Quasicircles of dimension $1 + k^2$ do not exist

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Abstract

A well-known theorem of Smirnov says that the Hausdorff dimension of a k-quasicircle is bounded above by $1 + k^2$. Here, we show that Smirnov's bound is not sharp. More precisely, we show that $D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{2.5})$, where $D(k)$ is the maximal dimension of a k-quasicircle and Σ^2 is the maximal asymptotic variance of the Beurling transform (taken over the unit ball of L^∞). The quantity Σ^2 was introduced in [5], and recently, Hedenmalm [12] proved that $\Sigma^2 < 1$. We will deduce the asymptotic expansion of $D(k)$ from a more general statement relating universal bounds for the integral means spectrum and the asymptotic variance of conformal maps. Our proof is essentially an improvement in the argument of Becker and Pommerenke [6] who gave the bound $D(k) \le 1 + 37\, k^2$ for small k.

1 Introduction

The Riemann mapping theorem states that any simply-connected domain $\Omega \subset \mathbb{C}$ is the image of a conformal map $f : \mathbb{D} \to \Omega$ (unless $\Omega = \mathbb{C}$ itself). Since simply-connected domains can be very wild, it is reasonable to expect that the complexity of the boundary $\partial\Omega$ is manifested in the complexity of the Riemann map. For instance, if Ω is bounded by a smooth curve, then $f \in C^\infty(\overline{\mathbb{D}})$. Here, we focus on domains with rough (but Jordan) boundaries $\partial\Omega$, whose Hausdorff dimensions are larger than 1. For such domains, the relationship between f and $\partial\Omega$ may be quantified using several geometric characteristics.

One notable characteristic is the *integral means spectrum* given by

$$\beta_f(p) = \limsup_{r\to 1} \frac{\log \int_{|z|=r} |f'(z)^p|\, d\theta}{\log \frac{1}{1-r}}, \qquad p \in \mathbb{C}.$$

The importance of the spectrum $\beta_f(p)$ lies in the fact that it is Legendre-dual to the multifractal spectrum of harmonic measure [20, 7]. Taking the supremum of $\beta_f(p)$ over bounded simply-connected domains, one obtains the *universal integral means spectrum*

$$B(p) = \sup \beta_f(p).$$

It is clear from Hölder's inequality that $B(p)$ is a convex function, with a minimum at $B(0) = 0$. Even though $B(p)$ is a central object in geometric function theory, apart from various estimates [15, 16], not much is rigorously known about its qualitative features. For instance, it is expected that $B(p) = B(-p)$ is an even function. However, while $B(2) = 1$ is an easy consequence of the area theorem, the statement "$B(-2) = 1$" is equivalent to the Brennan conjecture, which is a well-known and difficult open problem. Nor is it known whether $B(p) \in C^1$, let alone real-analytic. In this work, we study the behaviour of $B(p)$ near the origin, and as an application, we disprove the sharpness of Smirnov's bound for the dimension of quasicircles.

It will be convenient for us to work with conformal maps defined on the exterior unit disk $\mathbb{D}^* = \{z : |z| > 1\}$. Unless stated otherwise, we assume that conformal maps are in *principal normalization*, satisfying $\varphi(z) = z + \mathcal{O}(1/|z|)$ near infinity.

Let $\mathbf{\Sigma}_k$ be the collection of conformal maps that admit k-quasiconformal extensions to the complex plane with dilatation at most k. Maximizing over $\mathbf{\Sigma}_k$, we obtain the spectra $B_k(p) := \sup_{\varphi\in\mathbf{\Sigma}_k} \beta_\varphi(p)$.

We show:

Theorem 1.1.

$$\lim_{p\to 0} \frac{B_k(p)}{|p|^2/4} = \Sigma^2(k) := \sup_{\varphi\in\mathbf{\Sigma}_k} \sigma^2(\log \varphi'), \tag{1.1}$$

where

$$\sigma^2(g) = \lim_{R\to 1^+} \frac{1}{2\pi|\log(R-1)|} \int_{|z|=R} |g(z)|^2\, |dz| \tag{1.2}$$

denotes the asymptotic variance *of a Bloch function on the exterior unit disk.*

One advantage of Theorem 1.1 lies in the fact that asymptotic variance is much easier to estimate than the integral means spectrum. In dynamical cases, the two characteristics are linked by an explicit relation:

Theorem 1.2 (McMullen's identity, global version). *If $\partial\Omega$ is a Jordan curve, invariant under a hyperbolic dynamical system, e.g. a Julia set or a limit set of a quasi-Fuchsian group, then β_φ is real-analytic and*

$$(1/2) \cdot \beta''(p) = \sigma^2(\log \varphi'). \tag{1.3}$$

For a discussion, see [5].

Remark. For general domains, (1.3) need not hold: for instance, one can take a fractal domain which satisfies McMullen's identity and replace an arc $\gamma \subset \partial\Omega$ by a smooth curve. Then, $\beta''(p)$ does not change, but the asymptotic variance goes down by a definite factor, depending on the harmonic measure of γ. It is not hard to find examples with the inequality pointing in the other direction. Nevertheless, Theorem 1.1 says that McMullen's identity holds on the level of universal bounds.

Theorem 1.1 admits an infinitesimal analogue:

Theorem 1.3. *If $k \to 0$ and $k|p| \to 0$, then*

$$\lim \frac{B_k(p)}{k^2|p|^2/4} = \Sigma^2 := \sup_{|\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu)$$

where

$$\mathcal{S}\mu = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{\mu(\zeta)}{(\zeta - z)^2} \, |dz|^2 \tag{1.4}$$

is the Beurling transform *of μ.*

The quantity Σ^2 was first studied in [5], where it was established that $0.879 \le \Sigma^2 \le 1$, while recently, H. Hedenmalm managed to show [12] that actually $\Sigma^2 < 1$.

Remark. In light of Hedenmalm's estimate, Theorem 1.3 contradicts the conjecture "$B_k(p) = k^2p^2/4$ for all $k \in [0, 1)$ and $p \in [-2/k, 2/k]$" from [16, 24]. However, since we do not know whether or not $\lim_{k\to 1^-} \Sigma^2(k) \overset{?}{=} 1$, we cannot rule out Kraetzer's conjecture which asserts that "$B(p) = p^2/4$ for $p \in [-2, 2]$." It is currently known that $0.93 < \lim_{k\to 1^-} \Sigma^2(k) < (1.24)^2$. We refer the reader to [5, Section 8] for the lower bound and to [14, Corollary 2.3] for the upper bound.

1.1 Dimensions of quasicircles

We apply Theorem 1.3 to study dimensions of quasicircles. Let $D(k)$ denote the maximal dimension of a k-quasicircle, the image of the unit circle under a quasiconformal mapping of the plane.

In [6], Becker and Pommerenke showed that $1 + 0.36\,k^2 \le D(k) \le 1 + 37\,k^2$, if k is small. In his landmark work [4] on area distortion of quasiconformal mappings, K. Astala asked whether

$$D(k) \le 1 + k^2, \qquad 0 \le k < 1, \tag{1.5}$$

and if so, whether this bound was sharp. Using Astala's method, S. Smirnov [25] obtained the bound (1.5) but the sharpness remained open. Here, we improve upon the original argument of Becker and Pommerenke to show that Smirnov bound is not sharp for small k. István Prause informed me (private communication) that one can use the methods of [24, 25] to show that this implies that $D(k) < 1 + k^2$ for all $0 < k < 1$.

Theorem 1.4.

$$D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{2.5}).$$

In particular,

$$\lim_{k \to 0} \frac{D(k) - 1}{k^2} = \Sigma^2. \tag{1.6}$$

Remark. (i) The lower bound

$$\liminf_{k \to 0} \frac{D(k) - 1}{k^2} \ge \Sigma^2$$

was established in [5] by considering Julia sets of polynomial perturbations of $z \to z^d$.

(ii) In dynamical cases (when μ is invariant under a Fuchsian group or a Blaschke product), McMullen [22] showed that $2d^2/dt^2|_{t=0}\, \mathrm{H.\,dim}\, w^{t\mu}(\mathbb{S}^1) = \sigma^2(\mathcal{S}\mu)$. This is the original McMullen identity, presented in a somewhat extrinsic form.

The implication (Theorem 1.3 $\Rightarrow$ Theorem 1.4) follows from the relation

$$\beta_\varphi(p) = p - 1 \quad \Longleftrightarrow \quad p = \mathrm{M.\,dim}\, \varphi(\mathbb{S}^1), \qquad \varphi \in \mathbf{\Sigma}_k, \tag{1.7}$$

see [23, Corollary 10.18]. Here, two facts are tacitly being used: first, the work of Astala [3] shows that $D(k)$ may be characterized using Minkowski dimension in place of Hausdorff dimension. One may view Astala's result as a *fractal approximation* theorem: when evaluating $D(k)$, it suffices to take the supremum over certain Ahlfors regular k-quasicircles for which the Hausdorff and Minkowski dimensions coincide. However, Astala's fractals do not satisfy McMullen's identity (1.2).

Secondly, one can take a quasiconformal map that is conformal to one side and anti-symmetrize it in the spirit of [19, 25] to reduce its dilatation. More precisely, a Jordan curve γ is a k'-quasicircle if and only if it can be represented as $\gamma = \varphi(\mathbb{S}^1)$ with $\varphi \in \mathbf{\Sigma}_k$ where

$$k = \frac{2k'}{1+(k')^2}. \tag{1.8}$$

(This accounts for the discrepancy in the factor of 4 in Theorems 1.3 and 1.4.)

1.2 A sketch of proofs

As mentioned earlier, the proofs of Theorems 1.1 and 1.3 follow the argument of Becker and Pommerenke [6], except at the crucial point, we use an average bound for the non-linearity $n_\varphi = \varphi''/\varphi'$ instead of the supremum bound. That is, instead of invoking

$$|2(n_\varphi/\rho_*)(z)| \le 6k, \qquad \varphi \in \mathbf{\Sigma}_k, \tag{1.9}$$

where $\rho_*(z) = 2/(|z|^2-1)$, we use the box lemma:

Lemma 1.1. *For any $\epsilon > 0$, there exists R sufficiently large, so that*

$$⨍_B \left|\frac{2n_\varphi}{\rho_*}(z)\right|^2 \rho_*(z)|dz|^2 \,<\, \Sigma^2(k) + \epsilon + o(|z|-1), \tag{1.10}$$

for any conformal map $\varphi \in \mathbf{\Sigma}_k$ and every R-box B.

(An infinitesimal version of the box lemma will be given in Section 4, with a more precise error term.) Here, by a *box* in $\mathbb{D}^* = \{z : |z| > 1\}$, we mean an annular rectangle of the form $B = \{z : r_1 < |z| < r_2,\ \theta_1 < \arg z < \theta_2\}$, while an *$R$-box* is a box for which the hyperbolic distance between any two of the four corners is comparable to R. The proof of Lemma 1.1 uses fractal approximation techniques from [5, Section 6]. We will review these arguments in Sections 3 – 5.

In Section 6, we promote bounds on boxes to bounds on integral means:

Theorem 1.5. *Suppose $\varphi \in \mathbf{\Sigma}_k$ is a conformal map which satisfies*

$$m \le \frac{1}{k^2} \fint_B \left| \frac{2n_\varphi}{\rho_*}(z) \right|^2 \rho_*(z)|dz|^2 \le M, \quad 0 \le m \le M \le 36, \tag{1.11}$$

for any R-box sufficiently close to the unit circle. (Here, $R > 0$ is fixed.) Then,

$$m(1-\epsilon) \cdot \frac{k^2|p|^2}{4} - \mathcal{O}(k^4|p|^4) \le \beta_\varphi(p) \le M(1+\epsilon) \cdot \frac{k^2|p|^2}{4} + \mathcal{O}(k^4|p|^4),$$

provided $k|p| < \epsilon/(CR)$ and $0 < \epsilon < 1/2$.

Remark. (i) A non-zero lower bound on the box averages implies that the boundary $\partial\Omega$ is *uniformly wiggly* in the sense of Bishop and Jones [9]. This condition already entails that $\operatorname{H.dim}(\partial\Omega) > 1$. However, non-linearity bounds give much more quantitative estimates.

(ii) The relation $B_k(2/k) = 1$, e.g. see [24], suggests that in order to obtain bounds of the form $B_k(p) \le Ck^2|p|^2/4$ with $C < 1$, one must assume that the product $k|p|$ is small.

For applications to Minkowski dimension, one needs to cover the range $p \in [1, 2)$, which can be achieved by making k small. Alternatively, one can fix a map φ and tend $p \to 0$ to obtain bounds for $\limsup_{p\to 0} 4\,\beta_\varphi(p)/|p|^2$. Taking the supremum over $\mathbf{\Sigma}_k$ gives Theorem 1.1.

In dynamical cases, using the ergodicity of the geodesic flow, it is not hard to see that if $R > 0$ is large, then

$$\sigma^2(\log \varphi') - \epsilon < \fint_B \left| \frac{2n_\varphi}{\rho_*}(z) \right|^2 \rho_*|dz|^2 < \sigma^2(\log \varphi) + \epsilon, \tag{1.12}$$

This gives an elementary proof of McMullen's identity (Theorem 1.2) without the use of thermodynamic formalism.

1.3 Beltrami coefficients with sparse support

When studying thin regions of Teichmüller space, it is useful to consider Beltrami coefficients that are sparsely supported. For applications, see [8, 17, 21]. Suppose $\mu \in M(\mathbb{D})$ is a Beltrami coefficient supported on a "garden" $\mathcal{G} = \bigcup A_j$ where:

(1) Each A_j satisfies the *quasi-geodesic property* – i.e. is located within a bounded hyperbolic distance of a geodesic segment γ_j.

(2) *Separation property.* The hyperbolic distance $d_{\mathbb{D}}(\gamma_i, \gamma_j) > R$ is large.

In this setting, one can give an improved bound for the dimension of quasicircles:

Theorem 1.6. *If μ is a Beltrami coefficient with sparse support, then*

$$\mathrm{M.}\dim f^{t\mu}(\mathbb{S}^1) \le 1 + Ce^{-R}t^2,$$

if $0 \le t < t_0(R)$ is small.

1.4 Related results and open problems

In a recent work [13], Hedenmalm studied the notion of "asymptotic tail variance" of Bloch functions to show the estimate

$$B_k(p) \le k^2|p|^2/4 + \mathcal{O}(k^3), \qquad p \in \mathbb{C}.$$

In [18], joint with I. Kayumov, we give an additional characterization of Σ^2 in terms of Makarov's law of iterated logarithm. To conclude the introduction, we mention two open problems:

- Suppose Γ is a k-quasicircle of Hausdorff dimension $d > 1$. For any $\epsilon > 0$, either construct a "dynamical" k-quasicircle $\tilde{\Gamma}$ (like in Theorem 1.2) for which $\mathrm{H.}\dim \tilde{\Gamma} > d - \epsilon$, or explain why such a construction is impossible.
- Are the functions $B_k(p)$ and $D(k)$ differentiable on an interval $(-\epsilon, \epsilon)$? Is it true that $D(k) = \Sigma^2 k^2 + a_3 k^3 + o(k^3)$ for some $a_3 \in \mathbb{R}$?

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2 Preliminaries

In this section, we recall the definition of the universal Teichmüller space. Then, we discuss holomorphic families of conformal maps and describe infinitesimal analogues of the three cocycles in complex analysis.

2.1 Universal Teichmüller Space

The analysis of the spectrum $B(p)$ can be thought of as a global problem in the universal Teichmüller space $\mathcal{T}(\mathbb{D}^*)$. For the purposes of this paper, we view $\mathcal{T}(\mathbb{D}^*) = \bigcup_{0\le k<1} \mathbf{\Sigma}_k$ as the collection of conformal maps defined on the exterior unit disk with the *principal normalization*

$$\varphi(z) = z + \mathcal{O}(1/|z|), \qquad \text{as } z \to \infty,$$

which admit a k-quasiconformal extension to the complex plane (for some $0 \le k < 1$).

The *Bers embedding into the Bloch space* $b : \varphi \to b_\varphi := \log \varphi'$ expresses $\mathcal{T}(\mathbb{D}^*)$ as a bounded domain in a complex Banach space, thus giving $\mathcal{T}(\mathbb{D}^*)$ the structure of an infinite-dimensional complex manifold. More precisely, the image $b(\mathcal{T}(\mathbb{D}^*))$ is contained in a ball of radius 6 in the Bloch (semi)norm

$$\|g\|_{\mathcal{B}^*} := \sup_{z\in\mathbb{D}^*} \big|(|z|^2 - 1)g'(z)\big|.$$

In view of Royden's theorem which equates the Kobayashi and Teichmüller metrics, the sets $\mathbf{\Sigma}_k \subset \mathcal{T}(\mathbb{D}^*)$ are metric balls, justifying the study of the spectra $B_k(p)$.

Remark. (i) It is more standard to define the Bers embedding using Schwarzian derivatives. This endows $\mathcal{T}(\mathbb{D}^*)$ with the same complex structure; however, the metric closures in the ambient Banach spaces are different. The interested reader may consult the work of Astala-Gehring [1] for more information.

(ii) If $\mathbf{\Sigma}$ is the collection of all conformal maps in principal normalization, then $b(\mathbf{\Sigma})$ contains the closure of $\mathcal{T}(\mathbb{D}^*)$ as well as some isolated points. From the fractal approximation principle of Carleson and Jones [11], it follows that it is sufficient to take the supremum in the definition of $B(p)$ over $\mathcal{T}(\mathbb{D}^*)$. Therefore, from now on, we restrict our attention to maps in $\mathcal{T}(\mathbb{D}^*)$ and ignore the complement $\mathbf{\Sigma} \setminus \mathcal{T}(\mathbb{D}^*)$.

2.2 Holomorphic families

When discussing local problems, it is preferable to consider holomorphic families of conformal maps

$$\varphi_t \colon \mathbb{D}^* \to \mathbb{C}, \qquad \varphi_0(z) = z, \qquad \varphi_t(z) = z + \mathcal{O}(1/|z|), \qquad t \in \mathbb{D}.$$

There is not much difference in the two perspectives: according to the λ-lemma, each map φ_t, $t \in \mathbb{D}$ admits a $|t|$-quasiconformal extension to the complex plane. Conversely, if a conformal map φ has a k-quasiconformal extension H, then it may be naturally included into a holomorphic family $\{\varphi_t,\, t \in \mathbb{D}\}$ with $\varphi = \varphi_k$. This is done by letting H_t be the principal homeomorphic to the Beltrami equation $\overline{\partial} H_t = t\,(\mu/k)\,\partial H_t$ and restricting to the exterior unit disk.

The map $t \to \log \varphi_t'$ is a Banach-valued holomorphic map from $\mathbb{D}$ to the Bloch space of the exterior unit disk. From the Neumann series expansion for principal solutions to the Beltrami equation [2, p. 165],

$$\varphi_t' = \partial \varphi_t = 1 + t\mathcal{S}\mu + t^2 \mathcal{S}\mu\mathcal{S}\mu + \dots, \qquad |z| > 1, \tag{2.1}$$

it follows that the derivative of the Bers embedding at the origin is just

$$d/dt\Big|_{t=0} \log \varphi_t' = \mathcal{S}\mu. \tag{2.2}$$

In particular, this implies that $\mathcal{S}\mu \in \mathcal{B}^*$ and

$$\left\| \frac{\log \varphi_t'}{t} - \mathcal{S}\mu \right\|_{\mathcal{B}^*} = \mathcal{O}(|t|), \quad \text{for } |t| < 1/2. \tag{2.3}$$

Since the asymptotic variance is continuous in the Bloch norm, the function

$$t \to \sigma^2 \left(\frac{\log \varphi_t'}{t} \right)$$

extends continuously to $\sigma^2(\mathcal{S}\mu)$ at $t = 0$. For details, we refer the reader to [5, Section 8]. Taking the supremum over all $|\mu| \le \chi_{\mathbb{D}}$ shows that $\Sigma^2(k) = \Sigma^2 k^2 + \mathcal{O}(k^3)$. Similarly to (2.2), one observes that

$$(\mathcal{S}\mu)' = \frac{d}{dt}\Big|_{t=0} \frac{\varphi_t''}{\varphi_t'}, \qquad (\mathcal{S}\mu)'' = \frac{d}{dt}\Big|_{t=0} \left[\left(\frac{\varphi_t''}{\varphi_t'} \right)' - \frac{1}{2} \left(\frac{\varphi_t''}{\varphi_t'} \right)^2 \right] \tag{2.4}$$

are the infinitesimal forms of the non-linearity n_φ and the Schwarzian derivative s_φ respectively.

3 Locality of $(\mathcal{S}\mu)'/\rho_*$

In [5, Section 6], it was proved that

$$\Sigma^2 = \sup_{\mu \in M_{\mathrm{I}},\ |\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu), \tag{3.1}$$

where M_{I} is the class of Beltrami coefficients that are eventually-invariant under $z \to z^d$ for some $d \ge 2$, i.e. satisfying $(z^d)^*\mu = \mu$ in some open neighbourhood of the unit circle. The proof used a fractal approximation argument involving the periodization of Beltrami coefficients. Two facts where crucial to this scheme. First, the asymptotic variance of a Bloch function could be expressed as

$$\sigma^2(g) \;=\; 4 \cdot \limsup_{R \to 1^+} \fint_{R<|z|<2} \left| \frac{g'}{\rho_*}(z) \right|^2 \rho_* |dz|^2, \tag{3.2}$$

or

$$\sigma^2(g) \;=\; \frac{8}{3} \cdot \limsup_{R \to 1^+} \fint_{R<|z|<2} \left| \frac{g''}{\rho_*^2}(z) \right|^2 \rho_* |dz|^2, \qquad g \in \mathcal{B}^*. \tag{3.3}$$

The equivalence of (1.2), (3.2), (3.3) was established in [22, Section 6]. To be honest, McMullen only considered the case when the limit in either expression exists; however, the proof works equally well in this more general situation. The second fact needed to prove (3.1) was the following "localization" lemma:

Lemma 3.1. *Suppose μ is a Beltrami coefficient supported on the unit disk with $\|\mu\|_\infty \le 1$. Then,*

- (a) *For $|z| > 1$, $|((\mathcal{S}\mu)''/\rho_*^2)(z)| \le 6$.*
- (b) *If $d_{\mathbb{D}}(z^*, \operatorname{supp} \mu) \ge R$, then $|((\mathcal{S}\mu)''/\rho_*^2)(z)| \le Ce^{-R}$.*

(Here, $d_{\mathbb{D}}(\cdot, \cdot)$ denotes the hyperbolic distance between two subsets of the disk; while z^* denotes the reflection of the point z in the unit circle.)

Part (a) of Lemma 3.1 says that $(\mathcal{S}\mu)''$ is bounded as a holomorphic quadratic differential. Part (b) says that to determine the value of $(\mathcal{S}\mu)''/\rho_*^2$ at a point $z \in \mathbb{D}^*$, up to a small error, it suffices to know the values of μ in a neighbourhood of z^*. In particular, if μ_1 and μ_2 are two Beltrami coefficients that agree on a ball

$$B_{\mathrm{hyp}}(z, R) := \{\zeta : d_{\mathbb{D}}(\zeta, z) < R\}$$

of large radius, then the difference $|((\mathcal{S}\mu_1 - \mathcal{S}\mu_2)''/\rho_*^2)(z)|$ is small.

In this work, we will need a version of the above lemma concerning $(\mathcal{S}\mu)'(z)/\rho_*$. Since non-linearity is not invariant under $\mathrm{Aut}(\mathbb{D})$, it is convenient to work on the upper half-plane. Analogously to part (a) of Lemma 3.1, we have:

Lemma 3.2. *Suppose μ is a Beltrami coefficient supported on the upper half-plane with $\|\mu\|_\infty \le 1$. Then, for $z \in \overline{\mathbb{H}}$, $\left|(2(\mathcal{S}\mu)'/\rho_{\overline{\mathbb{H}}})(z)\right| \le 8/\pi$.*

Proof. The proof is by direct calculation:

$$\begin{aligned}
|(\mathcal{S}\mu)'(z)| &\le \frac{2}{\pi}\int_{\mathbb{H}} \frac{1}{|w-z|^3}\,|dw|^2, \\
&= \frac{2}{\pi}\int_{y_0}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^{3/2}}\,dxdy, \\
&= \frac{2}{\pi}\int_{y_0}^{\infty} \frac{x}{y^2\sqrt{x^2+y^2}}\bigg|_{-\infty}^{\infty} dy, \\
&= \frac{4}{\pi}\int_{y_0}^{\infty} \frac{dy}{y^2}, \\
&= \frac{4}{\pi y_0},
\end{aligned}$$

where $y_0 = |\operatorname{Im} z|$. Multiplying by 2 and dividing by the density of the hyperbolic metric gives the result. □

If $z \in \overline{\mathbb{H}}$ is far away from the reflection of the support of μ, one can give a better estimate. However, one needs to be a little careful since the straightforward analogue of part (b) is not true. For a point $x + iy \in \mathbb{H}$, define the “square”

$$Q_d(x+iy) := \big\{w : \operatorname{Re} w \in [x - dy, x + dy],\ \operatorname{Im} w \in [(1/d)y, dy]\big\}.$$

Lemma 3.3. *Under the assumptions of Lemma 3.2, if $\mu = 0$ on $Q_d(x + iy)$, then*

$$\left|\frac{2(\mathcal{S}\mu)'}{\rho_{\overline{\mathbb{H}}}}(x - iy)\right| \le C/d. \tag{3.4}$$

Proof. The lemma follows by estimating the contributions of the top, bottom, left and right parts of $\mathbb{H}\setminus Q_d(x+iy)$ separately and adding them up. We leave the details to the reader. □

4 Box Lemma

In this section, we prove the infinitesimal version of the box lemma:

Lemma 4.1. *If $R > 0$ is sufficiently large, then*

$$\fint_B \left| \frac{2(\mathcal{S}\mu)'}{\rho_*}(z) \right|^2 \rho_* |dz|^2 \;<\; \Sigma^2 + C/R + o(|z| - 1) \tag{4.1}$$

for any Beltrami coefficient μ with $|\mu| \le \chi_{\mathbb{D}}$ and R-box B in $\mathbb{D}^*$.

The proof is by contradiction: if one could find a Beltrami coefficient μ and a box B for which (4.1) did not hold, one would be able to periodize $\mu|_B$ to obtain a Beltrami coefficient μ_{per} with asymptotic variance greater than Σ^2.

We now give the details.

4.1 Box estimate

By a *box* in the upper half-plane, we mean a rectangle whose sides are parallel to the coordinate axes, with the bottom side located above the real axis. We say two boxes are *similar* if they differ by an affine scaling $L(z) = az + b$ with $a > 0$, $b \in \mathbb{R}$. We use $\overline{B} \subset \overline{\mathbb{H}}$ to denote the reflection of the box B with respect to the real axis.

A natural model for an R-box with $R = \log d$ in the upper half-plane is the rectangle

$$B = \bigl\{ w : \operatorname{Re} w \in [0, \theta d],\, \operatorname{Im} w \in [1/d, 1] \bigr\}, \quad \theta \in [1/2, 2].$$

Note that if $\theta = 1$ and $d \ge 2$ is an integer, one can place d similar boxes directly under B along the bottom edge. These 'standard' boxes are useful when discussing Beltrami coefficients eventually-invariant under $z \to z^d$. However, allowing d to be non-integer and $\theta \in [1/2, 2]$ allows some wiggle room, which is useful in some arguments. Using Lemma 3.3, it is not hard to show that:

Lemma 4.2. *Suppose μ_1 and μ_2 are two Beltrami coefficients on $\mathbb{H}$ with $\|\mu_i\| \le 1$, $i = 1, 2$. If additionally $\mu_1 = \mu_2$ on an R-box B, then*

$$\left| \fint_{\overline{B}} \left| \frac{2(\mathcal{S}\mu_1)'}{\rho_{\overline{\mathbb{H}}}}(z) \right|^2 \rho_{\overline{\mathbb{H}}} |dz|^2 \;-\; \fint_{\overline{B}} \left| \frac{2(\mathcal{S}\mu_2)'}{\rho_{\overline{\mathbb{H}}}}(z) \right|^2 \rho_{\overline{\mathbb{H}}} |dz|^2 \right| \le C_1/R. \tag{4.2}$$

Proof. To estimate (4.2), we partition $\overline{B}$ into $d\lfloor\theta d\rfloor$ similar cells, by subdividing the vertical side in d equal parts as measured in the $\rho_{\overline{\mathbb{H}}}$ metric and the horizontal side in $\lfloor\theta d\rfloor$ equal parts in the Euclidean metric. In the cell (i,j) in the i-th row and j-th column, Lemma 3.3 gives the estimate

$$|(\mathcal{S}\mu)'/\rho_{\overline{\mathbb{H}}}| < \frac{C}{\min\big(i,j,d-i,\lfloor\theta d\rfloor - j\big)}.$$

The lemma follows after squaring, summing over all the cells that make up $\overline{B}$ and dividing by $d\lfloor\theta d\rfloor$. □

4.2 Periodization

Given a box B, it is easy to tile the upper half-plane with similar boxes

$$\mathbb{H} = \bigcup_{j=1}^{\infty} \operatorname{clos} B_j, \qquad B_j = L_j(B), \qquad B_1 = B,$$

that have disjoint interiors. Furthermore, if μ is a Beltrami coefficient on B, we can form a Beltrami coefficient μ_{per} on $\mathbb{H}$ by requiring that $L_j^*\mu_{\mathrm{per}} = \mu$.

To prove Lemma 4.1, assume for the sake of contradiction that there is a box $B \subset \mathbb{H}$ and a Beltrami coefficient μ for which

$$⨍_B \left|\frac{2(\mathcal{S}\mu)'}{\rho_*}(z)\right|^2 \rho_*|dz|^2 \;>\; \Sigma^2 + C/R, \tag{4.3}$$

with $C > C_1$. According to Lemma 4.2, this would imply

$$⨍_{\overline{B}_j} \left|\frac{2(\mathcal{S}\mu_{\mathrm{per}})'}{\rho_{\overline{\mathbb{H}}}}(z)\right|^2 \rho_{\overline{\mathbb{H}}}|dz|^2 \;>\; \Sigma^2 + \epsilon, \quad j \ge 1. \tag{4.4}$$

4.3 Transferring to the unit disk

To transfer (4.4) to the unit disk, we would like exponentiate: $z \to \xi(z) = e^{2\pi i z}$. However, in order to do so, we must have a tiling invariant under the translation $z \to z+1$. While an exact tiling on $\mathbb{H}$ is not always possible by boxes similar to any given box, it is easy to construct a *packing* for which

$$\lim_{\epsilon\to 0} \left|\bigcup B_i \,\cap\, [i\epsilon, 1+\epsilon]\right| = 1.$$

Indeed, given a (non-periodic) tiling of the upper half-plane, it suffices to consider the integer translates of boxes B_i wholly contained in $\{0 < \operatorname{Re} z < 1\}$. Projecting to the unit disk gives a packing with

$$\lim_{r\to 1^-} \left| \bigcup \xi(B_i) \cap S_r \right| = 2\pi. \tag{4.5}$$

From the construction, μ_{per} descends to a Beltrami coefficient ν_{per} on the disk satisfying $\mu_{\text{per}} = \xi^* \nu_{\text{per}}$. (On the complement of the boxes, we take ν_{per} to be identically 0.) In terms of the Beurling transform, one is forced to pay a small error:

$$\left| \frac{2(\mathcal{S}\mu_{\text{per}})'}{\rho_{\overline{\mathbb{H}}}}(z) - \frac{2(\mathcal{S}\nu_{\text{per}})'}{\rho_*}(z) \right| \le o(\operatorname{Im} z), \qquad \text{as } \operatorname{Im} z \to 0. \tag{4.6}$$

To verify (4.6), one can use the following two facts:

(i) ξ is approximately a local isometry on $\{z \in \overline{\mathbb{H}} : |\operatorname{Im} z| < \delta\}$, when $\overline{\mathbb{H}}$ and $\mathbb{D}$ are equipped with their hyperbolic metrics.

(ii) ξ is approximately linear on a small ball $B(x, \delta)$ with $x \in \mathbb{R}$:

$$\left| \frac{1}{\xi'(x)} \cdot \frac{\xi(z) - \xi(w)}{z - w} - 1 \right| < \epsilon, \qquad z, w \in B(x, \delta).$$

Both of these properties are consequences of Koebe's distortion theorem, however, one needs to take into account that the exponential maps the real axis to the unit circle. The reader can consult [17, Section 2] for more details.

Putting (4.4) and (4.6) together shows that $\sigma^2(\mathcal{S}\nu_{\text{per}}) > \Sigma^2 + \epsilon$. This gives the desired contradiction, completing the proof of Lemma 4.1.

Remark. Lemma 4.1 is essentially sharp. To see this, observe that if μ is a Beltrami coefficient with $|\mu| \le \chi_{\mathbb{D}}$ and $\sigma^2(\mathcal{S}\mu) \ge \Sigma^2 - \epsilon$, then by the pigeon-hole principle, there exists a box B for which the integral in (4.1) is at least $\Sigma^2 - \epsilon$.

5 Locality of n_φ/ρ_*

In this section, we prove Lemma 1.1, the global version of the box lemma.

In view of the arguments of the previous section, it suffices to show an analogue of Lemma 3.3 :

Lemma 5.1. *Fix $0 < k < 1$. For any $\epsilon > 0$, there exists $R > 0$ sufficiently large so that if μ and ν are two Beltrami coefficients with $\mu \le k \cdot \chi_{\mathbb{D}}$, $\nu \le k \cdot \chi_{\mathbb{D}}$ and*

$$d_{\mathbb{D}}\Big(z^*, \operatorname{supp}(\mu - \nu)\Big) > R,$$

then,

$$\left| \frac{n_{w^\mu} - n_{w^\nu}}{\rho_*}(z) \right| \le \epsilon + o(|z| - 1), \qquad |z| > 1. \tag{5.1}$$

Since we do not require a quantitative error term, it suffices to give a simple compactness argument. It will be useful to consider *normalized* solutions $\tilde{w}^\mu$ of the Beltrami equation $\overline{\partial} w = \mu \, \partial w$ fixing $0, 1, \infty$. (The notation w^μ is reserved for principal solutions, defined for Beltrami coefficients that are compactly supported in the plane.) We will use the notation $\operatorname{dil.} \psi = \overline{\partial}\psi / \partial\psi$ to denote the dilatation of a quasiconformal mapping.

Lemma 5.2. *Suppose ψ_1 and ψ_2 are two k-quasiconformal mappings of the Riemann sphere, normalized to fix $0, 1, \infty$. For any $\epsilon > 0$, there exists R sufficiently large so that if $\operatorname{dil.} \psi_1 = \operatorname{dil.} \psi_2$ on $B_{\mathrm{hyp}}(w_0, R) = \{w \in \mathbb{H} : d_{\mathbb{H}}(w, w_0) > R\}$, then*

$$\left| \frac{n_{\psi_1} - n_{\psi_2}}{\rho_{\overline{\mathbb{H}}}}(\overline{w_0}) \right| < \epsilon. \tag{5.2}$$

Proof. Since non-linearity is invariant under compositions by an affine maps $z \to az + b$, $a > 0$, $b \in \mathbb{R}$, it suffices to prove the lemma with $w_0 = i$.

To the contrary, suppose that one could find sequences of Beltrami coefficients $\{\mu_n\}$ and $\{\nu_n\}$, with $\mu_n = \nu_n$ on $B_n = B_{\mathrm{hyp}}(i, 1/n)$, yet

$$\|\tilde{w}^{\mu_n} - \tilde{w}^{\nu_n}\|_{L^\infty(\overline{B})} > 1/n, \quad \text{where } \overline{B} = \{w \in \overline{\mathbb{H}} : d_{\overline{\mathbb{H}}}(w, -i) < 1\}. \tag{5.3}$$

Since the collection of normalized quasiconformal mappings with dilatation bounded by k forms a normal family, we can extract a convergent subsequence $\tilde{w}^{\mu_{n_j}} \to \tilde{w}$.

Factorization allows us to write $\tilde{w}^{\nu_n} = H_n \circ \tilde{w}^{\mu_n}$, where H_n is a normalized quasiconformal map with $\operatorname{supp}(\operatorname{dil.} H_n) \subseteq w^{\mu_n}(\mathbb{H} \setminus B_n)$ and $\| \operatorname{dil.} H_n\|_\infty < \frac{2k}{1+k^2}$. Since the supports of $\operatorname{dil.} H_{n_k}$ shrink to $\tilde{w}(\mathbb{R})$ which has measure 0, the only possible limit of H_{n_k} is the identity. This rules out (5.3), thus proving the lemma. □

5.1 Exponentiation

Like in Section 4, to transfer to the unit disk, we exponentiate. Given a normalized quasiconformal mapping φ fixing $0, 1, \infty$, we define its *exponential transform* as

$$\mathcal{E}_\varphi(w) := \frac{1}{2\pi i} \cdot \log\big(\varphi(\xi(w))\big). \tag{5.4}$$

The branch of logarithm above is chosen so that $\mathcal{E}_\varphi(w)$ fixes the origin, which ensures that $\mathcal{E}_\varphi(w)$ is also a normalized quasiconformal mapping. It is easy to see that

$$\mathcal{E}_\varphi(w+1) = \mathcal{E}_\varphi(w) + 1,$$

$$\operatorname{supp} \mu_{\mathcal{E}_\varphi} = e^{2\pi i w}(\operatorname{supp} \mu_\varphi), \quad \text{and} \quad |\mu_{\mathcal{E}_\varphi}(e^{2\pi i w})| = |\mu_\varphi(w)|.$$

We now show that exponentiation does not change the asymptotic features of non-linearity:

Lemma 5.3. *If φ is a normalized k-quasiconformal mapping that is conformal on $\mathbb{D}^*$, then*

$$\left| \frac{2n_{\mathcal{E}_\varphi}}{\rho_{\overline{\mathbb{H}}}}(w) - \frac{2n_\varphi}{\rho_*}(e^{2\pi i w}) \right| \le o(\operatorname{Im} w), \qquad \text{as } \operatorname{Im} w \to 0. \tag{5.5}$$

Proof. In view of the cocycle property of non-linearity, $n_{f\circ g}(z) = n_f(g(z))g'(z) + n_g(z)$, we have

$$n_{\mathcal{E}_\varphi}(w) = \Big[n_{\log}(\varphi(z))\varphi'(z) + n_\varphi(z)\Big]\xi'(w) + n_\xi(w),$$

where $z = \xi(w) = e^{2\pi i w}$. Since ξ is close to a local isometry between $(\overline{\mathbb{H}}, \rho_{\overline{\mathbb{H}}})$ and $(\mathbb{D}^*, \rho_*)$ when $|\operatorname{Im} w|$ is small, we have

$$\frac{n_{\mathcal{E}_\varphi}}{\rho_{\overline{\mathbb{H}}}}(w) \approx \frac{n_{\log}\big(\varphi(z)\big)}{\rho_*(z)} + \frac{n_\varphi}{\rho_*}(z) + \frac{n_\xi}{\rho_{\overline{\mathbb{H}}}}(w). \tag{5.6}$$

The third term is small because ξ is approximately linear; while the first term is small since $|n_{\log}(\varphi(z))| = 1/|\varphi(z)|$ is bounded on an annulus $\{z : 1 < |z| < 1 + \delta\}$. This gives (5.5). □

We now deduce Lemma 5.1 from Lemma 5.2:

Proof of Lemma 5.1. Since post-composition by a linear mapping does not change the non-linearity, we have $n_{w^\mu} = n_{\tilde{w}^\mu}$ and $n_{w^\nu} = n_{\tilde{w}^\nu}$. By the previous lemma, it suffices to compare the non-linearities of $\mathcal{E}_{\tilde{w}^\mu}(w)$ and $\mathcal{E}_{\tilde{w}^\nu}(w)$ where w is a pre-image of z under the exponential. However, since the exponential is nearly an isometry, the supports of $\operatorname{dil.} \mathcal{E}_{\tilde{w}^\mu}$ and $\operatorname{dil.} \mathcal{E}_{\tilde{w}^\nu}$ agree on a large neighbourhood of w. Thus, Lemma 5.1 follows from Lemma 5.2. □

A similar result involving Schwarzian derivatives was obtained in [10].

6 Estimating integral means

We first show that $D(k) \le 1 + 36\,k^2 + \mathcal{O}(k^3)$, for k small. We then explain how to use the box lemma to replace 36 with Σ^2. With some extra care, we refine the error term to $\mathcal{O}(k^{2.5})$.

6.1 Becker-Pommerenke argument

For convenience, we return to working with conformal maps defined on the unit disk. For a conformal map $\varphi : \mathbb{D} \to \mathbb{C}$, set

$$I_p(r) := \int_0^{2\pi} |\varphi'(re^{i\theta})^p| \, d\theta, \qquad p \in \mathbb{C}. \tag{6.1}$$

To estimate I_p, we use Hardy's identity (e.g. see [23, p. 174]) which says that if g is an analytic function in the disk, then

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\int_0^{2\pi} |g(re^{i\theta})| \, d\theta\right) = \int_0^{2\pi} \Delta |g(re^{i\theta})| \, d\theta.$$

Applying Hardy's identity to $\varphi'(z)^p$ gives

$$\frac{1}{r}\frac{d}{dr}(rI_p'(r)) \;=\; I_p''(r) + \frac{I_p'(r)}{r} \;=\; |p|^2 \int_{|z|=r} |\varphi'(z)^p| \left|\frac{\varphi''}{\varphi'}\right|^2 d\theta. \tag{6.2}$$

In particular, this shows that $rI_p'(r)$ is increasing, which implies that $I_p'(r) \geq 0$. Dropping the positive term $I_p'(r)/r$ in (6.2), we obtain

$$I_p''(r) \leq |p|^2 \int_{|z|=r} |\varphi'(z)^p| \left|\frac{\varphi''}{\varphi'}\right|^2 d\theta. \tag{6.3}$$

Replacing non-linearity by its supremum bound shows that

$$I_p''(r) \leq \frac{36|p|^2k^2}{(1-r^2)^2} I_p(r) \leq (1+\delta)\cdot\frac{9|p|^2k^2}{(1-r)^2} I_p(r), \quad r \in [r_0, 1), \tag{6.4}$$

where we can make $\delta > 0$ arbitrarily small, if we take r_0 is sufficiently close to 1. From the differential inequality (6.4) together with $I_p(r) \geq 0$, $I_p'(r) \geq 0$, it follows at once that

$$I_p(r) \leq C(\delta)\left(\frac{1}{1-r}\right)^{9|p|^2k^2(1+\delta)}, \qquad k \in [0,1), \quad p \in \mathbb{C}.$$

see Lemma 6.1(i) below. In other words, $\beta_\varphi(p) \leq 9|p|^2k^2$. Anti-symmetrization and the equation $\beta_\varphi(\mathrm{M.\dim}\,\partial\Omega) = \mathrm{M.\dim}\,\partial\Omega - 1$ yield the dimension bound $D(k) \leq 1 + 36\,k^2 + \mathcal{O}(k^3)$.

Remark. If we were given an infimum bound for non-linearity, we could instead work with $\tilde{I}_p(r) := rI_p(r)$ which satisfies $\tilde{I}_p(r) \geq 0$, $\tilde{I}_p'(r) \geq 0$ and

$$\tilde{I}_p''(r) \geq (1-\delta)\cdot\frac{|p|^2}{4(1-r)^2}\left[\int_{|z|=r} |\varphi'(z)^p| \left|\frac{2\varphi''}{\rho\varphi'}\right|^2 d\theta\right], \quad r \in [r_0, 1),$$

leading to lower bounds for integral means.

6.2 A differential inequality

To make use of (6.4), we used an elementary fact about differential inequalities. The proof is elementary, but the reader may consult [23, Proposition 8.7].

Lemma 6.1. (i) *Suppose $u(r)$ is a C^2 function on $[0,1)$ with*

$$u \geq 0 \qquad \text{and} \qquad u' \geq 0$$

satisfying

$$u''(r) \leq \frac{\alpha u}{(1-r)^2}, \qquad r \in [r_0, 1), \tag{6.5}$$

for some constant $\alpha > 0$. Then,

$$u(r) \;\leq\; v(r) \;=\; C\Big(\frac{1}{1-r}\Big)^{\beta}, \quad \text{where } \beta^2 + \beta = \alpha,$$

and $C > 0$ is sufficiently large so that $v(r_0) \geq u(r_0)$ and $v'(r_0) \geq u'(r_0)$.

(ii) *Conversely, if (6.5) is replaced by*

$$u''(r) \geq \frac{\alpha u}{(1-r)^2}, \tag{6.6}$$

then

$$u(r) \;\geq\; v(r) \;=\; c\Big(\frac{1}{1-r}\Big)^{\beta},$$

where $c > 0$ is sufficiently small so that $v(r_0) \leq u(r_0)$ and $v'(r_0) \leq u'(r_0)$.

Remark. When $\alpha > 0$ is small,

$$\beta = \alpha - \mathcal{O}(\alpha^2), \tag{6.7}$$

there is not much difference between α and β.

6.3 Averaging over annuli

Using the box lemma, we can give an improvement in the argument of Becker and Pommerenke. For this purpose, we consider the function

$$u(r) = \int_{\mathfrak{a}(r)} I_p(s)\, \frac{ds}{1-s}, \tag{6.8}$$

where

$$\mathfrak{a}(r) = \bigg(1 - (1-r),\, 1 - \frac{(1-r)}{e^R}\bigg), \qquad R > 0.$$

Since $\|\log \varphi'\|_{\mathcal{B}} \le 6$, we have $u(r) \asymp I_p$. In particular, to compute the integral means spectrum of φ, it suffices to measure the growth of $u(r)$. Let $A(r)$ be the annulus $\{z : |z| \in \mathfrak{a}(r)\}$. Integrating (6.3) and dividing both sides by $(1-r)^2$ gives

$$\frac{1}{(1-r)^2}\int_{\mathfrak{a}(r)} I_p''(s)(1-s)ds \le (1+\delta)\cdot\frac{|p|^2}{4(1-r)^2}\int_{A(r)} |\varphi'(z)^p|\left|\frac{2n_\varphi}{\rho}\right|^2\frac{ds}{1-s}\cdot d\theta, \quad (6.9)$$

provided $r \in [r_0, 1)$. A simple computation shows that the left hand side of (6.9) is simply $u''(r)$. We now estimate the right hand side:

Lemma 6.2. *Suppose that the average non-linearity of $\varphi \in \mathbf{\Sigma}_k$ over any R-box sufficiently close to the unit circle is bounded by M. Then,*

$$u''(r) \le \exp(CRk|p|)\cdot\frac{|p|^2}{4}\cdot\frac{Mu}{(1-r)^2}, \qquad r \in [r_0, 1), \quad (6.10)$$

for some r_0 sufficiently close to 1.

Proof. We partition the annulus $A(r)$ into R-boxes $B_1, B_2, \dots, B_{n(r)}$. (The number of boxes is roughly proportional to the hyperbolic length of $\{|z| = r\}$ divided by R, but we will not use this.) Since φ has a k-quasiconformal extension to the plane, $\|\log\varphi'\|_{\mathcal{B}} \le 6k$. This implies that the multiplicative oscillation of $|\varphi'|$ in a box B_j is at most

$$\operatorname{osc}_{B_j}|\varphi'| := \sup_{z_1, z_2 \in B_j}\log\frac{|\varphi'(z_1)|}{|\varphi'(z_2)|} \le C_1 Rk.$$

and so $\operatorname{osc}_{B_j}|\varphi'(z)^p| = C_1Rk|p|$. Therefore, if $k|p|$ is small, $|\varphi'(z)^p|$ is essentially constant on boxes, i.e.

$$\int_{B_j}|\varphi'(z)^p|\left|\frac{2n_\varphi}{\rho}\right|^2\frac{|dz|^2}{1-|z|} \approx |\varphi'(c_j)^p|\int_{B_j}\left|\frac{2n_\varphi}{\rho}\right|^2\frac{|dz|^2}{1-|z|}, \quad (6.11)$$

where c_j is an arbitrary point in B_j. Hence,

$$\fint_{B_j}|\varphi'(z)^p|\left|\frac{2n_\varphi}{\rho}\right|^2\frac{|dz|^2}{1-|z|} \le \exp(C_2Rk|p|)\cdot Mu(r).$$

Summing over all the boxes that make up the annulus $A(r)$ gives (6.10). □

6.4 Applications

We now prove Theorems 1.1 – 1.6. We begin with Theorem 1.1, which says that for a fixed $k \in (0,1)$,

$$\lim_{p \to 0} \frac{B_k(p)}{|p|^2/4} = \Sigma^2(k).$$

Proof of Theorem 1.1. We first choose an $R > 0$ sufficiently large so that the average non-linearity over R-boxes is at most $\Sigma^2(k) + \epsilon/3$. Lemma 6.2 implies that

$$u''(r) \le \Big(\Sigma^2(k) + 2\epsilon/3\Big) \cdot \frac{|p|^2}{4}, \qquad r \in [r_0, 1),$$

if $|p|$ is small. In view of Lemma 6.1(i), this shows that

$$u(r) \le C\Big(\frac{1}{1-r}\Big)^{|p|^2(\Sigma^2(k)+\epsilon)/4}, \qquad r \in [r_0, 1),$$

which implies a similar bound for $I_p(r)$. □

To prove Theorem 1.5, it suffices to explain how to utilize the lower bound for box averages. One can do this by reworking the proof of Lemma 6.2 with

$$\tilde{u}(r) = \int_{\mathfrak{a}(r)} \tilde{I}_p(s)\, \frac{ds}{1-s}$$

and

$$\frac{1}{(1-r)^2} \int_{\mathfrak{a}(r)} \tilde{I}_p''(s)(1-s)ds \ge (1-\delta) \cdot \frac{|p|^2}{4(1-r)^2} \int_{A(r)} |\varphi'(z)^p| \left|\frac{2n_\varphi}{\rho}\right|^2 \frac{ds}{1-s} \cdot d\theta.$$

McMullen's identity (Theorem 1.2) now follows from Theorem 1.5 and (1.12). To prove Theorem 1.6, it suffices to use Lemma 3.3 to give an improved upper bound for the average non-linearity over R-boxes, where R is the distance between the components A_j. The computation is somewhat similar to the one in [17, Section 6].

For small k, we can give a slightly more precise estimate. In view of (2.3), the right hand side of (6.9) is bounded by

$$\le \frac{(k^2 + Ck^3)|p|^2}{4(1-r)^2} \int_{A(r)} |\varphi'|^p \left|\frac{2\mathcal{S}\mu}{\rho}\right|^2 \frac{ds}{1-s} \cdot d\theta, \tag{6.12}$$

Choosing $R \asymp 1/\sqrt{k|p|}$, we get an error term of $\mathcal{O}\big((k|p|)^{2.5}\big)$. This proves Theorem 1.3. As mentioned in the introduction, this implies Theorem 1.4.

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