Intrinsic wildness of homeomorphisms. (M. Freedman, details and writeup by C. McMullen.)

Let $S^1 = \mathbb{R}/\mathbb{Z}$ and consider the annulus $A = [0,1] \times S^1$.

Theorem. Given a modulus of continuity $m(s) \to 0$ as $s \to 0$, there exists a homeomorphism $F : A \to A$ such that for every homeomorphism h of A, the modulus of continuity of hFh^{-1} is worse than $m(s)$.

Let us start out by constructing a wild map $f : [0,1] \to \mathbb{R}$. First choose a sequence of integers $a_n \to \infty$ rapidly. Then set

$$f(x) = \sum_0^\infty 2^{-n} \sin(2\pi a_n x).$$

Let

$$\operatorname{var}(f, \epsilon) = \sup\{|f(x) - f(y)| \,:\, |x - y| < \epsilon\}.$$

Note that along $[0,1]$, $f(x)$ oscillates about a_n times with laps of size 2^{-n} or larger. The same is true for $f \circ h$, for any homeomorphism $h : [0,1] \to [0,1]$. Thus there are two points separated by at most $1/a_n$ that form the endpoints of one lap. Therefore

$$\operatorname{var}(f \circ h, 1/a_n) \geq 2^{-n}.$$

On the other hand, if $f \circ h$ has modulus of continuity $m(s)$, then

$$\operatorname{var}(f \circ h, 1/a_n) \leq m(1/a_n).$$

So by choosing a_n large enough one can defeat any given modulus of continuity $m(s)$. (By ‘defeat we mean there exist $r_n \to 0$ such that

$$\operatorname{var}(f \circ h, r_n) > m(r_n)$$

for all n.)

Now define $F : A \to A$ by

$$F(x, t) = (x, t + f(x)).$$

Note that under iteration, we have

$$F^q(x, t) = (x, t + q f(x)).$$

Let $q = 2^n$. Then whenever $f(x)$ varies by size larger than 2^{-n} over an interval J, there is a subinterval J' such that F fixes the boundary of $J' \times S^1$ and performs a power of a Dehn twist on the interior.

Since $f(x)$ has at least a_n laps of size 2^{-n}, the map F^q fixes an ordered sequence of a_n circles $C_i = \{x_i\} \times S^1$, $x_1 < x_2 < \ldots$, and performs a nonzero power of a Dehn twist on the annulus between C_i and C_{i+1}. Two of these circles must be within distance $1/a_n$ of each other, by area considerations. Let L be the shortest arc joining them; we have $|L| < 1/a_n$. Since $F^q(L)$ must wrap at least once around A, we find that:

$$\mathrm{var}(F^q, 1/a_n) \geq 1/3.$$

The same reasoning applies to any conjugate of F by a homeomorphism.

On the other hand, if F has modulus of continuity $m(s)$, then

$$\mathrm{var}(F^q, 1/a_n) \leq m^q(1/a_n),$$

where m^q means we iterate $s \mapsto m(s)$ q times. So again, by choosing a_n large enough, we can defeat any modulus of continuity $m(s)$, and the argument applies not just to F but uniformly to all conjugates hFh^{-1}.