

Exercise A.66. Calculate the denominator of the left-hand side:

$$|z_j^{n-i} + z_j^{-(n-i)}| = 2 \cdot \Delta(\xi_1, \dots, \xi_n).$$

PROOF OF PROPOSITION A.64. Note that  $\zeta_j \cdot \xi_j(l) = \xi_j(l+1) - \xi_j(l-1)$ . Multiplying the numerator and denominator by  $\zeta_1 \cdots \zeta_n$ , the numerator becomes  $|\zeta_j(\lambda_i + n - i + 1) - \zeta_j(\lambda_i + n - i - 1)|$  and the denominator becomes

$$|\zeta_j(n - i + 1) - \zeta_j(n - i - 1)| = 2|\zeta_j(n - i + 1)|;$$

this is seen by noting that the bottom row of the matrix on the left is  $(\zeta_j(1) - \zeta_j(-1)) = (2\zeta_j(1))$ , and performing row reductions starting from the bottom row. The rest of the proof is the same as in the preceding proposition. The only change is when  $\lambda_n = 0$ , in which case the bottom row in the numerator matrix is the same as that in the denominator.  $\square$

Exercise A.67\*. Find a similar formula for

$$\frac{|z_j^{\lambda_i+n-i} - z_j^{-(\lambda_i+n-i)}|}{|z_j^{n-i} + z_j^{-(n-i)}|}.$$

## APPENDIX B

# On Multilinear Algebra

In this appendix we state the basic facts about tensor products and exterior and symmetric powers that are used in the text. It is hoped that a reader with some linear algebra background can fill in details of the proofs.

§B.1: Tensor product

§B.2: Exterior and symmetric powers

§B.3: Duals and contractions

### §B.1. Tensor Products

The *tensor product* of two vector spaces  $V$  and  $W$  over a field is a vector space  $V \otimes W$  equipped with a bilinear map

$$V \times W \rightarrow V \otimes W, \quad v \times w \mapsto v \otimes w,$$

which is universal: for any bilinear map  $\beta: V \times W \rightarrow U$  to a vector space  $U$ , there is a unique linear map from  $V \otimes W$  to  $U$  that takes  $v \otimes w$  to  $\beta(v, w)$ . This universal property determines the tensor product up to canonical isomorphism. If the ground field  $K$  needs to be mentioned, the tensor product is denoted  $V \otimes_K W$ .

If  $\{e_i\}$  and  $\{f_j\}$  are bases for  $V$  and  $W$ , the elements  $\{e_i \otimes f_j\}$  form a basis for  $V \otimes W$ . This can be used to construct  $V \otimes W$ . The construction is functorial: linear maps  $V \rightarrow V'$  and  $W \rightarrow W'$  determine a linear map from  $V \otimes W$  to  $V' \otimes W'$ .

Similarly one has the tensor product  $V_1 \otimes \cdots \otimes V_n$  of  $n$  vector spaces, with its universal multilinear map

$$V_1 \times \cdots \times V_n \rightarrow V_1 \otimes \cdots \otimes V_n,$$

taking  $v_1 \times \cdots \times v_n$  to  $v_1 \otimes \cdots \otimes v_n$ . (Recall that a map from the Cartesian product to a vector space  $U$  is *multilinear* if, when all but one of the factors  $V_i$  are fixed, the resulting map from  $V_i$  to  $U$  is linear.) The construction of tensor products is commutative:

$$V \otimes W \cong W \otimes V, \quad v \otimes w \mapsto w \otimes v;$$

distributive:

$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W);$$

and associative:

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W,$$

by  $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$ .

In particular, there are *tensor powers*  $V^{\otimes n} = V \otimes \cdots \otimes V$  of a fixed space  $V$ . By convention,  $V^{\otimes 0}$  is the ground field.

If  $A$  is an algebra over the ground field, and  $V$  is a right  $A$ -module, and  $W$  a left  $A$ -module, there is a tensor product denoted  $V \otimes_A W$ , which can be constructed as the quotient of  $V \otimes W$  by the subspace generated by all  $(v \cdot a) \otimes w - v \otimes (a \cdot w)$  for all  $v \in V, w \in W$ , and  $a \in A$ . The resulting map from  $V \times W$  to  $V \otimes_A W$  is universal for bilinear maps  $\beta$  from  $V \times W$  to vector spaces  $U$  that satisfy the property that  $\beta(v \cdot a, w) = \beta(v, a \cdot w)$ . This tensor product is also distributive.

## §B.2. Exterior and Symmetric Powers

The *exterior powers*  $\wedge^n V$  of a vector space  $V$ , sometimes denoted  $\text{Alt}^n V$ , come equipped with an alternating multilinear map

$$V \times \cdots \times V \rightarrow \wedge^n V, \quad v_1 \times \cdots \times v_n \mapsto v_1 \wedge \cdots \wedge v_n,$$

that is universal: for  $\beta: V \times \cdots \times V \rightarrow U$  an alternating multilinear map, there is a unique linear map from  $\wedge^n V$  to  $U$  which takes  $v_1 \wedge \cdots \wedge v_n$  to  $\beta(v_1, \dots, v_n)$ . Recall that a multilinear map  $\beta$  is *alternating* if  $\beta(v_1, \dots, v_n) = 0$  whenever two of the vectors  $v_i$  are equal. This implies that  $\beta(v_1, \dots, v_n)$  changes sign when two of the vectors are interchanged.<sup>1</sup> It follows that

$$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \text{sgn}(\sigma) \beta(v_1, \dots, v_n) \quad \text{for all } \sigma \in \mathfrak{S}_n.$$

The exterior power can be constructed as the quotient space of  $V^{\otimes n}$  by the subspace generated by all  $v_1 \otimes \cdots \otimes v_n$  with two of the vectors equal. We let

$$\pi: V^{\otimes n} \rightarrow \wedge^n V, \quad \pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$$

<sup>1</sup> This follows from the standard polarization: for two factors,  $\beta(v+w, v+w) - \beta(v, v) - \beta(w, w) = \beta(v, w) + \beta(w, v)$ .

denote the projection. If  $\{e_i\}$  is a basis for  $V$ , then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} : i_1 < i_2 < \cdots < i_n\}$$

is a basis for  $\wedge^n V$ . Define  $\wedge^0 V$  to be the ground field.

If  $V$  and  $W$  are vector spaces, there is a canonical linear map from  $\wedge^a V \otimes \wedge^b W$  to  $\wedge^{a+b}(V \oplus W)$ , which takes  $(v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_b)$  to  $v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$ . This determines an isomorphism

$$\wedge^n(V \oplus W) \cong \bigoplus_{a=0}^n \wedge^a V \otimes \wedge^{n-a} W. \quad (\text{B.1})$$

(From this isomorphism the assertion about bases of  $\wedge^n V$  follows by induction on the dimension.)

The *symmetric powers*  $\text{Sym}^n V$ , sometimes denoted  $S^n V$ , comes with a universal symmetric multilinear map

$$V \times \cdots \times V \rightarrow \text{Sym}^n V, \quad v_1 \times \cdots \times v_n \mapsto v_1 \cdots v_n.$$

Recall that a multilinear map  $\beta: V \times \cdots \times V \rightarrow U$  is *symmetric* if it is unchanged when any two factors are interchanged, or

$$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \beta(v_1, \dots, v_n) \quad \text{for all } \sigma \in \mathfrak{S}_n.$$

The symmetric power can be constructed as the quotient space of  $V^{\otimes n}$  by the subspace generated by all  $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$ , or by those in which  $\sigma$  permutes two successive factors. Again we let

$$\pi: V^{\otimes n} \rightarrow \text{Sym}^n V, \quad \pi(v_1 \otimes \cdots \otimes v_n) = v_1 \cdots v_n,$$

denote the projection. If  $\{e_i\}$  is a basis for  $V$ , then

$$\{e_{i_1} \cdot e_{i_2} \cdots e_{i_n} : i_1 \leq i_2 \leq \cdots \leq i_n\}$$

is a basis for  $\text{Sym}^n V$ . So  $\text{Sym}^n V$  can be regarded as the space of homogeneous polynomials of degree  $n$  in the variables  $e_i$ . Define  $\text{Sym}^0 V$  to be the ground field. As before, there are canonical isomorphisms

$$\text{Sym}^n(V \oplus W) \cong \bigoplus_{a=0}^n \text{Sym}^a V \otimes \text{Sym}^{n-a} W. \quad (\text{B.2})$$

The exterior powers  $\wedge^n V$  and symmetric powers  $\text{Sym}^n V$  can also be realized as subspaces of  $V^{\otimes n}$ , assuming, as we have throughout, that the ground field has characteristic 0. We will denote the inclusions by  $\iota$ , so we have

$$V^{\otimes n} \xrightarrow{\pi} \wedge^n V \xrightarrow{\iota} V^{\otimes n}, \quad V^{\otimes n} \xrightarrow{\pi} \text{Sym}^n V \xrightarrow{\iota} V^{\otimes n}.$$

The imbedding  $\iota: \wedge^n V \rightarrow V^{\otimes n}$  is defined by

$$\iota(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \quad (\text{B.3})$$

(This is well defined since the right-hand side is alternating.) The image of  $\iota$  is the space of anti-invariants of the right action of  $\mathfrak{S}_n$  on  $V^{\otimes n}$ :

$$(v_1 \otimes \cdots \otimes v_n) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad v_i \in V, \sigma \in \mathfrak{S}_n. \quad (B.4)$$

(The anti-invariants are the vectors  $z \in V^{\otimes n}$  such that  $z \cdot \sigma = \text{sgn}(\sigma)z$  for all  $\sigma \in \mathfrak{S}_n$ .) Moreover, if  $A = \iota \circ \pi$ , then  $(1/n!)A$  is the projection onto this anti-invariant subspace.<sup>2</sup> (Often the coefficient  $1/n!$  is put in front of the formula for  $\iota$ ; this makes no essential difference, but leads to awkward formulas for contractions.)

Similarly we have  $\iota: \text{Sym}^n V \rightarrow V^{\otimes n}$  by

$$\iota(v_1 \cdots v_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \quad (B.5)$$

The image of  $\iota$  is the space of invariants of the right action of  $\mathfrak{S}_n$  on  $V^{\otimes n}$ . If  $A = \iota \circ \pi$ , then  $(1/n!)A$  is the projection onto this invariant subspace.

The wedge product  $\wedge$  determines a product

$$\wedge^m V \otimes \wedge^n V \xrightarrow{\wedge} \wedge^{m+n} V, \quad (B.6)$$

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n},$$

which is associative and skew-commutative. This product is compatible with the projection from the tensor powers onto the exterior powers, but care must be taken for the inclusion of exterior in tensor powers, since for example  $v \wedge w$  is sent to  $v \otimes w - w \otimes v$  [not to  $\frac{1}{2}(v \otimes w - w \otimes v)$ ] by  $\iota$ . In general, the diagram

$$\begin{array}{ccc} \wedge^m V \otimes \wedge^n V & \xrightarrow{\wedge} & \wedge^{m+n} V \\ \downarrow \iota & & \downarrow \iota \\ V^{\otimes m} \otimes V^{\otimes n} & \xrightarrow{\quad} & V^{\otimes(m+n)} \end{array} \quad (B.7)$$

commutes when the bottom horizontal map is defined by the formula

$$(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_{m+n}) \mapsto \sum \text{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \otimes v_{\sigma(m+1)} \otimes \cdots \otimes v_{\sigma(m+n)}, \quad (B.8)$$

the sum over all "shuffles," i.e., permutations  $\sigma$  of  $\{1, \dots, m+n\}$  that preserve the order of the subsets  $\{1, \dots, m\}$  and  $\{m+1, \dots, m+n\}$ .

Similarly the symmetric powers have a commutative product  $(v_1 \cdots v_m) \otimes (v_{m+1} \cdots v_{m+n}) \mapsto v_1 \cdots v_m \cdot v_{m+1} \cdots v_{m+n}$ , with a similar compatibility. Note that  $v^2 \in \text{Sym}^2 V$  is sent to  $2v \otimes v$  in  $V \otimes V$ ,  $v^n \in \text{Sym}^n V$  to  $n!(v \otimes \cdots \otimes v)$  in  $V^{\otimes n}$ , and generally one has the analogue of (B.7), changing each "sgn( $\sigma$ )" to "1" in formula (B.8).

All these mappings are compatible with linear maps of vector spaces  $V \rightarrow W$ , and in particular commute with the left actions of the general linear group  $\text{GL}(V) = \text{Aut}(V)$  of automorphisms, or the algebra  $\text{End}(V) = \text{Hom}(V, V)$  of endomorphisms, on  $V^{\otimes n}$ ,  $\wedge^n V$ , and  $\text{Sym}^n V$ .

<sup>2</sup> It is this factor which limits our present discussion to vector spaces over fields of characteristic 0.

It is sometimes convenient to make algebras out of the direct sum of all of the tensor, exterior, or symmetric powers. The *tensor algebra*  $T^*V$  is the sum  $\bigoplus_{n \geq 0} V^{\otimes n}$ , with product determined by the canonical isomorphism  $V^{\otimes n} \otimes V^{\otimes m} \rightarrow V^{\otimes(n+m)}$ . The *exterior algebra*  $\wedge^* V$  is the sum  $\bigoplus_{n \geq 0} \wedge^n V$ , which is the quotient of  $T^*V$  by the two-sided ideal generated by all  $v \otimes v$  in  $V^{\otimes 2}$ . The *symmetric algebra*  $\text{Sym}^* V$  is the sum  $\bigoplus_{n \geq 0} \text{Sym}^n V$ , which is the quotient of  $T^*V$  by the two-sided ideal generated by all  $v \otimes w - w \otimes v$  in  $V^{\otimes 2}$ .

**Exercise B.9.** The algebra  $\text{Sym}^* V$  is a commutative, graded algebra, which satisfies the universal property that any linear map from  $V$  to the first graded piece  $C^1$  of a commutative graded algebra  $C$  determines a homomorphism  $\text{Sym}^* V \rightarrow C$  of graded algebras. Use this to show that  $\text{Sym}^*(V \oplus W) \cong \text{Sym}^* V \otimes \text{Sym}^* W$ , and deduce the isomorphism (B.2). Prove the analogous assertions for  $\wedge^* V$ , in the category of skew-commutative graded algebras. In particular, construct an isomorphism  $\wedge^*(V \oplus W) \cong \wedge^* V \hat{\otimes} \wedge^* W$ , where  $\hat{\otimes}$  denotes the skew-commutative tensor product: it is the usual tensor product additively, but the product has  $(a \otimes b) \cdot (c \otimes d) = (-1)^{\text{deg}(b)\text{deg}(c)}(a \cdot b) \otimes (c \cdot d)$  for homogeneous elements  $a$  and  $c$  in the first algebra, and  $b$  and  $d$  in the second. In particular, this proves (B.1).

### §B.3. Duals and Contractions

Although only a few simple contractions are used in the lectures, and most of these are written out by hand where needed, it may be useful to see the general picture.

If  $V^*$  denotes the dual space to  $V$ , there are contraction maps

$$c_j^i: V^{\otimes p} \otimes (V^*)^{\otimes q} \rightarrow V^{\otimes(p-1)} \otimes (V^*)^{\otimes(q-1)},$$

for any  $1 \leq i \leq p$  and  $1 \leq j \leq q$ , determined by evaluating the  $j$ th coordinate of  $(V^*)^{\otimes q}$  on the  $i$ th coordinate of  $V^{\otimes p}$ :

$$\begin{aligned} c_j^i(v_1 \otimes \cdots \otimes v_p \otimes \phi_1 \otimes \cdots \otimes \phi_q) \\ = \phi_j(v_i) v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_p \otimes \phi_1 \otimes \cdots \otimes \hat{\phi}_j \otimes \cdots \otimes \phi_q. \end{aligned} \quad (B.10)$$

More generally if  $I = (i_1, \dots, i_n)$  and  $J = (j_1, \dots, j_n)$  are two sequences of  $n$  distinct indices from  $\{1, \dots, p\}$  and  $\{1, \dots, q\}$ , respectively, there is a contraction

$$c_J^I: V^{\otimes p} \otimes (V^*)^{\otimes q} \rightarrow V^{\otimes(p-n)} \otimes (V^*)^{\otimes(q-n)} \quad (B.11)$$

which takes  $v_1 \wedge \cdots \wedge v_p \otimes \phi_1 \otimes \cdots \otimes \phi_q$  to

$$\prod_{\alpha=1}^n \phi_{j_\alpha}(v_{i_\alpha}) v_1 \otimes \cdots \otimes \hat{v}_{i_\alpha} \otimes \cdots \otimes v_p \otimes \phi_1 \otimes \cdots \otimes \hat{\phi}_{j_\alpha} \otimes \cdots \otimes \phi_q.$$

For example, if  $p = q = n$  and  $I = J = (1, \dots, n)$ , this contraction  $V^{\otimes n} \otimes (V^*)^{\otimes n} \rightarrow \mathbb{C}$  identifies  $(V^*)^{\otimes n}$  with the dual space of  $V^{\otimes n}$ .

Now  $(V^{\otimes n})^*$  consists of  $n$ -multilinear forms on  $V$ , and  $(\wedge^n V)^*$  consists of alternating  $n$  multilinear forms on  $V$ ; in particular,  $(\wedge^n V)^*$  is a subspace of  $(V^{\otimes n})^*$ ; this is the inclusion via  $\pi^*$ . The composite

$$\wedge^n(V^*) \rightarrow (V^*)^{\otimes n} \rightarrow (V^{\otimes n})^*,$$

where the first map is the inclusion  $\iota$  and the second is the isomorphism of the preceding paragraph, maps  $\wedge^n(V^*)$  isomorphically onto the subspace  $(\wedge^n V)^*$ . Explicitly,

$$\wedge^n(V^*) \xrightarrow{\cong} (\wedge^n V)^*,$$

$$\varphi_1 \wedge \cdots \wedge \varphi_n \mapsto [v_1 \wedge \cdots \wedge v_n \mapsto \sum \text{sgn}(\sigma) \varphi_{\sigma(1)}(v_1) \cdots \varphi_{\sigma(n)}(v_n) = \det(\varphi_j(v_i))].$$

This dual pairing  $\wedge^n V \otimes \wedge^n(V^*) \rightarrow K$  is often denoted  $\langle \cdot, \cdot \rangle$ .

There is a similar isomorphism of  $\text{Sym}^n(V^*)$  with  $\text{Sym}^n(V)^*$ , but without the signs “sgn( $\sigma$ ).”

**Exercise B.12.** If  $e_1, \dots, e_m$  is a basis for  $V$ , with  $e_i^*$  the dual basis for  $V^*$ , then  $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : 1 \leq i_1 < \cdots < i_n \leq m\}$  is a basis for  $\wedge^n V$ , and  $\{e_1^{i_1} \cdots e_m^{i_m} : i_\alpha \geq 0, \sum i_\alpha = n\}$  is a basis for  $\text{Sym}^n V$ . Show that, via the above isomorphisms, the dual bases for  $\wedge^n(V^*)$  and  $\text{Sym}^n(V^*)$  are

$$\{e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*\} \text{ and } \left\{ \frac{1}{\prod_{\alpha} (i_\alpha)!} (e_1^*)^{i_1} \cdots (e_m^*)^{i_m} \right\}.$$

There are related contractions, sometimes called internal products, and denoted  $\lrcorner$  and  $\llcorner$ , on exterior and symmetric powers. For the exterior powers they are maps:

$$\begin{aligned} \wedge^p V \otimes \wedge^{p+q}(V^*) &\rightarrow \wedge^q(V^*), & x \otimes \alpha &\mapsto x \lrcorner \alpha; \\ \wedge^{p+q} V \otimes \wedge^p(V^*) &\rightarrow \wedge^q(V), & x \otimes \alpha &\mapsto x \llcorner \alpha. \end{aligned} \tag{B.13}$$

These can be defined most simply as transposes of wedge products, i.e., they are determined by the identities

$$\langle z, x \lrcorner \alpha \rangle = \langle z \wedge x, \alpha \rangle \text{ for } z \in \wedge^q V,$$

and

$$\langle x \llcorner \alpha, \beta \rangle = \langle x, \alpha \wedge \beta \rangle \text{ for } \beta \in \wedge^q(V^*).$$

(The relation of this definition to the contraction maps  $c_j^I$  above is expressed in Exercise B.16.) Note that when  $q = 0$ , these contractions reduce to the previous duality pairing between  $\wedge^p V$  and  $\wedge^p(V^*)$ .

For symmetric powers, the internal products are defined similarly:

$$\begin{aligned} \text{Sym}^p V \otimes \text{Sym}^{p+q}(V^*) &\rightarrow \text{Sym}^q(V^*), & x \otimes \alpha &\mapsto x \lrcorner \alpha; \\ \text{Sym}^{p+q} V \otimes \text{Sym}^p(V^*) &\rightarrow \text{Sym}^q(V), & x \otimes \alpha &\mapsto x \llcorner \alpha. \end{aligned} \tag{B.14}$$

**Exercise B.15.** For  $v, w \in V$ , and  $\varphi, \psi \in V^*$ , show that

$$v \lrcorner (\varphi \wedge \psi) = \psi(v)\varphi - \varphi(v)\psi \quad \text{and} \quad (v \wedge w) \llcorner \varphi = \varphi(v)w - \varphi(w)v.$$

More generally, for if  $x = v_1 \wedge \cdots \wedge v_p$  and  $\alpha = \varphi_1 \wedge \cdots \wedge \varphi_{p+q}$ , with  $v_i \in V$  and  $\varphi_j \in V^*$ , then

$$(i) \quad x \lrcorner \alpha = \sum \text{sgn}(\sigma) \varphi_{\sigma(q+1)}(v_1) \cdots \varphi_{\sigma(q+p)}(v_p) \cdot \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(q)},$$

the sum over all permutations  $\sigma$  of  $\{1, \dots, p+q\}$  that preserve the order of  $\{1, \dots, q\}$ . If  $x = v_1 \wedge \cdots \wedge v_{p+q}$  and  $\alpha = \varphi_1 \wedge \cdots \wedge \varphi_p$ , then

$$(ii) \quad x \llcorner \alpha = \sum \text{sgn}(\sigma) \varphi_1(v_{\sigma(1)}) \cdots \varphi_p(v_{\sigma(p)}) \cdot v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(p+q)},$$

the sum over all permutations that preserve the order of  $\{p+1, \dots, p+q\}$ . Verify these formulas and use them to give formulas for these internal products in terms of standard bases. State and verify analogous formulas for symmetric powers. In particular, for  $v, w \in V, \varphi, \psi \in V^*$ ,

$$v \lrcorner (\varphi \cdot \psi) = \psi(v)\varphi + \varphi(v)\psi \quad \text{and} \quad (v \cdot w) \llcorner \varphi = \varphi(v)w + \varphi(w)v.$$

For example,  $v \lrcorner (\varphi^2) = 2\varphi(v)\varphi$  and  $(v^2) \llcorner \varphi = 2\varphi(v)v$ .

**Exercise B.16.** Using formula (ii) of the preceding exercise, show that the contraction map  $\llcorner$  may be given as  $1/p!q!$  times the composition of the maps

$$\wedge^{p+q} V \otimes \wedge^p(V^*) \rightarrow V^{\otimes(p+q)} \otimes (V^*)^{\otimes p} \rightarrow V^{\otimes q} \rightarrow \wedge^q V,$$

where the middle map is the contraction map  $c_j^I$  of (B.11), with  $I = J = \{1, \dots, p\}$ , and the other maps come from  $\iota$  and  $\pi$ . Prove the same formulas (with the same scalar factor) for the other internal products.

**Exercise B.17.** In the situation of formula (ii), suppose the  $v_i$  are independent, and let  $W$  be the  $(p+q)$ -dimensional subspace of  $V$  that they span; suppose the  $\varphi_i$  are independent, and let  $Z$  be the  $p$ -codimensional subspace of  $V$  of the common zeros of the  $\varphi_i$ . Show that  $x \llcorner \alpha = 0$  if  $\dim(W \cap Z) > q$ , and otherwise  $x \llcorner \alpha = u_1 \wedge \cdots \wedge u_q$  for some vectors  $u_i$  that span  $W \cap Z$ .

**Exercise B.18.** Prove the formulas

$$(x \wedge y) \lrcorner \alpha = x \lrcorner (y \lrcorner \alpha) \quad \text{and} \quad x \llcorner (\alpha \wedge \beta) = (x \llcorner \alpha) \llcorner \beta.$$

State and verify the analogous formulas for symmetric powers.

For a detailed development of these ideas, see [Bour, *Algebra*, Chap. 3].