Topics in Geometry and Dynamics

Problems

C. McMullen

1. (Arnold.) Estimate the average value of $\sin^{100} x$ with an error of less than 20%. (No calculators. Try to get an answer in less than 5 minutes.)

2. (Arnold.) Compute the 100th derivative of $f(x) = (x^2+1)/(x^3-x)$.

3. (Arnold.) Find the infimum of

$$\int_\Delta |df|^2\,|dz|^2,$$

   over all C^∞ functions $f : \overline{\Delta} \to \mathbb{R}$ such that $f|S^1$ is identically equal to 1, and $f(0) = 0$.

   (Hint: What does this have to do with the modulus of a cylinder?)

4. Let $K \subset [0,1] \subset \mathbb{C}$ be the Cantor middle–thirds set, and let B be the disk of radius 2 about $z = 0$. Show that

$$\inf_f \int_B |df|^2\,|dz|^2 > 0,$$

   where the infimum is over over all C^∞ functions $f : B \to \mathbb{R}$ such that $f = 1$ on ∂B and $f = 0$ on K.

5. Using the software `lim`, draw a tiling of the unit disk by hyperbolic $(2,4,8)$–triangles.

6. Let $X = \mathbb{H}/\Gamma$ be a compact hyperbolic surface of genus g. By applying the Poincaré series construction

$$\Theta(q) = \sum_\Gamma \gamma^*(q)$$

   to a suitable meromorphic quadratic differential $q(z)\,dz^2$ on $\mathbb{H}$, show that:

   > (*) For any $p \in X$ there exists a meromorphic quadratic differential on X with a simple pole at p and otherwise holomorphic.

7. Give a second proof of (*) using Riemann–Roch.

8. Let $Q(\mathbb{H})$ denote the Banach space of holomorphic quadratic differentials on the disk with $\|q\| = \int_{\mathbb{H}} |q|$, and similarly for $Q(X)$, $X = \mathbb{H}/\Gamma$ as above. Show that the Poincaré series map

$$\Theta : Q(\Delta) \to Q(X)$$

is surjective. (Hint: Let $F \subset \mathbb{H}$ be a compact fundamental domain for Γ, and let q be a holomorphic, Γ–invariant quadratic differential on the $\mathbb{H}$. Construct a suitable project $q_0 = \pi(q\chi_F)$ of the truncated differential to a holomorphic differential on $\mathbb{H}$, show that $q_0 \in Q(\mathbb{H})$, and show that $\Theta(q_0) = q$.)

9. Why is there an action of $\mathrm{SL}_2(\mathbb{R})$ on the moduli spaces of sections of K_X and K_X^2 (holomorphic and quadratic differentials), but not on the moduli space of sections of K_X^3 (cubic differentials)?

10. Let (X, ω) be the the golden 1-form of genus, obtained by gluing up an L–shaped polygonal with sides of length 1 and $\gamma = (1 + \sqrt{5})/2$. Show explicitly that the $\mathrm{SL}_2(\mathbb{R})$–orbit of (X, ω) contains the pentagonal form, i.e. a 1-form (Y, η) with a single zero and 5–fold holomorphic symmetry.

11. Give good estimates for the modulus of the annulus $A_r = \Delta - [r, r]$ when $r \in (0, 1)$ is near zero and near 1.

12. Express the exact value of $\mathrm{mod}(A_r)$ in terms of $a(r) = \int_{0,r} \omega$ and $b(r) = \int_r^1 \omega$, where $\omega = \sqrt{(z^2 - r^2)(z^2 - r^{-2})}\, dz$.

13. (Peter Winkler.) You are about to leave home and visit N cities by air, visiting each just once (you do not need to fly home). The price of a ticket from x to y satisfies $P(x, y) = P(y, x)$. One way to route your travel is by the Clinton strategy: when leaving x, choose your destination y to minimize $P(x, y)$, subject to the condion that you haven't visited y yet. Or you could use the Trump strategy: always choose y to maximize $P(x, y)$. (You can afford it!)

    Does the Trump trip ever cost less than the Clinton trip?

14. Let $f_c(z) = z^2 + c$ and let

$$M = \{c : \sup_n |f_c^n(0)| < \infty\}.$$

Prove that M is a compact subset of $\mathbb{C}$.

Challenge: prove that M is connected. (This is a theorem of Douady and Hubbard.)

15. Let

$$A = \{c \,:\, f_c \text{ has an attracting periodic cycle in } \mathbb{C}\}.$$

Prove that A is open and that $A \subset M$, using e.g. the Schwarz Lemma. Prove that $\overline{A}$ contains ∂M. Conclude that if $A = \mathrm{int}(M)$ then $\overline{A} = M$.

16. Prove that A is closed in the interior of M. In particular, if $\overline{A} = M$ then $A = \mathrm{int}(M)$. (Hint: prove that the number of attracting cycles of f_c is locally constant on the interior of M. You may wish to use the theory of holomorphic motions.)

17. Can any 4 lengths (L_i) be realized by a hyperbolic surface X of genus zero with 4 boundary components? Does the data (L_i) determine X uniquely? If not, is there a way to add conditions to make it determine X uniquely?

18. Give a conceptual proof that $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(AB)$ determine a representation of the free group into $\mathrm{SL}_2(\mathbb{R})$, up to conjugacy, provided it is irreducible. (Hint: think of $\det(A)$ as a quadratic form on $\mathrm{M}_2(\mathbb{R})$. The corresponding inner product is given by $(1/2)\,\mathrm{Tr}(AA^\dagger)$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\dagger = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Enough inner products determine a configuration of matrices.)

19. Prove the uniformization theorem for domains $\Omega \subset \mathbb{C}$. (Hint: think of the case of an annulus first. The general case follows the same pattern as the Riemann mapping theorem. Domains in the plane are much easier to uniformize than general Riemann surfaces.)

20. (Besikovitch.) Let $Q \subset \mathbb{C}$ be a quadrilateral with pairs of opposite sides (a, a') and (b, b'). Prove that, in the Euclidean metric, we have

$$\mathrm{area}(Q) \geq d(a, a') \cdot d(b, b').$$

Does this hold for other metrics?

21. Give an example of a K–quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 0$ and such that

$$K(f, 0) = \max |f(z)| \,:\, z \in S^1 \min |f(z)| \,:\, z \in S^1 > K.$$

(Hint: start with a linear stretch on a cylinder.)

22. Prove that there exists a unique hyperbolic triangle (up to congruence) with angles $\alpha, \beta, \gamma \geq 0$, for any triple of angles satisfying $\alpha+\beta+\gamma < \pi$. (Note that angles of zero are allowed.)

23. Prove subadditivity of the modulus: if $A_1, \ldots, A_n$ are disjoint, incompressible subannuli of an annulus A, then $\sum \operatorname{mod}(A_i) \leq \operatorname{mod}(A)$. When does equality hold? (In general a continuous map $f : A \to B$ is *incompressible* if $f_* : \pi_1(A) \to \pi_1(B)$ is injective.)

24. Prove that if $f : A \to B$ is a holomorphic covering map between annuli, then $\operatorname{mod}(B) = \deg(f) \cdot \operatorname{mod}(A)$.

25. Let $E_\alpha : \mathcal{T}_1 \to \mathbb{R}_+$ be the map that measures, on a given complex torus, the extremal length of the family Γ_α of simple closed curves in a fixed homotopy class α.

    Suppose α and β generate $H_1(\Sigma_1, \mathbb{Z})$. (i) What is the image of $\mathcal{T}_1$ under the map $\pi : \mathcal{T}_1 \to \mathbb{R}^2$ given by

    $$\pi(X) = (E_\alpha(X), E_\beta(X))?$$

    (ii) Is the map π injective? What happens when $E_\alpha(X)E_\beta(X) = 1$?

26. Compute the Poincaré metric on $A_R = \{z \,:\, 1 < |z| < R\}$.

27. Let $K \subset \mathbb{C}$ be a compact, connected set such that $\widehat{\mathbb{C}} - K$ is also connected (K is *full*). Let $\Delta_R = \{z \,:\, |z| < R\}$. Then for all R large enough, $\Delta_R - K$ is an annulus, and we can consider the 'renormalized modulus'

    $$m(K) = \lim_{R\to\infty} \operatorname{mod}(\Delta_R - K) - \log(R).$$

    Show this limit exists, and relate it to $|f'(\infty)|$ where

    $$f : \widehat{\mathbb{C}} - \overline{\Delta} \to \widehat{\mathbb{C}} - K$$

    is the Riemann mapping normalized so that $f(\infty) = \infty$.

28. Let G be the group of affine maps $f(x) = ax + b$ of $\mathbb{Z}/8$ to itself, with $a, b \in \mathbb{Z}/8$ and a odd. Let $H_1 \cong \mathbb{Z}/4$ be the subgroup where $b = 0$, and H_2 the subgroup where $b = 0$ when $a = \pm 1$ and otherwise $b = 4$.

    (i) Show that H_1 is not conjugate to H_2 in G.

    (ii) Show that for every conjugacy class C in G, we have $|C \cap H_1| = |C \cap H_2|$.

29. (Continuation.) Let X be a compact hyperbolic surface, let $Y \to X$ be a covering space with deck group G, and let $X_i = Y/H_i$. Show that X_1 and X_2 have the same length spectrum.

30. Give an example of a quasiconformal map $f : \mathbb{R} \to \mathbb{R}$ that is not absolutely continuous. (On $\mathbb{R}^n$, $n > 1$, quasiconformal maps preserve sets of Lebesgue measure zero.)

    (Bonus.) Find an f as above such that f sends a $E \subset \mathbb{R}$ of positive measure to a set $f(E)$ of Hausdorff dimension < 1.

31. Given an elementary proof that quasiconformal maps on $\mathbb{R}$ are Hölder continuous.

32. Let X be a compact Riemann surface of *topological* genus one. Why is the universal cover of X equal to $\mathbb{C}$ and not $\mathbb{H}$? (There are many answers. Try to find several. Can you show, for example, that the complement of a ball in the universal cover of X is an annulus of infinite modulus?)

33. (i) Define the complex conjugate $\overline{X}$ of a Riemann surface X.

    (ii) Show that $\tau \in \mathbb{H}$ lies on the geodesic stabilized by some hyperbolic element $A \in \mathrm{SL}_2(\mathbb{Z})$ iff the corresponding complex torus E is isogenous to $\overline{E}$. (This means there is a holomorphic covering map $E \to \overline{E}$.)

34. (i) Let $S = \{z \in \mathbb{H} : \operatorname{Re} z \in [0, 1]\}$. Use the map $g(z) = 2z + 1$ to glue together the edges of S. What does the resulting hyperbolic surface look like?

    (ii) Show that every pair of pants P with geodesic boundary can be constructed from a pair of ideal triangles T by gluing their boundaries together in pairs, and then taking the metric completion.

    Conclude, in particular, that $\operatorname{area}(P) = 2 \operatorname{area}(T) = 2\pi$.

    (iii) Show that a compact hyperbolic surface X of genus $g \geq 2$ *cannot* be constructed by gluing together ideal triangles along their edges, and then taking the metric completion.

    (iv) Show, however, that if $\gamma \subset X$ is a simple closed geodesic, then $X - \gamma$ can be constructed by gluing together ideal triangles.

35. Give an example of a sequence of K–quasiconformal maps such that $f_n \to \mathrm{id}$ on $\mathbb{C}$, but the complex dilatation μ_n of f_n does not converge weakly to zero.

36. Show, however, that if a sequence of K–quasiconformal maps $f_n : \mathbb{C} \to \mathbb{C}$ converges to f, uniformly on $\widehat{\mathbb{C}}$, and the dilatations μ_n of f_n converge to mu in L^1, then μ is the complex dilatation of f. (This lemma can be used to deduce the solvability of the Beltrami equation for L^∞ μ from the case of much smoother μ, e.g. from the case where μ is real–analytic.)

37. Let A with an annulus with $\pi_1(A) = \langle g \rangle$, and let L be a loop in general position representing g^n. Show that L has at least $n-1$ self–intersections, and that this bound is sharp.

    What happens on a Möbius band?

38. Let $\gamma \subset X$ be a primitive closed geodesic on a hyperbolic Riemann surface. Show that γ minimizes the number of self–intersections over all loops in the same homotopy class.

39. Let $I(\gamma)$ denote the minimum number of self–intersection of a homotpy class of loop γ on an oriented surface. Show that if γ is primitive, then

$$I(\gamma^n) = n^2 I(\gamma) + n - 1.$$

40. Let q be a meromorphic quadratic differential with a double order pole at p. Show that in a suitable local coordinate system with $z(p) = 0$, we have $q = dz^2/z^2$.

41. Let ω be a meromorphic 1-form with a double pole at p. Show that in a suitable local coordinate system, we have $\omega = (z^{-2} + a/z)dz$. Note that $a = \mathrm{Res}_p(\omega)$ is an invariant and hence ω does not have a unique local form.

42. Let ω be a meromorphic 1-form with a double pole at p. Show that in a suitable local coordinate system, we have $\omega = (z^{-2} + a/z)dz$. Note that $a = \mathrm{Res}_p(\omega)$ is an invariant and hence ω does not have a unique local form.

43. What is the length of the shortest closed geodesic on the hyperbolic Riemann surface $X = \widehat{\mathbb{C}} - \{0, 1, \infty\}$? (Hint: $X = \mathbb{H}/\Gamma(2)$.)

44. Let X be a punctured torus. What is the simplest example of a hyperbolic geodesic on X that is not simple?

45. Let X be a hyperbolic surface such that $\pi_1(X)$ is generated by two elements, but not one. Show that X is homeomorphic to a once–punctured torus or a triply–punctured sphere.

46. Let $\gamma \subset X$ be a closed hyperbolic geodesic with a unique point of self–intersection. Prove that $\pi(\gamma) \cong \mathbb{Z} * \mathbb{Z}$ injecdts into $\pi_1(X)$.

47. Show that the space of S closed subgroups of $\mathbb{R}^2$, in the geometric topology, is homeomorphic to the sphere S^4. (Cf. Pourezza–Hubbard.)

    *Show that the subset $N \subset S$ consisting of groups which are not isomorphic to $\mathbb{Z}^2$ is homeomorphic to S^2. Then, show that the pair (S, N) is homeomorphic to the suspension of (S^3, T), where T is the trefoil knot.

48. Let X be a hyperbolic Riemann surface of finite volume and genus zero. Show that X is isomorphic to the complement of a finite subset $E \subset \widehat{\mathbb{C}}$.

49. Let X be a hyperbolic Riemann surface with genus zero (every simple loop separates) and finitely many ends. Show that X embeds in $\widehat{\mathbb{C}}$. (Hint: use the uniformization theorem.)

50. Let $(X_n, x_n) \to (X, x)$ be a sequence of hyperbolic surfaces converging in the geometric topology. Prove that $\text{area}(X) \le \liminf \text{area}(X_n)$.

51. Let $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z} \subset \text{Aut}(\mathbb{C})$ be a discrete lattice acting by translations on $\mathbb{C}$, with compact quotient. Show that Λ is a geometric limit of a sequence cyclic subgroups $\langle g_n \rangle \subset \text{Aut}(\mathbb{C})$.

52. Show that when X has genus $g = 2$, every holomorphic quadratic differential $q \in Q(X)$ is the product of two 1-forms $\omega_1, \omega_2 \in \Omega(X)$.

    What happens for $g = 3$?

53. Make a physical model (e.g. out of paper) of the pillowcase, i.e. of $(\widehat{\mathbb{C}}, |q|)$ where $q = dz^2/(z(z^2-1))$. Then demonstrate, by manipulating your model, a degree two rational map of $\widehat{\mathbb{C}}$ to itself that preserves this metric up to scale. (This means you should produce a pillowcase of half the area, e.g. by cutting and folding the original pillowcase.)

54. Show that every finitely generated Fuchsian group is geometrically finite.

55. Show that the length spectrum of a geometrically finite hyperbolic 3-manifold is discrete.

56. Show there exists a finitely generated Kleinian group in which the length spectrum has infinitely multiplicity.

57. *Show there exists a finitely generated Kleinian group with indiscrete length spectrum.

58. Let $f(z)$ be a rational map of degree 2 or more. The (absolute) *multiplier* of a fixed point z of f is given by $|f'(z)|$. The multiplier spectrum $M(f)$ is the union over $n \geq 1$ of the fixed–point multipliers of f^n.

    Show there is a dense G_δ $U \subset S^1$ such that $M(f)$ is indiscrete for all $\lambda \in U$ (in fact 1 is not isolated in $M(f)$).

59. We say f is geometrically finite if $P(f) \cap J(f)$ is finite. (Here $P(f)$ is the closure of the strict forward orbits of the critical points of f, and $J(f)$ is the Julia set.) Show that if f is geometrically finite, then $M(f)$ is discrete.

60. Show that every proper holomorphic map $f : \Delta \to \Delta$ extends to a rational map on $\widehat{\mathbb{C}}$. Such rational maps are called *Blaschke products.* They are analogous to Fuchsian groups.

    Show that every Blaschke product is geometrically finite.

61. Show there exists a cubic rational map f such that $f(\overline{z}) = \overline{f(z)}$ and $f|\widehat{\mathbb{R}}$ is a homeomorphism with a critical point.

62. Show there is a cubic map f as above such that $M(f)$ contains a sequence of distinct points converging to 1.

63. **Show there is a cubic map f as above such that $f|\widehat{\mathbb{R}}$ is quasi-symmetrically conjugate to an irrational rotation (Swiatek.)

64. Let X be a square–tiled surface, and let X_θ be the surface obtained when each square is replaced by a rhombus with internal angle θ. Show that X_θ traces out a Teichmüller geodesic as θ ranges from 0 to π. What is its arclength parameterization?

65. Give an example of a quadratic differential $(X, q) \in Q\mathcal{M}_2$ with 2 double zeros, such that q is not the square of a holomorphic 1-form.

66. (i) Given an example of a real quadratic field with class number 2 or more. (ii) Give 2 matrices in $G = \mathrm{SL}_2(\mathbb{Z})$ that have the same trace but are not conjugate in G.

67. Show that a nonconformal Teichmüller mapping adapted to $q \in Q(X)$ is never differentiable near a zero of q.

68. Let $E = \mathbb{R}^2/\mathbb{Z}^2$. A point $\tau \in \mathbb{H}$ gives a metric on E by mapping the universal cover to $\mathbb{C}$ via $(x, y) \mapsto x + \tau y$ and then pulling back the Euclidean metric. We can do the same thing for $t \in \mathbb{R}$, except now we get a map to the real line and hence only a degenerate metric g_t on E. Show that if we take any pair of points $s \neq t \in \mathbb{R}$, and consider the family of metrics

$$g_\theta = \sin(\theta) g_s + \cos(\theta) g_t,$$

$\theta \in [0, \pi/2]$, , up to scale, then (E, g_θ) is conformally equivalent to (E, g_τ) with τ moving along the geodesic in $\mathbb{H}$ connecting s and t.

69. (i) Give a precise definition for the intersection form $\langle \alpha, \beta \rangle$ on $H = H_1(\Sigma_g, \mathbb{Z})$.

    (ii) Let α be a homologically nontrivial simple closed curve on Σ_g, and consider the automorphism of H defined by $f(\beta) = \beta + \langle \alpha, \beta \rangle \alpha$. Determine if f is represented by a right or left Dehn twist around α.

    (iii) Identity $\mathrm{SL}_2(\mathbb{Z})$ with the mapping-class group of a torus, and determine if $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ corresponds to a right or left Dehn twist.

70. More generally, consider $f \in \mathrm{Aff}(X, q)$ with $(X, q) \in Q\mathcal{M}_g$. Show that f is finite order, reducible or pseudo–Anosov according to whether $Df \in \mathrm{SL}_2(\mathbb{R})$ is finite order, parabolic or hyperbolic.

71. A *pentagonal form* is a form $(X, \omega) \in \Omega\mathcal{M}_2$ with a double zero and 5-fold symmetry. Show how to construct such a form from two regular pentagons in the plane.

72. Show that the $\mathrm{SL}_2(\mathbb{R})$ orbit of the golden table contains a pentagonal form. (The golden table is the 1-form of genus 2 (X, ω) constructed from a symmetric L–shaped polygon with sides of length 1 and γ, $\gamma^2 = \gamma + 1$.)

73. (i) Give an example of two conformal metrics λ, μ on $\mathbb{H}$, both of constant curvature -1, such that $\lambda < \mu$.

    (ii) Show that if λ, μ are two conformal metrics of constant curvature $+1$ on $\widehat{\mathbb{C}}$, and $\lambda \leq \mu$ everywhere, then $\lambda = \mu$.

    Why is the case of $\widehat{\mathbb{C}}$ different from the case of $\mathbb{H}$?

74. Let α, β be a pair of simple closed curves on a topological torus Σ_1, with geometric intersection number $i(\alpha, \beta) = n > 0$. Describe precisely how to associate a quadratic differential (X_t, q_t) in $Q\mathcal{T}_1$ to the

pair $(\alpha, t\beta)$, $t \in \mathbb{R}_+$, and in particular describe the familly of lattices $\Lambda_t \subset \mathbb{C}$ such that $(X_t, q_t) = (\mathbb{C}/\Lambda_t, dz^2)$.

75. The purpose of this exercise is to give several polygonal representations of the same Riemann surface.

    Fix $g \geq 2$ and let X be the hyperelliptic Riemann surface of genus g defined by $y^2 = x^{2g+2} - 1$.

    Show that X is the unique hyperelliptic Riemann surface with $\mathrm{Aut}(X) \cong \mathbb{Z}/2 \times \mathbb{Z}/(2g+2)$.

    Give polygon models for the eigenforms on X of the order $2g+2$ automorphism $T(x, y) = (\zeta x, y)$, where $\zeta = \exp(2\pi i/(2g+2))$.

76. A *Strebel differential* $(X, q) \in Q\mathcal{M}_g$ is a quadratic differential such that all the leaves of $\mathcal{F}(q)$ are closed. In this cylinders of $\mathcal{F}(q)$ dtermine a finite measured lamination $\alpha = \sum h_i \alpha_i$ on X, i.e. a weighted system of disjoint simple closed curves.

    Suppose that $\mathcal{F}(-q)$ is also a Strebel differential, with associated finite lamination β. Show that α and β fill the surface X , in the sense that

    $$i(\alpha, \gamma) + i(\beta, \gamma) > 0$$

    for all simple closed curves γ.

77. Construct filling curve systems on surfaces that realize the Coxeter diagrams A_n, D_n and E_n.

78. Compute the dilatation of the pseudo–Anosov mapping $\tau_A \tau_B^{-1}$ based on the A_n diagram.

79. Compute the period of the finite–order mapping $\tau_A \tau_B$ based on the E_n diagram, $n = 6, 7, 8$.

80. Construct filling curve systems for the Euclidean Coxeter diagram $\widetilde{A_n}$, with n even. Describe the corresponding square–tiled surfaces directly.

81. A *cylinder* for $(X, q) \in Q\mathcal{M}_g$ is a region $U \subset X$ such that $(U, |q|)$ is isometric to a Euclidean cylinder. In this case U is foliated by closed geodesics of $|q|$ with constant slope.

    Show that every (X, q) contains a cylinder, and indeed that the set of slopes of cylinders is dense in S^1. (Hint: first construct a maximal immersed rectangle with given slope; use the fact that (X, q) has finite area and finitely many zeros.)

82. (Perron method.) Let $T \subset \mathbb{C}$ be the triangle with vertices $\{0, 1, i\}$, and let $E_0 = \{z \,:\, |z| < 2\}$. Let $E_0 \supset E_1 \supset E_2 \supset \ldots$ be a descending sequence of convex compact sets such that $E_i \supset T$ and $\operatorname{diam}(E_i - E_{i+1}) \leq 1$. What is the least possible n such that $E_n = T$?

83. Let $L_n \subset \mathbb{R}^3$ be a sequence of disjoint lines with $L_n \to \infty$. Show that the fundamental group of the complement of $\bigcup L_n$ in $\mathbb{R}^3$ is a free group on infinitely many generators.

84. Consider a compact complex threefold of the form $FM = \mathrm{SL}_2(\mathbb{C})/\Gamma$. Show directly that M is not a Kähler manifold.

85. Let $\gamma \subset \mathbb{H} \cong \mathcal{T}_1$ be the geodesic stabilized by a hyperbolic metric $A \in \mathrm{SL}_2(\mathbb{Z})$. What special properties do the complex tori underlying the points of γ enjoy? (Hint: compare these tori to CM points, i.e. tori with extra endomorphisms.)

86. Let $f \in \mathrm{Aff}^+(X, q)$ be an affine automorphism with derivatrive $Df \in \mathrm{PSL}_2(\mathbb{R})$. Show that f is finite order, reducible or pseduo–Anosov according to whether Df is elliptic, parabolic or hyperbolic.

87. Let $M = (\mathbb{C}^2 - \{0\})/(z \mapsto 2z)$ be the Hopf manifold. Find the group $\mathrm{Aut}(M)$ of all holomorphic automorphisms of f. (Hint: lift an automorphism, show it extends to $z = 0$, and then consider its power series expansion at the origin.)

    Show that every $f \in \mathrm{Aut}(M)$ has entropy $h(f) = 0$.

88. Prove that, up to sign, the E_8 diagram describes the lattice $v^\perp$ in $\mathbb{Z}^{1,8}$, where $v = (-3, 1^8) = (-3, 1, 1, 1, 1, 1, 1, 1, 1)$. (The inner product on $\mathbb{Z}^{1,8}$ is given by a matrix with diagonal entries $(1, -1^8)$.

89. Find the A_6 configuration of cylinders in the flat surface constructed for billiards in a regular septagon.

90. Find a filling pair of multicurves $A, B \subset \Sigma_g$ that generate the Ward Teichmüller curve, which comes from a $(1, 2, n-3)$ triangle.