Ergodic Theory, Geometry and Dynamics

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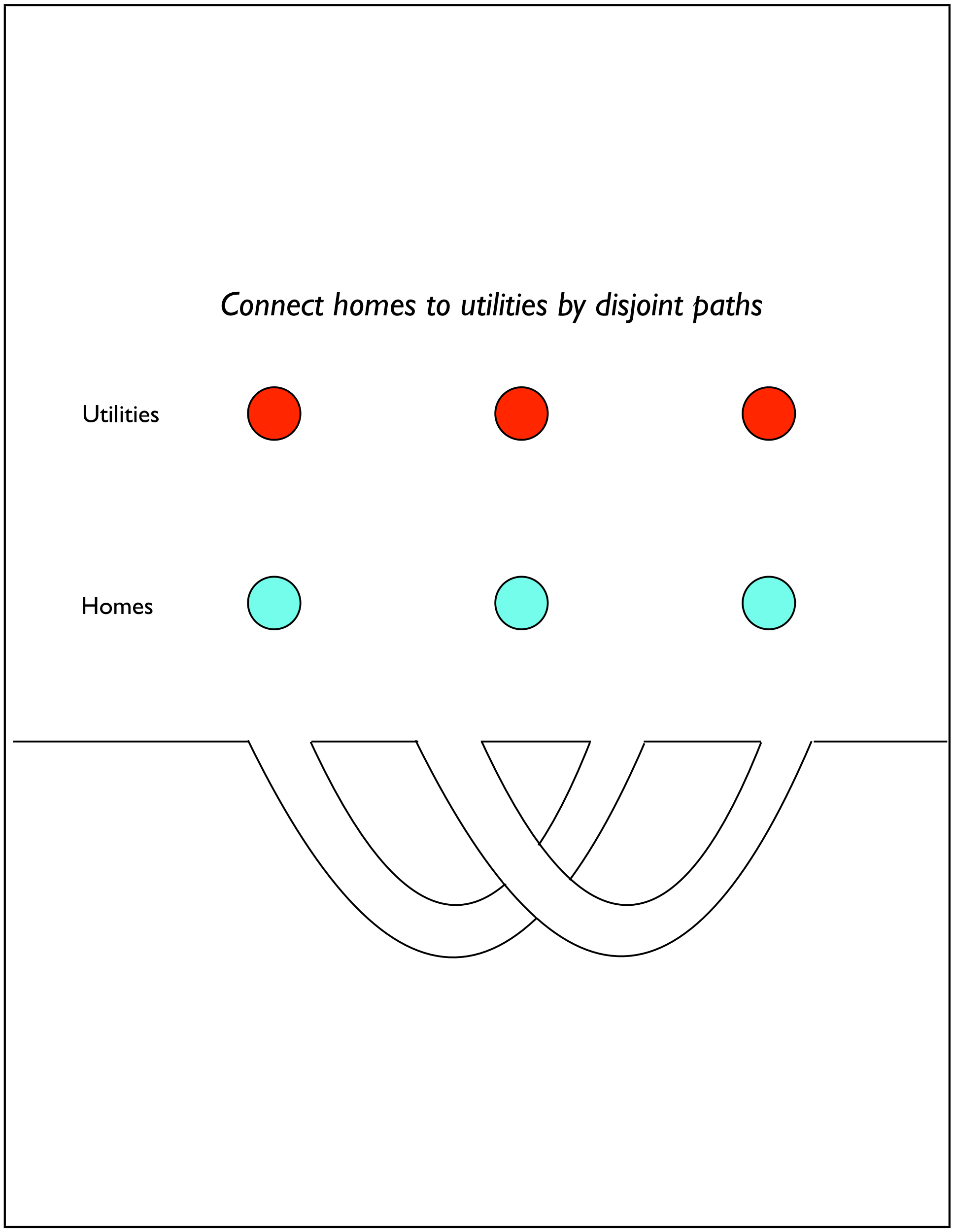
Comments, Supplementary Notes, Miscellany

1 Outline of course

1. Basic ergodic theory: Recurrence; the ergodic theorem, L^2 and pointwise.
2. Spectrum of unitary operators. Lebesgue spectrum, point spectrum. Spectrum of a measure preserving transformation.
3. Abelian groups, characters, Pontryagin duality. Irrational rotation, translations in general, automorphisms of groups. Hyperbolic toral automorphism.
4. Topological measure theory: the space of invariant measures; unique ergodicity; the Hopf argument.
5. Spherical and hyperbolic geometry. Groups: SO(3), SU(2), SO$(2,1)$, $\mathrm{SL}_2(\mathbb{R})$, $SU(1,1)$. Subgroups: K, A and N. Inner products, distances, dihedral angles; many models for $\mathbb{H}$ and $\mathbb{H}^n$.
6. Dynamics on hyperbolic surfacesr: mixing of geodesic and horocycle flows. Proof of Mostow rigidity.
7. Minimality. Behavior of horocycles. The philosophy of double cosets. A glimpse of Ratner's theory: planes in hyperbolic 3–mnanifolds.
8. Unique ergodicity of the horocycles flow.
9. Passage to $\mathrm{SL}_3(\mathbb{R})$. Oppenheim conjecture.
10. Property T for $\mathrm{SL}_3(\mathbb{Z})$.

2 The utility graph $K_{3,3}$



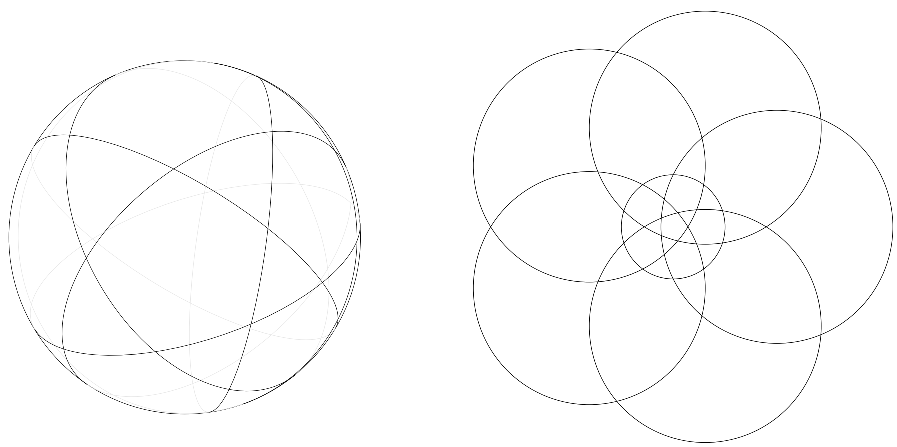
3 A configuration of circles with A_5 symmetry.

What is the simplest configuration of circles in $\widehat{\mathbb{C}}$ with A_5 symmetry?

The 12 circles corresponding to the 12 faces of the dodecahedron give one example. However these circles can all be expanded about their centers until they come together in pairs and create an even more symmetric configuration of 6 circles. These 6 circles are also invariant under 'reflection through the equator', which the 12 circles are not.

The 6 circle configuration $\mathcal{C}$ has a remarkably simple description in $\mathbb{C}$. Namely it consists of the unit circle C_0 plus 5 circles C_i symmetrically placed around it, one of them centered at $z = 2$.

Each C_i cuts C_0 in a pair of antipodal points. Altogether they chop C_0 into 10 arcs of equal length $\pi/5$. Each $\pi/5$ arc is one edge of an equilateral spherical triangle. Altogether, there sphere is cut by the C_i into 20 such triangles, and 12 pentagons.



The unit circle, C_0 centered at z=0; a circle C_1 centered at z=2 and meeting C_0 in a pair of opposite points; and the rotations of C_1 by 1/5th of a revolution, give six circles invariant under A_5.

Figure 1.

To 'compute' this configuration, it is useful to have a formula for the internal angle θ of an equilateral spherical triangle with a given edge length

L. These two quantities are related by

$$\cos\theta = \frac{1-\cos(L)}{\tan L \sin L}.$$

Note that for L small we get $\cos\theta \approx 1/2$ and hence θ is approximately $60°$.

For $L = \pi/5$ we find $\cos\theta = 1/\sqrt{5}$.

In the case at hand, θ is also the angle at A of the right triangle ABC, where A is the center of the circle C_1, B is the origin and $C = i$ is one of the points where C_0 meets C_1. This just means that ABC is a right triangle with short sides of length 1 and 2 and a hypotenuse of length $\sqrt{5}$, so that $|AC|/|BC| = 1/\sqrt{5}$.