THE ASYMPTOTIC NUMBER OF UNLABELLED REGULAR GRAPHS

BÉLA BOLLOBÁS

Over ten years ago Wright [4] proved a fundamental theorem in the theory of random graphs. He showed that if $M = M(n)$ is such that almost no labelled graph of order n and size M has two isolated vertices or two vertices of degree $n-1$, then the number of labelled graphs of order n and size M divided by the number of unlabelled graphs of order n and size M is asymptotic to $n!$. The result is best possible, since if the above ratio is asymptotic to $n!$ then almost no labelled graph of order n and size M has a non-trivial automorphism. The aim of this paper is to prove the analogue of Wright's theorem for regular graphs.

Random regular graphs have not been studied for long. The main reason for this is that until recently there was no asymptotic formula for the number of labelled regular graphs: such a formula was found by Bender and Canfield [1]. Even more recently, in [2] a model was given for the set of labelled regular graphs which makes the study of random regular graphs fairly accessible. In particular, a result in [3] implies that, for $r \geqslant 3$, almost every labelled r-regular graph has only the identity as its automorphism.

Let $r \geqslant 3$ be fixed and let $n \to \infty$ in such a way that $rn = 2m$ is even. Denote by $\mathscr{L}_r = \mathscr{L}_{n,r}$ the set of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. Write $\mathscr{U}_r = \mathscr{U}_{n,r}$ for the set of unlabelled r-regular graphs of order n. Put $L_r = |\mathscr{L}_r|$ and $U_r = |\mathscr{U}_r|$. We know from [1] that

$$L_r \sim e^{-(r^2-1)/4} \frac{(2m)!}{2^m m!} (r!)^{-n}.$$

The aim of this paper is to prove the following result.

THEOREM. *If $r \geqslant 3$ then*

$$U_r \sim L_r/n! \sim e^{-(r^2-1)/4} \frac{(2m)!}{2^m m!} (r!)^{-n}/n!, \tag{1}$$

where $m = rn/2$.

Proof. The second relation is the result of Bender and Canfield [1] and so we have to prove that $U_r \sim L_r/n!$. Throughout the proof all relations $(\sim, o, O)$ refer to the passage of n to ∞ and our inequalities are claimed to hold if n is sufficiently large. We shall write $c_1, c_2, \ldots$ for positive constants.

Let $W = \bigcup_{i=1}^{n} W_i$ be a fixed set with rn elements partitioned into n sets of r elements each. Call a partition of W into $m = rn/2$ pairs a *configuration*. Clearly there are $N(m) = (2m)!/2^m m!$ configurations. Given a configuration F, denote by $\phi(F)$ the graph with vertex set $V = \{1, 2, \ldots, n\}$ in which ij is an edge if and only if F

Received 14 September, 1981.

J. London Math. Soc. (2), 26 (1982), 201–206

has a pair with one element in W_i and the other in W_j. Clearly every $G \in \mathcal{L}_r$ is of the form $G = \phi(F)$ for $(r!)^n$ configurations F, so that $L_r \leqslant N(m)/(r!)^n$. What we shall need from the asymptotic formula for L_r is that if n is sufficiently large then

$$L_r \geqslant c_1 N(m)/(r!)^n$$

for some positive constant c_1. (In fact every $0 < c_1 < e^{-(r^2-1)/4}$ will do.)

It is convenient to turn $\mathcal{L}_r$ into a *probability space* by giving all graphs $G \in \mathcal{L}_r$ the same probability, namely L_r^{-1}. Let us estimate the probability that a random graph $G \in \mathcal{L}_r$ contains a fixed set of u edges. Clearly there are at most r^{2u} sets of u pairs that are mapped by ϕ into our u edges. There are $N(m-u)$ configurations that contain u given pairs, and so the above probability is at most

$$\begin{aligned} P_u &= r^{2u}N(m-u)/\{c_1 N(m)\} = \frac{1}{c_1} r^{2u}N(m-u)/N(m) \\ &= \frac{1}{c_1} r^{2u}\{(2m-1)(2m-3)\dots(2m-2u+1)\}^{-1} \\ &\leqslant \frac{1}{c_1} r^{2u}((2m-2)!)^{-u/(2m-1)} \leqslant \left(\frac{c_2}{n}\right)^u . \end{aligned} \tag{2}$$

Let us now see what we have to show to prove (1). Consider the symmetric group Σ_n acting on V. For $\omega \in \Sigma_n$ let $\mathcal{L}(\omega)$ be the set of graphs in $\mathcal{L}_r$ invariant under ω and put $L(\omega) = |\mathcal{L}(\omega)|$. Then

$$\sum_{\omega \in \Sigma_n} L(\omega) = \sum_{G \in \mathcal{L}_r} a(G),$$

where $a(G)$ is the order of the automorphism group of G. Since $\mathcal{L}_r$ has precisely $n!/a(G)$ graphs isomorphic to a given graph $G \in \mathcal{L}_r$,

$$U_r = \sum_{G \in \mathcal{L}_r} (n!/a(G))^{-1} = \frac{1}{n!} \sum_{G \in \mathcal{L}_r} a(G) = \frac{1}{n!} \sum_{\omega \in \Sigma_n} L(\omega).$$

For the identity permutation $1 \in \Sigma_n$ we have $L(1) = L_r$ so that (1) holds if and only if

$$\sum_{\substack{\omega \in \Sigma_n \\ \omega \neq 1}} L(\omega)/L_r = \sum_{\substack{\omega \in \Sigma_n \\ \omega \neq 1}} P(\omega) = o(1), \tag{3}$$

where $P(\omega) = L(\omega)/L_r$ is the probability that $G \in \mathcal{L}_r$ is invariant under ω.

We partition the sum in (3) into several parts. Denote by $M(s, s_2, s_3)$ the set of permutations in Σ_n that move s vertices and have s_2 2-cycles and s_3 3-cycles. Note that if $M(s, s_2, s_3) \neq \varnothing$ then $2s_2 + 2s_3 \leqslant s$. Furthermore, $M(0,0,0) = \{1\}$, $M(1,0,0) = \varnothing$ and, rather crudely,

$$|M(s, s_2, s_3)| \leqslant n^s/\{s_2!s_3!\}. \tag{4}$$

The gist of our proof of (1) is an appropriate upper bound on $\max\{P(\omega) : \omega \in M(s, s_2, s_3)\}$. Though the proof of this upper bound is rather simple, unfortunately it is somewhat cumbersome to present. We prepare the ground with a simple but useful remark concerning representations of sets as unions of other sets.

Let $\mathscr{A}$ be a collection of disjoint finite sets containing m_j sets of size j, $j = 1, 2, \ldots, t$. Denote by $f(b; \mathscr{A}) = f(b; m_1, m_2, \ldots, m_t)$ the number of b-sets that are unions of some of the sets in $\mathscr{A}$. Then

$$f(b; 0, 0, \ldots, 0, m_h, m_{h+1}, \ldots, m_t) \leqslant f(b; b - h\lfloor b/h \rfloor, 0, 0, \ldots, 0, m'_h) = \binom{m'_h}{\lfloor b/h \rfloor} \quad (5)$$

where $m'_h = \left\lfloor \sum_h^t i m_i / h \right\rfloor$.

The equality is obvious, and so we prove the inequality. Let $\mathscr{A}$ be a collection of disjoint sets consisting of m_j j-sets for $h \leqslant j \leqslant t$, but containing no j-sets for $j < h$. For each $A_i \in \mathscr{A}$ let C_i be an h-set so that $\mathscr{C}' = \{C_i : A_i \in \mathscr{A}\}$ consists of disjoint h-sets. Let $\mathscr{C}''$ be a collection of $m'_h - \sum_h^t m_j$ disjoint h-sets and $b - h\lfloor b/h \rfloor$ singletons so that $\mathscr{C} = \mathscr{C}' \cup \mathscr{C}''$ consists of disjoint sets. We have to show that $f(b; \mathscr{A}) \leqslant f(b; \mathscr{C})$. To see this, suppose that a b-set B is the union of some sets in $\mathscr{A}$. Let B' be a b-set which is the union of those C_i for which $A_i \subset B$ and some sets in $\mathscr{C}''$. Then the map $B \to B'$ is one-to-one and so $f(b; \mathscr{A}) \leqslant f(b; \mathscr{C})$, proving (5).

Let us fix a permutation $\omega \in M(s, s_2, s_3)$, $s \geqslant 2$. We introduce the following notation: S is the set of vertices moved by ω, S_2 is the set of vertices in 2-cycles of ω, S_3 is the set of vertices in 3-cycles of ω and, with a slight abuse of notation, $S_4 = S \setminus S_2 \cup S_3$. As before, we write $s_i = |S_i|/i$ so that $s = 2s_2 + 3s_3 + 4s_4$. (Note that s_4 need not be an integer.) We also put $S_1 = V \setminus S$.

Let us consider the orbits of (the permutation induced by) ω acting on those elements of $V^{(2)}$, the set of unordered pairs of elements of V, that have at least one element in S. For simplicity we say that an unordered pair $(x, y) \in V^{(2)}$ *joins* S_k *to* S_l if $x \in S_k$ and $y \in S_l$.

Denote by $m_h(k, l)$ the number of orbits of size h consisting of pairs joining S_k to S_l. By definition $m_h(k, l) = m_h(l, k)$. It is easily checked that

$$\begin{aligned}
&m_h(k,1) = 0 \quad \text{unless } h \geqslant h(k,1) = k, && m_h(3,4) = 0 \quad \text{unless } h \geqslant h(3,4) = 6,\\
&m_h(2,3) = 0 \quad \text{unless } h = h(2,3) = 6, && m_h(2,2) = 0 \quad \text{unless } h = h(2,0) = 1,\\
&m_h(2,4) = 0 \quad \text{unless } h \geqslant h(2,4) = 4, && \qquad\qquad \text{or } h = h(2,2) = 2,\\
&m_h(3,3) = 0 \quad \text{unless } h = h(3,3) = 3, && m_h(4,4) = 0 \quad \text{unless } h \geqslant h(4,0) = 2.
\end{aligned}$$

Futhermore,

$$m_1(2,2) \leqslant s_2 \qquad \text{and} \qquad 2m_2(4,4) + 3m_3(4,4) \leqslant 2s_4 .$$

To see the last statement note that if x is in a k-cycle but xy belongs to an orbit of size less than k, then $k = 2l$ and $y = \omega^l x$, that is, the pair (x, y) joins diametrically opposite vertices of an even cycle.

In order to give the notation some consistency we write $h(3,0) = 1$ and $h(4,4) = 4$. For $1 \leqslant k \leqslant l$, $l \neq 2$, a (k, l)-*orbit* (or (l, k)-*orbit*) is an orbit of size at least $h(k, l)$ consisting of pairs joining S_k to S_l. Furthermore, for $k \geqslant 2$ a $(k, 0)$-*orbit* is an orbit of size less than $h(k,k)$ consisting of pairs joining S_k to S_k. The $(k,0)$-orbits are the exceptional orbits of pairs joining vertices of S_k in the sense that altogether they contain at most $ks_k/2 = |S_k|/2$ edges, while the $(k;k)$-orbits have at least $ks_k(ks_k-2)/2$ edges.

Now we turn to our task of estimating $P(\omega)$, $\omega \in M(s_1, s_2, s_3)$. If $G \in L_e(\omega)$ then the set of edges of G cannot split orbits. Suppose that G has $h(i,j)k_{ij}$ edges in the set of (i,j)-orbits. We call (k_{ij}) the *set of parameters* of G. Note that these parameters have the following properties:

$$k_{ij} = k_{ji},$$

$$h(i,j)k_{ij} \text{ is a non-negative integer},$$

$$\sum_{j=0}^{4} h(i,j)k_{ij} + h(i,0)k_{i0} + h(i,i)k_{ii} = ris_i, \qquad i = 2,3,4. \tag{6}$$

The last relation holds since an edge in an (i,j)-orbit contributes 1 to the sum of degrees of the vertices in S_i unless $j = 0$ or i, when it contributes 2.

We shall call an array (k_{ij}) having the above properties a *permissible array*. When (6) is written out in full, a permissible array (k_{ij}) satisfies

$$2k_{20}+2k_{21}+4k_{22}+6k_{23}+4k_{24} = 2rs_2,$$

$$3k_{31}+6k_{32}+6k_{33}+6k_{34} = 3rs_3,$$

$$4k_{40}+4k_{41}+4k_{42}+6k_{43}+8k_{44} = 4rs_4. \tag{6'}$$

Given a permissible array (k_{ij}) we write $P(\omega;(k_{ij}))$ for the probability that a graph $G \in \mathscr{L}_r$ belongs to $\mathscr{L}(\omega)$ and has (k_{ij}) as its set of parameters. Then

$$P(\omega) = \sum P(\omega;(k_{ij})), \tag{7}$$

where the summation is over all permissible arrays.

Let us estimate $P(\omega;(k_{ij}))$ for a fixed permissible array (k_{ij}). Denote by $s(i,j)$ the number of pairs in the (i,j)-orbits of ω. Clearly

$$s(i,j) \leqslant 12 s_i s_j$$

with s_0 defined to be 1. The set of $h(i,j)k_{ij}$ edges consisting of full (i,j)-orbits can be selected in at most

$$f(i,j) = f(h(i,j)k_{ij}; m_1(i,j), m_2(i,j), \dots, m_t(i,j)) \leqslant \binom{\lfloor s(i,j)/h(i,j)\rfloor}{\lfloor k_{ij}\rfloor}$$

ways. Here the inequality is a consequence of (5) and the fact that $m_l(i,j) = 0$ for $l < h(i,j)$. By (2) the probability of a graph $G \in \mathscr{L}_r$ containing a given set of

$$\begin{aligned} k &= \sum h(i,j)k_{ij} \\ &= k_{20}+2k_{21}+2k_{22}+6k_{23}+4k_{24}+3k_{31}+3k_{33}+6k_{34}+2k_{40}+4k_{41}+4k_{44} \end{aligned} \tag{8}$$

edges incident with S is at most $(c_2/n)^k$. Hence

$$P(\omega\,;(k_{ij})) \leqslant (c_2/n)^k \prod f(i,j)\,, \tag{9}$$

where the product is over all unordered pairs (i,j) with $\max(i,j) \geqslant 2$.

We should like to bound from above the right-hand side of (9) as (k_{ij}) runs over all permissible arrays. Though this is an elementary problem in optimization theory, the manipulations in a standard solution are offputting so we prefer a different approach.

We claim that

$$(c_2/n)^k \prod f(i,j) \leqslant n^{-s(1+\varepsilon)} s_2^{s_2} s_3^{s_3} \tag{10}$$

where $\varepsilon = 2^{-4}$. To prove (10) we shall separate the large elements k_{ij} from the small ones. Set

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } k_{ij} \geqslant 10^{-4} s/r\,, \\ 0 & \text{if } k_{ij} < 10^{-4} s/r\,. \end{cases}$$

Then

$$(c_2/n)^k \{\prod f(i,j)\} n^{s-10^{-2}s} s_2^{-s_2} s_3^{-s_3} \leqslant n^{s-k} n^{\sum \varepsilon_{1j}k_{1j}} s^{\sum \varepsilon_{ij}k_{ij}} s_2^{-s_2} s_3^{-s_3} = A\,,$$

say, where $\sum \varepsilon_{1j}k_{1j} = \sum_{j=2}^{4} \varepsilon_{1j}k_{1j}$ and the summation in the exponent of s is over $2 \leqslant i \leqslant j \leqslant 4$.

Recall that $s = 2s_2+3s_3+4s_4$ so that by (6′) and (8) we have

$$\begin{aligned} s-k = {} & k_{20}\left(\frac{2}{r}-1\right) + k_{21}\left(\frac{2}{r}-2\right) + k_{22}\left(\frac{4}{r}-2\right) \\ & + k_{23}\left(\frac{12}{r}-6\right) + k_{24}\left(\frac{8}{r}-4\right) + k_{31}\left(\frac{3}{r}-3\right) + k_{33}\left(\frac{6}{r}-3\right) \\ & + k_{34}\left(\frac{12}{r}-6\right) + k_{40}\left(\frac{4}{r}-2\right) + k_{41}\left(\frac{4}{r}-4\right) + k_{44}\left(\frac{8}{r}-4\right). \end{aligned}$$

Hence if $s = n^\alpha$, by (6′) we find that

$$\begin{aligned} \frac{\log A}{\log n} \leqslant {} & c_3 \frac{s}{\log n} + s-k + \sum_{j=2}^{4} \varepsilon_{1j}k_{1j} + \alpha \sum_{2 \leqslant i \leqslant j \leqslant 4} \varepsilon_{ij}k_{ij} \\ & -\alpha \sum_{j=0}^{4} \varepsilon_{2j}h(2,j)k_{2j}/(2r) - \alpha \sum_{j=0}^{4} \varepsilon_{3j}h(3,j)k_{3j}/(3r) \\ \leqslant {} & c_3 \frac{s}{\log n} - k_{20}\frac{r-2}{r} - k_{21}\left(2\frac{r-1}{r}-1\right) - k_{22}\left(2\frac{r-2}{r} - \alpha + \frac{2}{r}\alpha\right) \\ & -k_{23}\left(6\frac{r-2}{r} - \varepsilon_{23}\alpha\left(1-\frac{3}{r}-\frac{2}{r}\right)\right) - k_{24}\left(4\frac{r-2}{r} - \alpha + \frac{2}{r}\alpha\right) \end{aligned}$$

$$- k_{31}\left(3\frac{r-1}{r} - 1\right) - k_{33}\left(3\frac{r-2}{r} - \alpha + \frac{\alpha}{r}\right) - k_{34}\left(6\frac{r-2}{r} - \alpha + \frac{2\alpha}{r}\right)$$

$$- k_{40}\frac{2r-4}{r} - k_{41}\left(4\frac{r-1}{r} - 1\right) - k_{44}\left(4\frac{r-2}{r} - \alpha\right)$$

$$\leqslant c_3\frac{s}{\log n} - \frac{r-2}{r}\sum k_{ij} \leqslant c_3\frac{s}{\log n} - \frac{r-2}{12}s \leqslant s/13\,.$$

This implies (10) since $1/13 - 10^{-2} > 2^{-4}$.

Now the proof is easily completed. For a permutation $\omega \in M(s, s_2, s_3)$ there are at most $(6rs)^{12}$ permissible arrays and so by (7), (9) and (10)

$$P(\omega) \leqslant (6rs)^{12} n^{-s(1+\varepsilon)} s_2^{s_2} s_3^{s_3}\,.$$

Therefore by (4)

$$\sum_{\omega \in M(s, s_2, s_3)} P(\omega) \leqslant c_4^s n^{-\varepsilon s}$$

$$\sum_{\substack{\omega \in \Sigma_n \\ \omega \neq 1}} P(\omega) \leqslant \sum_{s=2}^{n} s^2 c_4^s n^{-\varepsilon s} \leqslant \sum_{s=2}^{n} n^{-\varepsilon s/2} = o(1)\,.$$

This proves (3), and so the proof of our theorem is complete.

It is worth noting that the proof is considerably more pleasant for $r \geqslant 5$. In that case we do not have to consider the number of 2-cycles and 3-cycles in a permutation $\omega \in \Sigma_n$; it suffices to use the number of vertices moved by ω as the only parameter, and to note that at most n^s permutations move precisely s vertices.

In this paper we considered regular graphs but in fact the proof can be adapted to graphs with more general degree sequences. In particular, the following result can be obtained without any additional work. Let $\Delta \geqslant 3$ be a constant. Then the number of labelled graphs with degree sequences $3 \leqslant d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n \leqslant \Delta$ divided by the number of unlabelled graphs having the same degree sequence is asymptotic to $n!$.

References

1. E. A. Bender and E. R. Canfield, 'The asymptotic number of labeled graphs with given degree sequences', *J. Combin. Theory Ser.* A, 24 (1978), 296–307.
2. B. Bollobás, 'A probabilistic proof of an asymptotic formula for the number of labelled regular graphs', *European J. Combin.*, 1 (1980), 311–316.
3. B. Bollobás, 'Distinguishing vertices of random graphs', *Ann. Discrete Math.*, to appear.
4. E. M. Wright, 'Graphs on unlabelled nodes with a given number of edges', *Acta Math.*, 126 (1971), 1–9.

Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge,
16 Mill Lane,
Cambridge CB2 1SB.