Course Notes
Math 275, Harvard University
C. McMullen

1 Topics

Description. A survey of fundamental results and current research. Topics may be chosen from the several interacting areas described below.

Riemann surfaces and Teichmueller theory.

- Hyperbolic surfaces
- The Poincare' metric on a plane region
- Fenchel-Nielsen coordinates
- Complex projective structures
- Quasifuchsian groups
- Quasiconformal mappings
- Extremal length
- Bers embedding
- Teichmueller's theorem
- The Weil-Petersson metric
- Kaehler hyperbolicity
- Earthquakes
- Geodesic currents

Iteration on Teichmueller space.

- The mapping-class group
- Chararacterization of rational maps
- 3-manifolds that fiber over the circle
- The Theta conjecture

- Gluing acylindrical manifolds
- Geometrization of Haken manifolds
- The Mordell-Shafarevich conjecture

Hyperbolic 3-manifolds.

- Knot complements
- Reflection groups
- Mostow rigidity
- Margulis tubes
- Hyperbolic volume
- Dehn filling
- Ahlfors finiteness theorem
- Bers area theorem
- Sullivan bound on cusps
- Limit sets and Hausdorff dimension
- The bifurcation current
- Ratner-Shah rigidity of immersed planes

Conformal dynamics.

- Julia sets
- Montel's theorem
- Classification of stable regions
- No wandering domains
- Holomorphic motions
- Bifurcations and stability
- Hausdorff dimension and measures

- Critically finite rational maps
- Families of rational maps
- Heights and periodic points
- Solvability of the quintic

Dynamics on moduli spaces.

- Billiards
- Geodesic and horocycle flows over moduli space
- Recurrence and unique ergodicity
- Entropy
- Curves systems and pseudo-Anosov maps
- Teichmueller curves
- Regular polygons
- L-shaped tables
- Jacobians with real multiplication
- Rigidity of VHS (Schmid)
- Real multiplication and torsion packets (Moeller)
- Cubic curves in the plane
- Totally geodesic surfaces

Reflections groups, entropy, algebraic dynamics.

- Coxeter groups
- Lattices
- Glue
- Entropy and homology
- Entropy in holomorphic dynamics

- Manifestations of Lehmer's number
- Dynamics on K3 surfaces
- Dynamics on rational surfaces
- Triangle groups
- Braid groups and Hodge theory
- The clique polynomial

2 Lectures

Lecture 1. Riemann surfaces. Uniformization. Hyperbolic surfaces. Geodesics minimize geometric intersection number $i(\alpha, \beta)$. Marked Riemann surfaces; $\mathrm{M}_g = \mathcal{T}_g / \operatorname{Mod}_g$. Pairs of pants, Fenchel–Nielsen coordinates, $\mathcal{T}_g \cong (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$. Construction of $\overline{\mathcal{M}}_g$ using cuffs of zero length.

The Schwarz Lemma: every holomorphic map $f : X \to Y$ is either (a) a locally isometric covering map or (b) a contraction, for the hyperbolic metric.

Lecture 2. Estimates for the hyperbolic metric on a plane region. If U is simply–connected, then ρ_U and the $1/d$ metric agree to within a factor of two. Relation to the Koebe 1/4 theorem. Proof of 1/4 by the area theorem.

Application: any attracting cycle of a quadratic polynomial $f(z) = z^2 + c$ attracts a critical point. Thus if f has an attracting cycle, it has only one, and it can be found by iterating the critical point. If $f^n(c) \to \infty$ then f has no attracting cycles. So to have an attracting cycle, $\langle f^n(0) \rangle$ must be bounded. Conjecture (main problem in the field): if the critical orbit is bounded, can it be perturbed slightly so there is an attracting cycle?

Lecture 3. The doubly–connected Riemann surfaces $\mathbb{C}^*$, Δ^* and $A_R = \{z : 1 < |z| < R\}$. Proof that any X with $\pi_1(X) \cong \mathbb{Z}$ is one of these. The modulus of an annulus;

$$\operatorname{mod}(A_R) = \frac{\log R}{2\pi} = \frac{\pi}{\log \alpha} = \frac{\pi}{L(\gamma \subset A_R)},$$

where $A_R \cong \mathbb{H}/\langle g \rangle$, $g(z) = \alpha z$, $\alpha > 1$. Extremal length; $\operatorname{mod}(A) = \lambda(\Gamma_h) = 1/\lambda(\Gamma_c)$, $h, c =$ height and circumference. Key calculation: (x circumference, y height):

$$cL(h, \rho) \leq \int 1 \cdot \rho \, dx \, dy \leq (hc)^{1/2} A(\rho)^{1/2},$$

so $L_\rho^2/A_\rho \leq h/c$; thus $\mathrm{mod}(A) = h/c$. Estimates for the modulus of the annulus $A(a,b)$ between the squares of 'radius' n and $n+1$.

Lecture 4. Classification of triply–connected surfaces; pairs of pants. $\langle v, w\rangle = \pm\cosh d(\gamma_v, \gamma_w)$. Key calculation: with $C_i \leq -1$,

$$\det\begin{pmatrix} 1 & -C_1 & -C_2 \\ -C_1 & 1 & -C_3 \\ -C_2 & -C_3 & 1\end{pmatrix} = 1 - 2C_1C_2C_3 - C_1^2 - C_2^2 - C_3^2 \leq -4 < 0,$$

so this is a quadratic form of signature $(2,1)$ (why not $(0,3)$?), and hence arbitrary pairs of pants exist.

Lecture 5. Incomplete hyperbolic structures: take the strip $S = \{z : 0 \leq \mathrm{Re}(z) \leq 1\}$ and glue iy to $2iy + t$. The resulting hyperbolic surface X_t?

Beurling's criterion for an extremal metric: the probability measure ρ^2/A_ρ is in the convex closure of the measures $\rho|\gamma/L_\rho$ for a suitable set of geodesics. We may assume $A_\rho = 1$. Then for any other metric τ, we have

$$L_\tau \leq \int_\gamma \tau = L_\rho \langle \tau/\rho, \rho|\gamma/L_\rho\rangle,$$

which implies that

$$\frac{L_\tau}{L_\rho} \leq \langle \tau/\rho, \rho^2\rangle \leq \int_X (\tau/\rho)\rho^2 \leq \left(\int_X (\tau/\rho)^2\rho^2\right)^{1/2} = A_\tau^{1/2},$$

which gives $L_\tau/A_\tau^{1/2} \leq L_\rho = L_\rho/A_\rho^{1/2}$.

Examples on a quadrilateral, annulus, torus, and $\mathbb{RP}^2$.

Quasiconformal maps: analytic and geometric definitions. K–quasiconformal maps distortion extremal length by at most K.

One-dimensional quasiconformal maps. Example of a homeomorphism (Euclidean earthquake) that is conformal a.e. but not conformal. Solvability of the Beltrami equation.

The moduli space of annuli is $\mathbb{R}_+$ with metric $(1/2)dm/m$. The Teichmüller maps are given by the affine stretch (Grötzsch).

The space $\mathcal{T}(\mathbb{R}^2) = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{H}$ of complex structures on $\mathbb{R}^2$. The metric $d(A,B) = \frac{1}{2}\log K(B^{-1}A)$. (The order is important.) Proof that this agrees with twice the hyperbolic metric (consider $d(i, iy)$).

The unit disk model for $\mathcal{T}(\mathbb{R}^2)$: $\mu \in \Delta$ corresponds to $\overline{\partial}_\mu = \partial - \mu\overline{\partial}$. The Beltram equation $f_{\overline{z}} = \mu f_z$. The disk bundle over X; the space of measurable Beltrami differentials $M(X)$; the 'measurable Riemann mapping theorem'.

The Teichmüller space $\mathcal{T}_1$. Horocycles and extremal lengths of loops. Proof that distance on $\mathcal{T}_1$ coincides with hyperbolic distance. Proof by the Riemann mapping theorem that $\mathcal{M}_1 = \mathbb{H}/\Gamma(2)$ is isomorphic to $X = \widehat{\mathbb{C}} - \{0, 1, \infty\}$. The hyperbolic metric on X.

The space of annuli as the imaginary axis is $\mathcal{T}_1$.

Teichmüller spaces with $g > 1$; lengths and eigenvalues as moduli.

Theorem. The functions $\log \ell_\gamma(X)$ are uniformly Lipschitz on $\mathcal{T}_g$.

The heat kernel. On Euclidean space it is given by

$$K_t(x, y) = \frac{1}{(4\pi t)^{d/2}} \exp(-|x - y|^2/4t)\cdot$$

The functions $f_t = K_t * f_0$ satisfy $df_t/dt = \Delta f$.

Lecture 6.

Theorem (Selberg trace formula). On a compact hyperbolic surface, the trace of K_t, given by

$$\int_X \mathrm{tr}(\exp(-t\Delta)) = \sum \exp(-t\lambda_n),$$

can be expressed in terms of the area of X and its length spectrum, with terms of the form $\exp(-\ell_n^2/t)$. Small eigenvalues matter when t is large, short lengths when t is small.

Cor. The length spectrum determines the spectrum of the Laplacian, and vice–versa.

The reduced trace of K_t on a compact hyperbolic surface X is the same as the reduced trace summed over all its primitive annular covers. On a cover with core curve of length L, the reduced trace is:

$$\mathrm{Tr}_0(K_t) = \frac{1}{2}(\pi t)^{-1/2} e^{-t/4} \sum_{n=-\infty}^{\infty}{}' \frac{L}{\sinh(nL/2)} \exp(-n^2 L^2/(4t)).$$

Theorem. The locus in $\mathcal{M}_{g,n}[r]$ where the length of the shortest closed geodesic is $\geq r > 0$ is compact. The theme of short geodesics.

Theorem: For X in $\mathcal{M}_g$, the spectrum of Δ_X determines X up to finitely many choices. (You can almost hear the shape of a drum).

Proof: (Wolpert) Use the fact that $\mathcal{M}_g(r)$ is compact.

Theorem: There are isospectral examples.

Proof (Sunada): Covering constructions.

Lecture 7. Hyperbolic preliminaries: (a) the collar theorem. (b) short geodesics are simple.

Proof of (b): take the covering space corresponding to a puncture and apply the Schwarz lemma.

Mumford's compactness criterion. Bers' constant C_g. The hairy torus: $\sqrt{g}$. Proof via Fenchel–Nielsen coordinates. Proof via geometric limits: a lower bound on the injectivity radius gives an upper bound on the diameter.

Lecture 8. Examples of geometric limits. Dehn filling. Volumes of hyperblic manifolds. Infinitely generated limits. Second proof of Mumford's thoerem. Proof of Wolpert's theorem. The theme of short geodesics.

Lecture 9. Quasiconformal maps: analytic and geometric definitions. K–quasiconformal maps distortion extremal length by at most K.

One-dimensional quasiconformal maps. Example of a homeomorphism (Euclidean earthquake) that is conformal a.e. but not conformal. Removability of sets with Hausdorff dimension $E < 1$.

Equicontinuity of quasiconformal maps. Hölder continuity via the modulus of the paths separating two points from infinity.

Motion of points on the triply–punctured sphere: $d(x, f(x)) \le \log K(f)$.

Quasicircles, bounded turning, compactness, Hausdorff topology. Example: The snowflake. Exercise: $\mathrm{H.}\dim(Q) < 2$.

Lecture 10. The space $\mathcal{T}(\mathbb{R}^2) = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{H}$ of complex structures on $\mathbb{R}^2$. The metric $d(A,B) = \frac{1}{2}\log K(B^{-1}A)$. (The order is important.) Proof that this agrees with twice the hyperbolic metric (consider $d(i, iy)$).

The unit disk model for $\mathcal{T}(\mathbb{R}^2)$: $\mu \in \Delta$ corresponds to $\overline{\partial}_\mu = \partial - \mu\overline{\partial}$.

The Beltram equation $f_{\overline{z}} = \mu f_z$. The disk bundle over X; the space of measurable Beltrami differentials $M(X)$; the 'measurable Riemann mapping theorem'. The solution to $\overline{\partial} v = \mu$ is given by $v = \mu * (1/\pi)(1/z)$. Example of $v(z) = \overline{z}$ for $|z| < 1$, $1/z$ for $|z| > 1$. If μ is bounded then v is Zygmund.

We similarly have $f_z = (-1/\pi)(1/z^2) * f_{\overline{z}}$. Contractivity proof of solution to $\mu(f) = \mu$.

The tangent and cotangent spaces to Teichmüller space: classical version: $H^1(X, K^{-1})$ and $Q(X)$.

The geometry of quadratic differentials. Examples: rectangle, cylinder, torus, pillowcase; Lattès maps. Connect sums of tori.

Local straightening: if $\mathrm{ord}_p(q) \ge -1$ then in local coordinates, $q = z^n\, dz^2$.

Teichmüller's theorem.

The complex structure on Teichmüller space as a quotient of $M(X)$.

3 Other thoughts

Poisson summation. This method is used frequently in analytic number theory, and it is necessary to show that counting functions for lattices are related to theta functions.

It its most basic form, Poisson summation is that statement that for a Schwartz function $f : \mathbb{R} \to \mathbb{C}$, we have

$$\sum_{\mathbb{Z}} \widehat{f}(n) = \sum_{\mathbb{Z}} f(0).$$

Conceptually, there is a third function that intervenes here, namely the pushforward for f to a function on $S^1 = \mathbb{R}/\mathbb{Z}$ defined by

$$F(x) = \sum_{\mathbb{Z}} f(x+n).$$

Then $\widehat{F}(t)$ is a function on $\mathbb{Z}$, which is now the dual of S^1. With suitable normalizations, we have $\widehat{F}(n) = \widehat{f}(n)$, and then the formula above just says that $F(0) = \sum_{\mathbb{Z}} \widehat{F}(n)$.

Variants of the Apollonian packing. The image of the Apollonian packing is ubiquitious these days, as an example of the fashionable topic of a thin group. But in fact any hyperbolic 3-manifold of infinite volume gives an example of a thin group, and even those which are small variations on the Apollonian packing deserve to be better known.

To describe these variations, one first needs to know what the Apollonian packing is, as a 3–manifold. Namely, it is the limit set of the fundamental group of a pared manifold of the form $M = (H_g, P_0 \cup P_1 \cup P_2)$, where $g = 2$, H_g is a handelbody of genus g, and P_i are annuli on ∂H_g describing parabolic loci. The first annuls P_0 is chosen to be the commutator, so that $H_g - P_0$ is isomorphic to $\Sigma_1, 1 \times I$, where $\Sigma_{1,1}$ is a torus with one boundary component. The second annulus P_1 can then be taken as one of the standard simple loops a in $\pi_1(\Sigma_{1,1}) = \langle a, b\rangle$. The third annulus resides on a second copy of $\Sigma_{1,1}$, and represents a second simple closed curve S on $\Sigma_{1,1}$. It is uniquely determined by its homology class $b = (p, q) \in H_1(\Sigma_1, \mathbb{Z})$, in coordinates where $a = (1, 0)$. The second generator of homology is only well–defined up to a multiple of the first, so (p, q) and $(p + kq, q)$ determine the same pared manifold; thus the data describing $M = M_{p/q}$ is a rational number $p/q \in \mathbb{Q}/\mathbb{Z}$.

Now the manifold $M_{0,1}$ is the Apollonian gasket. It is also part of the Whitehead link complement, i.e. if we glue its boundary components together we get $S^3 - W$.

So the first type of variant is the family of manifolds $M_{p/q}$. These enumerate cusps on the boundary of Maskit's embedding of Teichmüller space, abundantly illustrated in *Indra's Pearls*.

Graph complements in S^3. See Hodgson *et al*, Hyperbolic Graphics of Small Complexity, for more on these.

Sprinkling points on the pillowcase. Consider $\widehat{\mathbb{C}}$ as the double of a square, with vertices ending up at $(-1, 0, 1, \infty)$. Problem: sprinkle points evenly on the sphere with respect to this metric. Solution: note that $\pm i$ are fixed points of rotation of the square. Consider the map

$$f(z) = \left(\frac{z-i}{z+i}\right)^2 .$$

This is a Lattès map for multiplication by $1+i$ on the square torus. It sends $(0, \infty)$ to 1 and $(-1, 1)$ to -1. The iterated preimages of zero correspond to $(1+i)^{-n}\mathbb{Z}[i]$ and give the desired points.

Measurable Riemann mapping theorem. For a Riemannian surface (M, g), harmonic functions quickly lead to conformal coordinates. That is, if $d * df = 0$, then by integrating df and $*df$ to u and v we get a complex coordinate $z = u + iv$. This shows by familiar methods that the Beltrami equation can be solved locally for smooth μ.

For general L^∞ μ, start with smooth μ_n that converge to μ pointwise a.e. and also atisfy $|\mu_n| \geq k < 1$. By equicontinuity of quasiconformal maps, we get smooth F_n converging to a quasiconformal map F. Letting $f_n = \partial F_n$, we have $\overline{\partial} F_n = \mu_n f_n$ and $\overline{\partial} F_n \to \overline{\partial} F$ weakly, and $f_n \to \partial F$ weakly. So it suffices that $\mu_n f_n \to \mu f$ weakly.

This is a general fact. More precisely we have:

Lemma 3.1 *Suppose $\nu_n \to 0$ pointwise, $\|\nu_n\|_\infty \leq 1$, and $\|f_n\|_2 \leq M$. Then $\nu_n f_n \to 0$ weakly.*

Proof. For any $g \in L^2$ we have

$$|\langle \nu_n f_n, g\rangle| \leq M \cdot \|\nu_n g\|_2,$$

and the latter tends to zero by the dominated convergence theorem. ∎

Now write

$$\mu f - \mu_n f_n = \mu(f - f_n) + (\mu - \mu_n) f_n .$$

Since $\|\mu\| \leq 1$, and μ does not depend on n, the first term tends to zero weakly. The second converges to 0 weakly by the Lemma.

Measured foliations on the torus. We can regard a point τ in the upper halfplane as determining a metric on the plane $\mathbb{R}^2$ with coordinates (a, b) using the map

$$\phi_\tau : \mathbb{R}^2 \to \mathbb{C}$$

given by

$$\phi_\tau(a, b) = a + b\tau.$$

(Of course this descends to give a complex structure to the torus $\mathbb{R}^2/\mathbb{Z}^2$). The metric is given by

$$|\phi_\tau(a, b)|^2 = |a + b\tau|^2.$$

In the limit where τ becomes real, this quadratic form becomes degenerate and it gives $\mathbb{R}^2$ a measured foliation. For any two points $\tau_1, \tau_2 \in \mathbb{R}$ we take the weighted sum of these two degenerate forms to get a metric on $\mathbb{R}^2$ again. In this metric the two foliations are by definition orthogonal. Choosing weights $s, t \geq 0$ with $s^2 + t^2 = 1$, the metric comes from the map

$$\phi_{s,t}(a, b) = s(a + \tau_1 b) + it(a + \tau_2 b) = a(s + it) + b(s\tau_1 + it\tau_2).$$

This corresponds to the point in $\mathbb{H}$ given by

$$\tau_{s,t} = \frac{s\tau_1 + it\tau_2}{s + it} = (s - it)(s\tau_1 + it\tau_2) = (s^2\tau_1 + t^2\tau_2) + ist(\tau_2 - \tau_1).$$

(using the fact that $s^2 + t^2 = 1$). This is the same as

$$\tau_{s,t} = (t + is)^2(\tau_2 - \tau_1)/2 + (\tau_2 + \tau_1)/2,$$

which means $\tau_{s,t}$ moves along the hyperbolic geodesic from τ_1 to τ_2.

Uniqueness of the Teichmüller geodesic. The idea is that if f is a pseudo–Anosov map that fixes 2 points $\mu_\pm$ in $\mathbb{P}\mathcal{ML}$, then any third fixed point ξ satisfies

$$i(f(\xi), \mu_\pm) = \lambda^{\pm} i(\xi, \mu_\pm),$$

so one of the intersection pairings must vanish. This plus unique ergodicity shows ξ is μ_+ or μ_-.

From quadratic differentials to measured foliations. Near a non-singular point, a smooth measured foliation is conveniently expressed by a *degenerate*, positive quadratic form $Q(v)$ on the real tangent space to a surface. The vectors *tangent* to the foliation are those satisfying $Q(v) = 0$; and the *transverse measure* of a path p is obtained by integrating $Q(v)^{1/2}$, where $v = dp/dt$.

For a holomorphic quadratic differential, the Q corresponding to the horizontal foliation is simply

$$Q = (\mathrm{Im}(\sqrt{q}))^2.$$

For example, if $q = dz^2$, then $Q(x+iy) = y^2$. The point here is the $\sqrt{q}$ is naturally a 1-form, but only well-defined up to sign; nevertheless, the square of its imaginary part is positive and well–defined.

This also makes clear that a measured foliation locally gives a real–valued function on a surface, up to sign and translation by a constant. If we take two transverse foliations, the two associated functions can be combined to make a complex coordinate $z = f_1 + if_2$, and then dz^2 gives a quadratic differential. These can be pieced together provided we have global transversality.

The idea of using *laminations* is that geodesics are either coincident or transverse, so a point in $Q\mathcal{T}_g$ can be reconstructed from a pair of measured laminations.

Ultrahyperbolic metrics and Royden's theorem. Let $\rho = \rho_\Delta = 2|dz|/(1-|z|^2)$ denote the unique complete hyperbolic metric on the unit disk $\Delta = \Delta$.

One way to make an *incomplete* hyperbolic metric α on the unit disk is to shrink ρ so that there is a path to S^1 of finite length. One can also make ρ *more negatively curved* by shrinking the metric. Ahlfors' version of the Schwarz lemma says that, conversely, the *only* way to make ρ into a metric with $K(\alpha) \leq -1$, possibly incomplete, is to shrink it.

Theorem 3.2 *For any metric α on Δ with curvature $K(\alpha) \leq -1$, we have $\alpha \leq \rho$.*

As a prelimary, we note that the curvature of a metric $\alpha = \alpha(z)\,|dz|$ is given naturally by

$$K(\alpha) = -\frac{\Delta \log \alpha}{\alpha^2}\cdot$$

(Check this for $\alpha(z) = 1/y$; we have $(d/dy^2)(-\log y) = 1/y^2$.) Note the minus sign! To check ths gin intuitively, note that at a local minimum of α we have negative curvature because nearby spheres are longer than in Euclidean space.

Proof. First suppose $\alpha(z)$ is smooth on the closed unit disk. Since $\rho(z) = 2/(1-|z|^2) \to \infty$ near S^1, the ratio α/ρ attains its maximum at an interior point $p \in \Delta$. At this point the Hessian is a non–positive quadratic form, and hence its trace satisfies

$$\Delta \log(\alpha/\rho) \leq 0,$$

i.e. at the point p we have

$$0 < \Delta \log \alpha \leq \Delta \log \rho.$$

(The sign is checked below.) From $K(\alpha) \leq K(\rho) < 0$ at p we get

$$\frac{\Delta \log \alpha}{\alpha^2} \geq \frac{\Delta \log \rho}{\rho^2} > 0,$$

which also checks the sign. This shows:

$$1 \geq \frac{\Delta \alpha}{\Delta \rho} \geq \frac{\alpha^2}{\rho^2},$$

so $\alpha/\rho \leq 1$ at its maximum, and thus $\alpha \leq \rho$ everywhere.

To remove the smoothness assumption, replace ρ with the Poincaré metric ρ_r on the disk of radius $r < 1$, and let $r \to 1$. ■

Corollary 3.3 *The Teichmüller metric agrees with the Kobayashi metric on $\mathcal{T}_{g,n}$.*

Proof. Teichmüller disks give an isometric copy of the hyperbolic plane through every point in every complex direction. Because of this, the pullback of the Teichmüller metric to the upper halfplane under a holomorphic map has curvature ≤ -1, so we get contraction by theorem above. Taken together these facts imply $d_K = d_T$. ■

Topological mixing. Once unique ergodicity of the invariant laminations or foliations $\mathcal{F}, \mathcal{G}$ of a pseudo–Anosov map is shown, one can establish the following:

Theorem 3.4 *Let $f : \Sigma \to \Sigma$ be a pseudo–Anosov mapping–class, with stretch factor $\lambda > 1$. Then for any pair of (simple) closed curves α and β, we have:*

$$\lim_{n\to\infty} \frac{i(\alpha, f^n(\beta))}{\lambda^n} = \frac{i(\alpha, \mathcal{F}) i(\beta, \mathcal{G})}{i(\mathcal{F}, \mathcal{G})}.$$

Here $f(\mathcal{F}) = \lambda \mathcal{F}$, so $\mathcal{F}$ is the attracting foliation.

Note that

$$i(\mathcal{F}, f^n\beta) = i(f^{-n}\mathcal{F}, \beta) = \lambda^{-n} i(\mathcal{F}, \beta) \to 0,$$

so in the limit we expect $f^n(\beta)$ to converge in $\mathbb{P}\mathcal{ML}$ to a lamination with zero intersection number with $\mathcal{F}$, and hence (by unique ergodicity) to a multiple of $\mathcal{F}$. The multiple is determined by comparing $i(\beta, \mathcal{G})$ to $i(\mathcal{F}, \mathcal{G})$.

The model Teichmüller map/geodesic. The model for a Teichmüller map $f_t : X_0 \to X_t$, with $d(X_0, X_t) = t$, coming from an A–orbit in $Q\mathcal{T}_g$, i.e. a family (X_t, q_t), is the following:

locally we can choose coordinates so that $q_t = dz^2$; then:

$$f_t(z) = \cosh(t)z + \sinh(t)\overline{z}.$$

Note that is general $Jf = |f_z|^2 - |f_{\overline{z}}|^2$, which motivates our choice of coefficients. We also have:

$$Df_t = \cosh(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

so $K(Df_t) = \exp(2t)$ which gives

$$d(X_0, X_t) = (1/2)\log K(f_t) = t.$$

We also have, evidentally,

$$\mu(f_t) = \tanh(t)\frac{d\overline{z}}{dz},$$

which gives the arclength parameterization of $[0, 1)$ in the metric $dx/(1-x^2)$.

Strebel differential with one cylinder. Consider a cylinder A with the usual quadratic differential q, foliating A by closed loops. Pinch the top of bottom of A to create a surface of genus zero with 4 boundary components of the same length in the $|q|$ metric. Glue these together in pairs so that the pinch points are identified. The result is a Strebel differential of one cylinder, $(X, q) \in Q\mathcal{M}_2$, with 2 double zeros. The gluing can be done so that the core curve is null homologous. Then q is not the square of a holomorphic 1-form.

An affine stretch that is not a Teichmüller map. There are examples of twisted quadratic differentials which give rise to maps that locally appear to be Teichmüller maps but are not globally.

One of the simplest examples arises by taking the map $f(x+iy) = Kx+iy$ on $\mathbb{C}^*$, and observing that f descends to a map on $E = \mathbb{C}^*/\langle\alpha\rangle$ for any $\alpha \in \mathbb{C}^*$, $|\alpha| \neq 1$. The quotient space is a complex torus and f is even isotopic to the identity.

The issue here is that the differential dz^2 on $\mathbb{C}^*$ does not descend to E. We get a foliation $\mathcal{F}$ of E, but no transverse invariant measure. The foliation $\mathcal{F}$ actual has two closed leaves and all others spiral towards these. Because of the absence of a transverse measure, we can not set up an extremal length problem which is optimized by f.

Multicurves and quadratic differentials. Consider two multicurves $A = \sum \alpha_i A_i$ and $B = \sum \beta_i B_i$ on Σ_g. Assume that these multicurves fill the surface, in the sense that $i(A, C) + i(B, C) > 0$ for any simple closed curve C. Then there exists a unique point $(X, q) \in Q\mathcal{M}_g$ with $\mathcal{F}(q) = A$ and $\mathcal{F}(-q) = B$.

To describe q, we first realize A and B by hyperbolic geodesics so their intersection minimally. At each intersection point of A_i and B_j we construct a rectangle with dimensions $\alpha_i \times \beta_j$. Then we glue rectangles togeether along their edges whenever they are joined by an arc of $A \cup B$. Because of the assumption that $A \cup B$ fills the surface, its complementary components are $2k$-gons, and each of these becomes a simple pole ($k = 1$), a regular point ($k = 2$) or a zero of order $k - 2$ of the quadratic differential q.

Decomposition into cylinders of equal modulus. Next we show that the weights or heights $h = (\alpha_i, \beta_j)$ can be chosen in a unique way, up to scale, so that the moduli of all the cylinders of q are the same. For this purpose it is useful to put the curves (A_i, B_j) together into a single family (C_k), and let $M_{ij} = i(C_i, C_j)$. Then the circumference of C_i is given by $c_i = \sum_j M_{ij} h_j$. If all the cylinders have modulus $m = h_i / c_i$, then we get

$$(1/m) h_i = c_i = \sum_j M_{ij} h_i,$$

In other words, the desired heights come from the Perron–Frobenius eigenvector of M, and $\mu 1/m$ is the leading eigenvalue, $\mu = \rho(M)$.

For these heights we obtain $(X, q) \in Q\mathcal{M}_g$ such that there are right Dehn twists $\tau_A, \tau_B \in \mathrm{Aff}(X, q)$ with

$$D\tau_A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad D\tau_B = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}.$$

In particular we have:

Theorem 3.5 *The group* $\mathrm{PSL}(X, q)$ *is a lattice if* $\rho(M) \leq 2$.

Example. It is often more convenient to work with the rectangular incidence matrix $C_{ij} = i(A_i, B_j)$. Then $\rho(M)^2 = \rho(CC^t)$. For the A_4 diagram we have $C = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and hence

$$\rho(M)^2 = \rho\left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right) = \gamma^2;$$

since $\gamma < 2$, we get a lattice.

Coxeter diagrams. The same holds whenever the incidence graph of (A, B) is a spherical or affine Coxeter diagram. Indeed, the inner product associated to a Coxeter diagram with edge weights m_{ij} is

$$B_{ij} = -2\cos(\pi/m_{ij})$$

which is $2, 0, -1$ for $m_{ij} = 1, 2, 3$. This is consistent with the relation $(s_i s_j)^{m_{ij}} = \mathrm{id}$ in the corresponding Coxeter group. For the simply–laced case, we just get

$$B = 2I - M,$$

where M is the adjacency matrix of the Coxeter graph. Since M is Perron–Frobenius, we have $B \gg 0$ ($B \geq 0$) iff $\rho(M) < 2$ ($\rho(M) = 2$).

Thus Thurston's construction leads to a lattice iff the corresponding Coxeter diagram is spherical or affine. (Leininger.) The spherical cases are the most interesting, they correspond to A_n, D_n, E_6, E_7 and E_8. In the affine cases, $\mathrm{PSL}(2, q)$ is commensurable to $\mathrm{PSL}(2, \mathbb{Z})$, so the resulting differential is square–tiled.

The Torelli group. The kernel of the natural map $\mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is the *Torelli group* Tor_g.

For $g = 1$ it is trivial.

For $g = 2$ it is a free group on infinitely many generators (Mess). In fact it is generated freely by Dehn twists around separating loops. To see this, one identifies Tor_2 with the fundamental group of the complement in Siegel space $\mathfrak{H}_2$ of the set R of (marked) Abelian varieties that are polarized products of elliptic curves. Each component of R is a totally geodesic copy of $\mathbb{H} \times \mathbb{H}$, and then one can use embedded Morse theory to describe the homotopy type of $\mathfrak{H}_2 - R$.

For $g \geq 3$, the group is finitely generated and not free, as shown by Dennis Johnson.

Entropy and holomorphic maps. Gromov–Yomdin show that for a holomorphic automorphism f of a Kähler manifold M, we have $h(f) = \log \rho(f|H^*(M))$. The key here is that we know exactly how the volume of the graph of f grows, using the Kähler form.

This fails for non–Kähler manifolds. A nice example is provided any compact hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$. Its frame bundle carries a complex structure: we can write

$$FM = \mathrm{SL}_2(\mathbb{C})/\Gamma,$$

so it is a complex threefold. Now the geodesic flow for time 1 on FM certainly has positive entropy, but it is also homotopic to the identity, so the log spectral radius of its action on cohomology is zero.

Open problem: show that FM contains no compact curve of genus $g \geq 2$.

Geodesics in $\mathbb{H}$ as RM points. Let $\gamma \subset \mathbb{H}$ be the geodesic stabilized by a hyperbolic matrix $A \in \mathrm{SL}_2(\mathbb{Z})$. Then A acts as a pseudo–Anosov map $f_A : E \to E$ for each $E \in \gamma \subset \mathcal{T}_1$. We can normalize f_A so it is also a group automorphism. Then the entire ring $\mathbb{Z}[A]$ embeds in $\mathrm{End}(E)$. Each $B \in \mathbb{Z}[A]$ gives a map f_B that stabilizes a pair of orthogonal geodesics on E. In particular, the elements of $\mathbb{Z}[A]$ of trace zero act anticonformally on E. Fixing one of these, one can characterize γ as the locus in $\mathcal{T}_1$ where a given matrix $B \in \mathrm{M}_2(\mathbb{Z})$ is realized by an antiholomorphic endomorphism of E. Thus γ_A can be compared to a CM point $\tau_B \in \mathcal{T}_1$, which is the locus where a given matrix can be realized as *conformal* endomorphism of E. the CM locus in