2\pi i\phi)$ with irrational ϕ), or a finite sequence of finite-fold bifurcations ending on an ∞-fold bifurcation.

I conjecture that, in either case, $\mathcal{J}(\lambda)$ tends to a curve, which is also the limit of

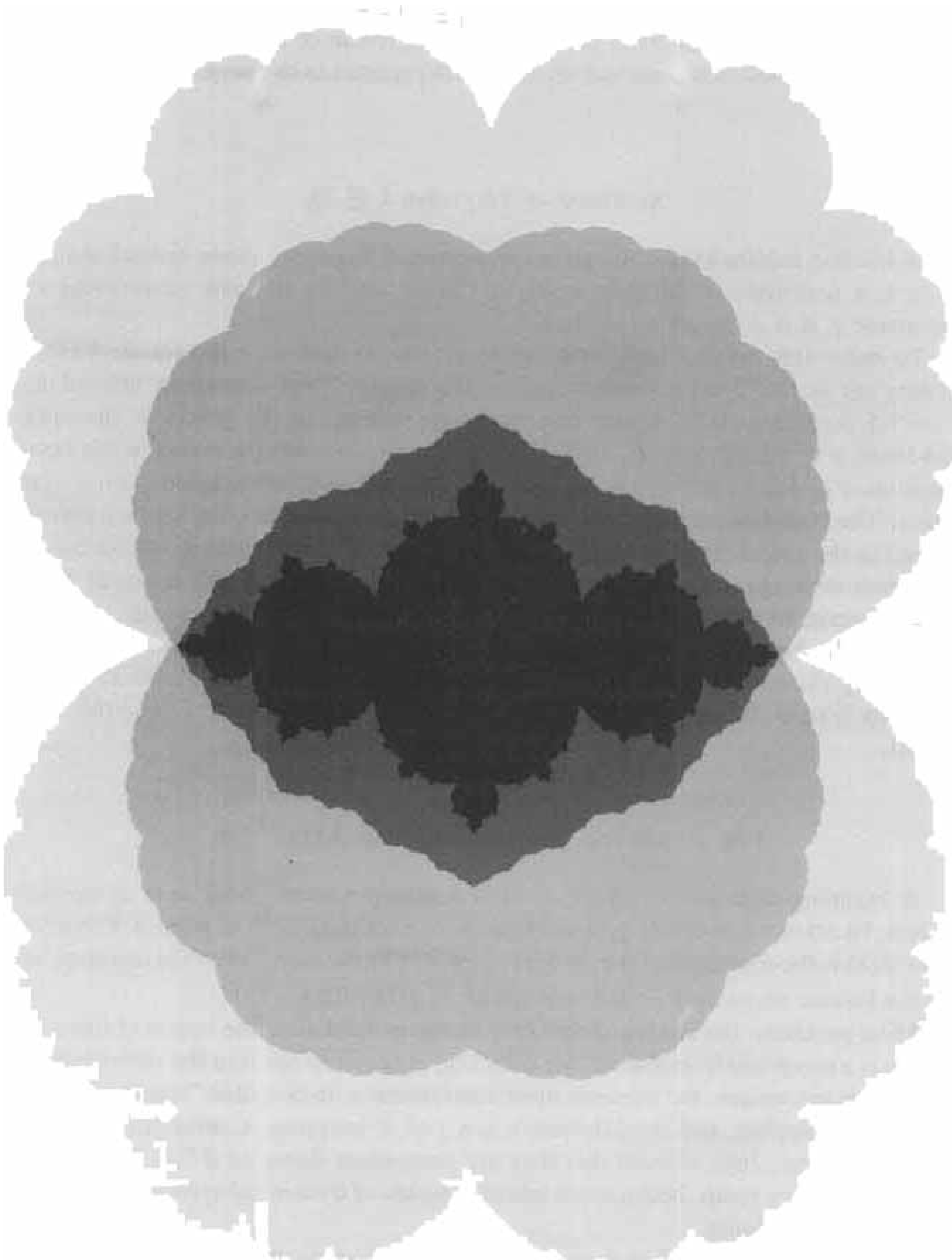

FIGURE 3. Composite $\mathcal{J}(\lambda)$ for several real valued λ's: $\lambda = 1$ (whole picture), $\lambda = 1.5$ (from mid-grey to black), $\lambda = 2.5$ (dark grey to black) $\lambda = 2.9$ (black only). The shape for $\lambda = 3$ is called the San Marco shape in the text. The two points that are common to all four diagrams are $z = 0$ and $z = 1$.

$\mathcal{J}^*(\lambda)$. In the case of the real irrational points, which lie on the λ segment [3.569,4], this conjecture is known to be true, since $\mathcal{J}(\lambda)$ is a subset of the real line. It is known that the connectedness of $\mathcal{J}(\lambda)$ depends upon the value of λ. An interesting open question arises: How is the connectedness of $\mathcal{J}(\lambda)$ related to the properties of nonreal values of λ?

The Shape of $\mathcal{J}(\lambda)$ when $\lambda \notin Q$

In his first publication on the global properties of iteration, Fatou remarked that, when λ is real and >4, $\mathcal{J}(\lambda)$ is nearly a Cantor set.[2] In the new terminology of Reference 5, it is a linear fractal dust.

To make this result perspicuous, let us digress to note that the classic Cantor ternary set is the $\mathcal{J}$-set corresponding to the special "tent" mapping defined by $z \rightarrow 1.5 - 3\,|z - 0.5|$. Under this mapping, indeed, all the points in the open mid-third, $z \in]1/3, 2/3[$, yield $z_1 < 0$, hence $\lim z_n \rightarrow -\infty$. All the points in the open segments, $z \in]1/9, 2/9[$ or $z \subset]7/9, 8/9[$, yield $z_1 > 0$, but $z_2 < 0$, hence again $z_n \rightarrow -\infty$, et cetera. The complement of these excluded segments is $\mathcal{J}(\lambda)$. Its kth ternary points, defined as the endpoints of a segment excluded at the kth stage, yield $z_n = 0$ for $n \geq k$, and hence converge to an unstable limit point. Among the nonternary points of $\mathcal{J}(\lambda)$, some converge to an unstable cycle of period >1, while others are ergodic.

Using the definition in Reference 8, this $\mathcal{J}(\lambda)$ is a *fractal dust*. It is a dust because it is totally discontinuous, so that its topological dimension is $D_T = 0$. On the other hand, its fractal dimension is $D = \log 2/\log 3 > 0$. Because $D > D_T$, this is a fractal.

THE z-FRACTAL $\mathcal{J}(\lambda)$ AS FRACTAL ATTRACTOR

A mapping such as $z \rightarrow \lambda z(1 - z)$ is routinely viewed today as a dynamical system. Its attractor is dull (e.g., a single point or a finite number of points). However, since $\mathcal{J}(\lambda)$ is the repeller set for $z \rightarrow \lambda z(1 - z)$, it is by the same token the attractor set for the inverse mapping $z \rightarrow 0.5 + \epsilon\sqrt{0.25 - z/\lambda}$, with $\epsilon = \pm 1$.

More precisely, the last statement only becomes valid after the notion of dynamic system is appropriately extended. An extension is required because the above inverse mapping is not unique, but depends upon a parameter ϵ, to be called "label"; hence, it is a 1 to 2 mapping, and the kth iterate is a 1 to 2^k mapping. Considering all these iterates together, Julia showed that they are everywhere dense on $\mathcal{J}^*(\lambda)$.[4] But this is not a satisfactory result, because the intuitive notion of dynamical systems demands a single-valued mapping.

To achieve this goal, I propose that one set a discrete dynamical system in the product space of the complex plane **C** by the label-set made of two points + and $-$. We take it that the ϵ_n sequence proceeds according to its own rules, independently of the z_n sequence, while the z_n sequence is ruled by the ϵ_n sequence. For example, the ϵ_n sequence may be a Bernoulli process of independent random throws of a fair coin, or a more general ergodic random sequence. The conclusion seems inescapable (though I have not tested the details) that any ergodic sequence ϵ_n generates a trajectory whose projection of the **C** plane is dense on $\mathcal{J}^*(\lambda)$.

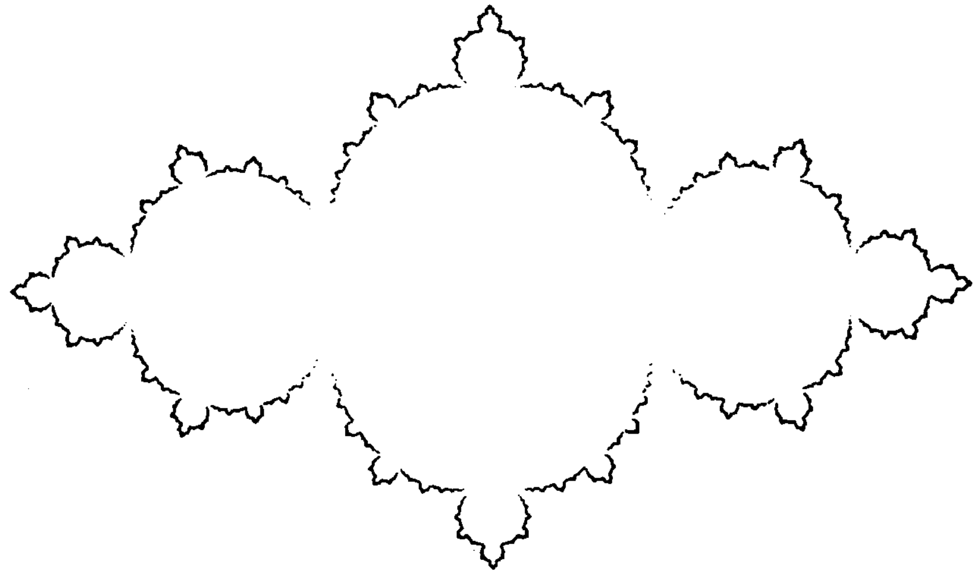

FIGURE 4. The 64 000 first positions of a dynamic system attracted to the San Marco fractal shape.

For $\lambda = 2$, when $\mathcal{J}^*(\lambda)$ is a circle, the invariant measure is known to be uniform. For $\lambda = 4$, when $\mathcal{J}^*(\lambda)$ is $[0, 1]$, the invariant measure is readily seen to be the real axis projection of a uniform measure on a circle; hence, it has the "arc-cosine" density $\pi^{-1}[x(1 - x)]^{-1/2}$. Both are found empirically to be very rapidly approximated by sample dynamical paths. See also the approximation of the San Marco shapes in FIGURE 4. On the other hand, the most interesting cases, where $\mathcal{J}^*(\lambda)$ is extremely convoluted, as in FIGURE 2, involve a complication. The limit measure ($\equiv$ invariant measure) on $\mathcal{J}^*(\lambda)$ is extremely uneven. The tips of the deep "fjords" require very special sequences of the ϵ_n to be visited, and hence are visited extremely rarely compared to the regions near the figure's outline.

A DIGRESSION CONCERNING "STRANGE ATTRACTORS"

Lately, the term "attractor" has often been associated with the adjective "strange," and the reader may legitimately wonder whether strange and fractal attractors have anything in common. Indeed, they do.

First Point

The fractal (Hausdorff-Besicovitch) dimension D has been evaluated for many strange attractors, and found to exceed strictly their topological dimension. Hence, these attractors (and presumably other ones, perhaps even all strange attractors) are fractal sets. The D of the Smale attractor is evaluated in Reference 8. And the Saltzman-Lorenz attractor with $v = 40$, $\sigma = 16$, and $b = 4$ yields $D \simeq 2.06$; this result was obtained independently by M. G. Velarde and Ya. G. Sinai, who report it in private conversations but neither of whom has, to my knowledge, published it. (Last

minute addition: A preprint by H. Mori and H. Fujisaka confirms my value of D for the Smale attractor and the Velarde-Sinai value of D for the Saltzman-Lorenz attractor. For the Hénon mapping with $a = 1.4$ and $b = 0.3$, it finds that $D = 1.26$.) The fact that $D \cong 2.06$ is very close to 2, but definitely above 2, means that the Saltzman-Lorenz attractor is definitely not a standard surface, but that it is not extremely far from being one.

Since the relevance of D in this context may puzzle those who only know of fractal dimension as a measure of the irregularity of continuous curves, let me point out that in this instance, D is not a measure of irregularity but of the way smooth surfaces pile upon each other—a variant of the notion of fragmentation, which is also studied in Reference 5.

Let us also recall from Reference 5 that the Hausdorff-Besicovitch discussion was not the sole candidate for fractal dimension, but was selected because (1) it is the most thoroughly studied, (2) it has theoretical virtues, and (3) in most instances, the choice does not matter, because diverse reasonable alternative dimensions yield identical values. In an interesting further development in the same direction, a relation has recently been conjectured to exist, and verified empirically on examples, between a strange attractor's fractal dimension and its Lyapunov numbers (preprints by H. Mori and H. Fujisaka and by D. A. Russell, J. D. Hanson, and E. Ott.)

Second Point

One is tempted, conversely, to ask whether the fractal attractors I study are "strange." It depends which meaning is given to this last word. Using its old-fashioned "meaning," as a milder synonym to "monstrous," "pathological," and other epithets once applied to fractals, the answer is "Yes, but why bother to revive a term whose motivation has vanished when fractals were shown, by Reference 5, to be no more strange than coastlines or mountains." Unfortunately, the term "strange" has since acquired a technical sense, one so exclusive that the Saltzman-Lorenz attractor must be called "strange-strange." In this light, many fractal attractors of my dynamic systems are not strange at all. Indeed "strangeness" reflects nonstandard topological properties, with the nonstandard fractal properties mentioned above coming along as a seemingly inevitable "overhead." In this sense (1) a topological circle (intuitively, a closed curve without double points) is not strange, however crumpled it may be; hence, (2) the fractal attractors $\mathcal{J}^*(\lambda)$ for $|\lambda - 2| < 1$ are surely not strange.

However, the fractal attractors associated with other rational mappings I have studied are topologically peculiar.[6] Thus, the answer to our question is confused. But it is not an important question: the term "strange" has, in my opinion, exhausted its usefulness, and ought to be abandoned.

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