Math 55b Take-Home Final

Solutions

Part I.

1. Given $1 \le p < \infty$, let $E_p \subset C^1[0,1]$ denote the space of functions such that $f(0) = 0$ and $\int_0^1 |f'(x)|^p \, dx < 1$. Show that the closure of E_p in $C[0,1]$ is compact iff $p > 1$.

   Proof. Suppose $p > 1$. Then by Hölder's inequality all $f \in E_p$ satisfy $|f(x)-f(y)| \le |x-y|^{1/q}$, where $1/p+1/q = 1$; so by Arzela-Ascoli, the closure of E_p is compact. For $p = 1$, this is false; e.g. E_1 contains the sequence of functions $f_n(x) = x^n/2$, which does not have a uniformly convergent subsequence. ∎

2. Let $f(x) \ge 0$ be a smooth, compactly supported function on $\mathbb{R}^3$. The gravitional force at $p \in \mathbb{R}^3$ coming from the mass distribution $f(x)\,|dx|$ is given by the vector

$$F(p) = \int_{\mathbb{R}^3} \frac{x-p}{|x-p|^3} f(x)\,|dx|.$$

   (a) Show that $F(x) = \nabla\phi$ for a suitable function $\phi(x)$.

   (b) Show that if $f(x)$ vanishes on a neighborhood of p, then ϕ is harmonic near p.

   (c) More precisely, show that $\Delta\phi = Cf$ for some constant C.

   (d) Show that the gravitational force is zero inside a hollow, spherically symmetric planet.

   Answer. (a,b) Use the fact that $x/|x|^3 = -\nabla(1/|x|)$, and $1/|x|$ is harmonic away from $x = 0$. (c) It is easy to see that $F(p) \approx -mp/|p|^3$ when p is large, where $m = \int f$ is the total mass. Thus the flux through a large sphere is $\approx -4\pi m = -4\pi \int f$. But the fluxes through all spheres enclosing the support of f are the same, by Stokes' theorem. So we can conclude that $\int_B \nabla \cdot F = \int_B \Delta\phi = -4\pi \int_B f$ for any ball B, which implies $\Delta\phi = -4\pi f$. (d) By symmetry we have $F(p) = h(|p|)p$. If $B(0,r)$ is contained inside the hollow planet, then by Stokes' theorem the flux of F through the sphere $|x| = r$ is zero (it is equal to the mass of $B(0,r)$), and hence $h(r) = 0$.

3. Let $f \in C^1(\mathbb{R})$ be a function such that $\|f\|_\infty$ and $\|f'\|_\infty$ are both bounded. Define $I : C[0,1] \to C[0,1]$ by $I(u) = v$ where

$$v(x) = \int_0^x f(u(t))\, dt.$$

Show that the differential equation $u'(x) = f(u(x))$ has a unique solution on $[0,1]$ with $u(0) = 0$, by showing:

(a) $I^n(u)$ converges uniformly, for any $u \in C[0,1]$, to a function g satisfying $I(g) = g$;

(b) The fixed point g of I is unique; and

(c) The fixed points of I correspond bijectively to solutions to the given differential equation.

Answer. Suppose $v = I(u)$. Then $v(0) = 0$ and $|v'(x)| \le |f| = O(1)$ so $\|v\| = O(1)$. Now if $v_i = I(u_i)$, $i = 1, 2$, then

$$|v_1(x) - v_2(x)| \le |x|(\sup |f'|)\|u_1 - u_2\|,$$

so I is a contraction on $C[0,a]$ when a is sufficiently small. This shows $I^n(u)|[0,a]$ converges uniformly, and a similar argument gives convergence on $C[0,1]$, and uniqueness of the fixed point.

4. Suppose $f(z)$ is analytic on the unit disk $\Delta \subset \mathbb{C}$, $f(0) = 0$ and $\operatorname{Re} f(z) \le 1$ for all z.

(a) What is the largest possible value $M(r)$ for $|\operatorname{Im} f(z)|$ on the circle $|z| = r < 1$?

(b) Let $f_n : \Delta \to \mathbb{C}$ be analytic functions with $\operatorname{Re} f_n \le 1$ and $f_n(0) = 0$, and suppose $\operatorname{Re} f_n$ converges uniformly on the unit disk. Prove that $\operatorname{Im} f_n$ converges uniformly on the disk $|z| \le r$ for each $r < 1$.

(c) Give an example where $\operatorname{Re} f_n$ converges uniformly on Δ but $\operatorname{Im} f_n$ does not.

Answer. (a) By the Schwarz lemma the worst case comes from the Möbius transformation $A : (\Delta, 0) \to (L, 0)$, where $L = \{z : \operatorname{Re} z \le 1\}$. This map is given by $A(z) = 2z/(1+z)$, which maps $[-r, r]$ to $[-2r/(1-r), 2r/(1+r)]$. Thus $A(B(0,r))$ is a ball of radius $M(r) = 2r/(1-r^2)$. For (b), apply (a) to $f_n - \lim f_n$. An example of (c) is given by $f_n(z) = A((1 - 1/n)z)$.

5. Let $f(x) = \int_0^x dt/\sqrt{t(1-t^2)}$ for $x \in [0,1]$.

(a) Show that there is a unique analytic function $F(z)$ defined on $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ such that $F(z_n) \to f(x)$ whenever $z_n \to x \in [0,1]$.

(b) Show that $S = F(\mathbb{H})$ is an open square in $\mathbb{C}$, and that $F : \mathbb{H} \to S$ is a homeomorphism.

Answer. (a) One can take

$$F(p) = \int_0^p \frac{dz}{\sqrt{z(1-z^2)}},$$

using the fact that the denominator only vanishes on the real axis to choose a consistent square-root in $\mathbb{H}$. (b) Clearly F extends continuously to $\mathbb{R}$, and its argument is constant on the intervals

$$(-\infty,-1), (-1,0), (0,1), (1,\infty).$$

Thus these intervals are sent homeomorphically to straight lines. By computing $\arg F$ along the real axis, we see these lines form the edges of a square. By the argument principle, F is $1-1$ on $\mathbb{H}$ and its image is the interior of the square.

Part II. Mark each of the following assertions True (T) or False (F).

1. **T.** A smooth map $f : U \to \mathbb{C}$, $U \subset \mathbb{C}$, is analytic iff for all 1-forms α on $\mathbb{C}$, $f^*(*\alpha) = *f^*(\alpha)$.

2. **F.** If α, β are k-forms on $\mathbb{R}^n$, $k > 0$, then $\alpha\beta = -\beta\alpha$.

3. **F.** If $f_n(z)$ are analytic functions and $f_n \to f$ uniformly on a domain U, then f is analytic and $f_n' \to f'$ uniformly on U. (Consider $f_n(z) = z^n/n$ on $U = \Delta$.)

4. **F.** Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous map such that the zeros of $f(z) - a$ are isolated for every $a \in \mathbb{C}$. Then f is an open map ($f(U)$ is open whenever U is open). (Consider the map $f(x, y) = (x^2, y)$.)

5. **F.** If E is any subset of $\mathbb{R}^n$, then the boundary of the boundary of the interior of E is empty. (For example, if $n = 1$ and $E = [0, 1]$, then $\partial\partial \operatorname{int} E = \{0, 1\}$.)

6. **T.** If $f_n \in C[0, 1]$ converges pointwise to 0, and $|f_n(x)| \le 1$ for all n, x, then $\int_0^1 f_n(x)\, dx \to 0$.

7. **T.** Suppose $f_n \in C[0, 1]$ converges uniformly to 0, and $\alpha_n \in C[0, 1]$ are monotone increasing functions with $\alpha_n(1) - \alpha_n(0) = 1$. Then $\int_0^1 f_n d\alpha_n \to 0$.

8. **F.** Suppose v is a smooth vector field on $\mathbb{R}^3$. Then $\nabla \times \nabla \times v = 0$.

9. **F.** There exists a sequence of nonempty, disjoint, closed intervals $I_i \subset [0, 1]$ such that $\bigcup I_i = [0, 1]$. (Let $E = \overline{\bigcup \partial I_i}$. If the intervals are disjoint and cover, then E has no isolated points, so E is uncountable. Thus E contains a point p which is not in $\bigcup \partial I_i$. But clearly p is also not in $\bigcup \operatorname{int} I_i$, so $\bigcup I_i \neq [0, 1]$.)

10. **T.** The analytic function defined on the unit disk by $f(z) = \sum n^5 z^n$ extends to a rational function on the Riemann sphere. (The operator $Dg = zg'(z)$, applied 5 times to $1/(1 - z)$, gives f.)