Topics in Geometry and Dynamics

Problems

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1. Consider the powers of 2, $x_n = 2^n$ for $n = 0, 1, 2, \ldots$, written in base 10. What proportion of these numbers begin with the digit 1?

2. Let $x \in [0,1]$ be chosen at random, and let its continued fraction expansion be $x = 1/(a_1 + 1/(a_2 + \cdots))$. What proportion of the a_i's are 1?

3. Prove that the quadratic form $q(x) = x_1^2 + x_2^2 - p(x_3^2 + x_4^2)$ does not represent zero, when $p = 3$ or more generally when p is a prime congruent to $3 \bmod 4$.

4. Show that the closure of a subgroup of the Lie group $\mathrm{GL}_n(\mathbb{R})$ is itself a Lie group.

5. Let $f : S^3 \to S^3$ be a rotation of infinite order. What are the possibilities for the closure of an orbit of f?

6. Give an example of a compact convex set $K \subset \mathbb{R}^3$ whose extreme points do not form a closed set. Show that no such example is possible in dimension two.

7. Let $f : X \to X$ be a homeomorphism of a compact metric space to itself, and let $M \subset C(X)^*$ be the set of invariant probability measures for f, in the weak* topology. (i) Show that M is a compact, convex set. (ii) Show that the set of ergodic measures correspond to the extreme points of M.

8. Give an example of a compact, convex set in $\mathbb{R}^3$ whose extreme points do not form a closed set.

9. Given an explicit example of a compact, convex set K in $\ell^2(\mathbb{Z})$, containing more than one point, whose extreme points are dense in K. Is such an example possible in $\mathbb{R}^n$?

10. Let $X = \mathbb{R}^2/\mathbb{Z}^2$. Given $A \in \mathrm{GL}_2(\mathbb{Z})$, let $f_A : X \to X$ be the linear automorphism of the torus induced by A.

    Find all the ergodic, f_A-invariant measures for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Show that the set of ergodic probablility measures is not closed in the space of all invariant measures.

11. What is the spectral measure on S^1 and the multiplicity function, for the unitary operator on $L^2(\mathbb{R}^2/\mathbb{Z}^2)$ coming from f_A, $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$?

12. Show that the set of ergodic probablility measures for f_A is not closed when $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

13. Show that $U = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ satisfies $U^{2m} = \pm I \bmod f_m$, where f_m is the mth Fibonacci number.

14. Let $\epsilon > 0$ be the golden ratio, satisfying $\epsilon^2 = \epsilon + 1$. Show that the discriminant of the ring $\mathbb{Z}[\epsilon^m]$ is $5f_m^2$, where f_m is the mth Fibonacci number. How does this relate to the previous problem?

15. Let $f : \Sigma_2 \to \Sigma_2$ be the full shift on 2 symbols. Show that the set of ergodic probability measures is *dense* in the space of all invariant measures.

16. Show that the 'adding machine', the map $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ given by $f(x) = x + 1$, is minimal and uniquely ergodic.

17. Show that the solvable Lie group $AN \subset G = \mathrm{SL}_2(\mathbb{R})$ consisting of the upper triangular matrices is *not* unimodular.

18. Show directly that there is no (finite or σ-finite) measure on the circle $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ that is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$ by Möbius transformations. How is this related to the preceding problem?

19. Let $T : S^1 \to S^1$ be an irrational rotation, where $S^1 = \mathbb{R}/\mathbb{Z}$. Show that there exists a universal constant C (independent of T) such that $d(T^q(0), 0) \le C/q$ for infinitely many $q > 0$. (Challenge: find the best value of C.)

20. (Challenge.) Show there exists a universal constant $C > 0$, independent of T, such that the balls the form $B(T^q(0), C/q)$, $q > 0$ fail to cover S^1. (The best value of C is not known; cf. Cassels, 1952, and [GL, p.577].)

21. Prove von Neumann's ergodic theorem using the spectral theorem for a unitary operator. (You need only handle the case where U is an automorphism of H.)

22. What does the image on the course home page depict?

23. Let $T : S^1 \to S^1$ be an irrational rotation, and let $n, \epsilon > 0$ be given. Show there is a measurable set $E \subset S^1$ such that the sets $E, T(E), \ldots, T^n(E)$ are disjoint have cover all of S^1 save a set of measure $< \epsilon$.

24. What is the spectral measure for the operator shift operator on $\ell^2(\mathbb{Z})$? For an irrational rotation, acting on $L^2(S^1)$? For a hyperbolic element $A \in \mathrm{SL}_2(\mathbb{Z})$ acting on $L^2(S^1 \times S^1)$?

25. (i) Show that for any two disjoint circles C_1 and C_2 in the complex plane, there exists a Möbius transformation A such that $A(C_1)$ and $A(C_2)$ are both centered at $z = 0$. (ii) Explain how to construct $A^{-1}(0)$ in terms of the planes in hyperbolic 3-space bounded by C_1 and C_2.

26. Let $T : X \to X$ be an ergodic automorphism of a space of finite measure. Show that if $m(A) > 0$, then $B = \bigcup_{i=1}^{\infty} T^i(A)$ has full measure.

27. (Challenge/open problem.) Let $T : S^1 \to S^1$ be an irrational, let $U \subset S^1$ be an open set, and let $S_n(x) = |\{i : 0 \le i \le n \quad \text{and} \quad f^i(x) \in U\}/n$.

    Give a direct proof that $S_n(x) \to |U|/|S^1|$ a.e., without using the ergodic theorem.

28. Let $U \in \mathcal{B}(H)$ be a unitary operator whose spectral measure μ is absolutely continuous (with respect to Lebesgue measure on S^1). Prove that U is 'mixing' in the sense that $\langle U^i f, g\rangle \to 0$ as $i \to \infty$, for all $f, g \in H$.

29. Let μ and μ' be probability measures on S^1 in the same measure class (so they have the same sets of measure zero). Let T and T' be the operators given by $f(\lambda) \mapsto \lambda f(\lambda)$ on $H = L^2(S^1, \mu)$ and $H' = L^2(S^1, \mu')$ respectively. Show there is an isometric isomorphism between H and H' sending the action of T to the action of T'.

30. Let $T \in \mathcal{B}(H)$ be a unitary operator on H. Show that if λ is in the spectrum $\sigma(T)$, then for any $\epsilon > 0$ there exists an $f \in H$ with $\|f\| = 1$ and
$$\|Tf - \lambda f\| < \epsilon.$$
    In this case we say f is an *almost eigenvector* for T.

31. Consider the shift operator T acting on $\ell^2(\mathbb{Z})$. Given $\lambda \in S^1$ and $\epsilon > 0$, find an explicit almost eigenvector for T with eigenvalue λ.

32. Consider the action of the linear map $T = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on $L^2(S^1 \times S^1)$. Given $\lambda \in S^1$ and $\epsilon > 0$, find an explicit almost eigenvector for T with eigenvalue λ.

33. When is the sequence αn^2, $n \in \mathbb{Z}$, uniformly distributed on $S^1 = \mathbb{R}/\mathbb{Z}$?

34. Let T be an automorphism of a probability space (X, μ). Given $f \in L^1(X)$, let $F = \lim S_n(f)$. Show that F is the unique function in L^1 such that (a) F is T-invariant and (b) $\int_A F = \int_A f$ for any T-invariant set A.

35. Let T be an irrational rotation of S^1. Show there exists an $f \in C(S^1)$ with $\int f = 0$ such that f is not a coboundary, i.e. such that f does not have the form $f = g - g \circ T$ for some $g \in C(S^1)$.

36. The strong law of large numbers is easy to prove for *bounded*, independent, identically distributed random variables X_i. (a) Suppose $E(X_i) = 0$, $|X_i| \le 1$ and let $S_n = (X_1 + \ldots + X_n)/n$. Show that $E(S_n^2) = O(1/n)$. (b) Show that if $\sum 1/n_k < \infty$, then $S_{n_k} \to 0$ almost surely. (c) Let $n_k = \lfloor \sqrt{k} \rfloor$. Show S_n varies only a little for $\sqrt{k} < n < \sqrt{k+1}$, and use (b) to show $S_n \to 0$ almost surely.

37. Let $f : G \to H$ be a measurable homomorphism between a pair of Lie groups. Show that f is smooth. (Hint: first assume $G = \mathbb{R}$).

38. Let $N \subset G$ be a closed, connected subgroup of a unimodular Lie group. Show that G/N carries a smooth, G–invariant measure iff N is unimodular.

39. *(Winkler.) The next season of Survivor will be played as follows. At each tribal council, each person on the island writes down the name of another person the island, *possibly themselves*, chosen at random. Then *all* the named people are voted off the island. If and when only one person is left, they receive a million dollars. Let p_n be the probability that someone wins, starting with n people on the island.

    (i) Plot, using a computer, the values of p_n for $N \le n \le Ne$, with $N = 50$ and $N = 200$.

    (ii) Explain why these two plots are almost the same.

**(iii) Prove that $\lim p_n$ does not exist. (This represents a failure of the 'Law of Large Numbers'.) For a hint, see Prodinger's article *How to select a loser*.

40. *True or false: any rotation–invariant linear function $\phi : L^\infty(S^1) \to \mathbb{R}$ is a multiple of Lebesgue measure, i.e. $\phi(f) = C \int_{S^1} f(x)\, dx$.

41. Let $B = \{(x, y) \; : \; x^2 + y^2 < 1\} \subset \mathbb{RP}^2$ be the Minkowski model for the hyperbolic plane. Give a formula for the hyperbolic metric (as a Riemannian metric) on B in these coordinates.

42. Relate the hyperbolic distance function $d(p, q)$ on B to the cross-ratio of the ordered points (p', p, q, q'), where p', q' are the points where the line through p and q meets $S^1 = \partial B$.

43. Give an explicit formula for the natural covering map / group homomorphism $f : \mathrm{SU}(2) \to \mathrm{SO}(3)$.

44. There is no canonical conformal metric on the Riemann sphere $\widehat{\mathbb{C}}$, since its group of automorphisms $\mathrm{SL}_2(\mathbb{C})$ is noncompact. However the $(2, 3, 5)$ orbifold, as a Riemann surface, carries a *unique* conformal metric of curvature $+1$. How is this possible? What happens with the (n, n) orbifold, $n > 1$?

45. Prove that in the Minkowski model, if $\langle p, p\rangle = -1$ and $\langle q, q\rangle = 1$, then $\langle p, q\rangle = \pm \sinh d(p, \gamma_q)$, where γ_p is the oriented geodesic determined by q. Explain how the sign is related to the orientation of γ_q and to the choice of one of the two sheets of the hyperboloid defined by $\langle p, p\rangle = -1$.

46. Characterize horocycles in the Klein model for $\mathbb{H}^2$.

47. Give formulas for distances in S^2 and $\mathbb{H}^2$ in terms of the Hermitian inner products on $\mathbb{C}^2$ of signatures $(2, 0)$ and $(1, 1)$.

48. Draw some pictures with `lim`, available at

    `math.harvard.edu/~ctm/programs`.

49. What is the image in $\widehat{\mathbb{C}}$ of the circle $C_p = p^\perp \subset S^2$, $p = (x, y, z)$, under stereographic projection?

50. Construct and draw, explicitly, a finite collection of circles in $\widehat{\mathbb{C}}$ that is invariant under a copy of A_5 in $\mathrm{Aut}(\widehat{\mathbb{C}})$.

51. The area of a lune of angle θ (the region between two great circles on the unit sphere S^2) is clearly 2θ. Use this fact to prove the Gauss-Bonnet theorem for a spherical triangle T: the area of T coincides with its excess angle (the sum of its interior angles, minus π).

52. Let S be an affine 2-dimensional subspace of $\mathbb{PR}^{2,1}$, and let $L_S = \mathcal{H} \cap S$, thought of as a subset of $\mathcal{H} \cong \mathbb{H}$. Show that S is either the empty set, a point, a hyperbolic circle, a horocycle, a geodesic, or a parallel of a geodesic. In particular, S has constant curvature.

53. Show that every conic tangent to $S^1 = \partial B^2 \subset \mathbb{RP}^2$, and meeting B^2, arises as L_S for some S.

54. Let S be an affine 2-dimensional subspace of $\mathbb{PR}^{2,1}$, and let $G_S = \mathcal{G} \cap S$, thought of as a family of geodesics in $\mathbb{H}$. Describe the families G_S that arise in this way, geometrically. (For example, G_S might be all the geodesics through a given point in $\mathbb{H}$.)

55. Let $[a,b] \cup [b,c]$ be the union of two geodesic segments in $\mathbb{H}^2$, each of length L, with bending angle $\pi > \beta > 0$ at b. (We have $\beta = 0$ if the segments lie on a straight line.) Let γ_a be the geodesic orthogonal to $[a,b]$ at a, and similarly for γ_c. Determine the greatest angle $B(L)$ such that $\gamma_a \cap \gamma_c = \emptyset$ for all $\beta \leq B(L)$.

    Now let $\gamma = \bigcup_{i=-\infty}^{i=\infty} [a_i, a_{i+1}]$ be an infinite broken geodesic, comprised of segments of length L with all bending angles less than $B' < B(L)$. Show that γ is a quasigeodesic; more precisely, show that γ is a $K(B', L)$–quasigeodesic, for an explicit function $K(B', L)$.

56. What is the length of the shortest closed geodesic(s) on the triply–punctured sphere, $X = \mathbb{H}/\Gamma(2)$? Draw a picture of the homotopy class of this geodesic.

57. Prove that triangles T in $\mathbb{H}^2$ are thin: there exists an $R > 0$ such that if S_1, S_2, S_3 are the edges of T, then $B(S_1 \cup S_2, R) \supset S_3$. What is best value of R when T is an ideal triangle? Show that this value of R works for all triangles.

58. Let $L \subset \mathbb{R}^n$ be a unimodular lattice. A *greedy basis* is obtained by first picking a shortest vector $v_1 \in L$, then a shortest vector v_2 linearly independent from v_1, up to v_n.

    (i) For what values of n can we insure that (v_i) forms a basis for L over $\mathbb{Z}$?

(ii) Show there is a function $C_n(r)$ such that $|v_n| \leq C_n(|v_1|)$.

59. Let X be a compact hyperbolic surface of genus g. Let $\gamma_1, \ldots, \gamma_{3g-3}$ be a maximal set of disjoint simple closed geodesic, chosen by the greedy algorithm: γ_1 is the shortest simple closed geodesic, and γ_i is the shortest among those disjoint from $\gamma_1, \ldots, \gamma_{i-1}$.

    (i) Given an explicit upper bound $L_{g,1}$ on the length $L(\gamma_1)$.

    (ii) Give an explicit upper bound $L_{g,2}$ on $L(\gamma_2)$.

    *(iii) Given an explicit upper bound $L_{g,3g-3}$ for $L(\gamma_{3g-3})$.

    *(iv) Show that we can find X such that $L(\gamma_{3g-3})$ is on the order of $\sqrt{g}$ (or more).

    (See Buser's book.)

60. Let $\Gamma \subset \mathrm{SO}(n,1)$ be a discrete group. Its *limit set* is defined by
$$\Lambda = \overline{\Gamma p} \cap S^{n-1}_\infty,$$
    for any $p \in \mathbb{H}^n$.

    (i) Show that the definition of Λ does not depend on p, and that Λ is invariant under Γ.

    (ii) Show that $|\Lambda| \leq 2$ iff Γ is elementary (meaning Γ contains an abelian subgroup with finite index).

    (iii) Suppose from now on the $|\Lambda| > 2$. Show that the action of Γ on Λ is minimal (every orbit is dense).

    (iv) Show that Λ is perfect (it has no isolated points).

    (v) Show that Λ is the smallest closed, nonempty, Γ-invariant subset of the sphere.

    (vi) Show that fixed points of elements of Γ are dense in Λ.

    (vii) Show that the action of Γ on $\Omega = S^{n-1}_\infty - \Lambda$ is properly discontinuous; more precisely, show that Ω/Γ is an orbifold of dimension $(n-1)$.

61. Let $f : X \to Y$ be a homeomorphism between a pair of compact hyperbolic surfaces, let $\widetilde{f} : \Delta \to \Delta$ be its lift to the universal cover, and let $F : S^1 \to S^1$ be the continuous extension of $\widetilde{f}$ to the circle.

    (i) Show that F is Hölder continuous.

(ii) Show that the Hölder exponent of F controls the ratios of lengths of corresponding geodesics on X and Y.

(iii) Show that if F is Lipschitz, then all lengths agree and therefore F is a Möbius transformation.

62. Let $\Gamma, \Gamma' \subset \mathrm{PSL}_2(\mathbb{R})$ be Fuchsian groups, and suppose $f : S^1_\infty \to S^1_\infty$ is a homeomorphism conjugating Γ to Γ'. Prove that f is differentiable at every parabolic fixed point of Γ.

63. Let $G = \mathrm{PSL}_2(\mathbb{R})$, and suppose $g_n \to \mathrm{id}$ in $G - N$. Show that for any $a \in A$, there exist $u_n, u'_n \in N$ such that $u'_n g_n u_n \to a$.

64. Let $f : X_1 \to X_2$ be a homeomorphism between a pair of compact hyperbolic surfaces $X_i = \mathbb{H}/\Gamma_i$, $i = 1, 2$. Let $F : S^1_\infty \to S^1_\infty$ be the boundary values of a lift of f to the universal covers of its domain and range. Show that F is *quasi-symmetric*. This means there exists a $k > 1$ such that for any pair of adjacent arcs on the circle of the same length, $I = [a, b]$ and $I' = [b, c]$, the length ratio of their images satisfies
$$\frac{1}{k} \le \frac{|f(I)|}{|f(I')|} \le k.$$

65. Let $f : \mathbb{R} \to \mathbb{H}$ be a path parameterized by arclength, satisfying
$$d(f(s), f(t)) \ge D(|s - t|),$$
where $D(r) \to \infty$ as $r \to \infty$. Suppose the image of f lies within a bounded distance of a geodesic γ. Prove that f is a quasigeodesic, in the sense that there is a $k > 1$ such that
$$d(f(s), f(t)) \ge |s - t|/k - 1$$
for all $s, t \in \mathbb{R}$.

66. Consider f as above such that $D(r) = r^\alpha - C$, $0 < \alpha < 1$. Prove that the image of f lies within a bounded distance of a geodesic (and hence f is a quasigeodesic).

67. Show there is an f as above with $D(r) \approx \log(r)$ (for larger r) such that the image of f does *not* lie within a bounded distance of a geodesic.

68. Explain the fallacy in the following 'proof' that the horocycle flow for a compact hyperbolic surface X is uniquely ergodic: (i) by mixing of

the geodesic flow, any large circle $S^1(x,r)$ in nearly equidistributed in $\mathrm{T}_1 X$; and (ii) horocycles are limits of spheres as $r \to \infty$, so they too are uniformly distributed. (Hint: (i) is true even when X has finite volume, but in that case the horocycle flow is *not* uniquely ergodic.)

69. Using Ratner's theorem, find all the possibilities for $\overline{Hx}$ in $\Gamma\backslash G$, where

$$\Gamma_0 = \mathrm{SL}_2(\mathbb{Z}) \subset G_0 = \mathrm{PSL}_2(\mathbb{R}),$$

$G = G_0 \times G_0$, $\Gamma = \Gamma_0 \times \Gamma_0$, and H is the image of G_0 in $G_0 \times G_0$ under the diagonal embedding.

Interpret your answer in terms of pairs of unimodular lattices or elliptic curves.

70. Let Γ be a countable discrete group acting by homeomorphisms on a complete metric space X. Suppose the orbit Γp is closed. Prove that Γp is discrete.

71. Let d and ρ denote the hyperbolic distance and metric on the unit disk Δ. Let $\rho'(z)|dz| = \rho(z)/d(0,z)^2|dz|$.

(i) Show that the metric completion of (Δ, ρ') is naturally homeomorphic to $\Delta \cup S^1$.

(ii) Show that the Hausdorff dimension of the metric completion is infinite.

72. Let $N(X,L)$ denote the number of primitive, oriented closed geodesics on a compact hyperbolic surface $X = \mathbb{H}/\Gamma$ with length $\leq L$. Show that $\lim_{L\to\infty}(\log N(L))/L = 1$.

73. Let $Y \to X$ be a degree two covering of a hyperbolic surface X. Then by the prime number theorem in hyperbolic geometry, $N(Y,L)$ and $N(X,L)$ are both asymptotic to $L/\log(L)$. But only 'half' of the closed geodesics on on X lift to Y. How are these two assertions compatible? Can you explain the case of a general finite covering $Y \to X$?

74. Let $X = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. To show the horocycle flow on $T_1 X$ is minimal, following Hedlund, prove the following assertions. We say a horocycle $H \subset \mathbb{H}$ is *transitive* if ΓH is dense in the space of all horocycles; equivalently, if H gives a dense horocycle orbit in $T_1 X$.

(i) There exists a transitive horocycle. (Use Baire category).

(ii) If horocycles H_1 and H_2 rest on the same point $Q \in S^1_\infty$, and H_1 is transitive, then so is H_2.

(iii) If H rests on a fixed point Q of a hyperbolic element in Γ, then H is transitive. (Consider the closure of ΓH and use (i) and (ii).

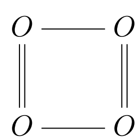
(iv) Suppose ΓH contains a sequence of horocycles $H_1, H_2, \ldots$ that tend to infinity, in the sense that they eventually enclose every compact subset of $\mathbb{H}$. Then H is transitive. (Show that $\overline{\Gamma H}$ contains a horocycle of the type considered in (iii).)

(v) Let F be a compact fundamental domain for Γ. Since X is compact, for any horocycle H and $R > 0$, there exists a $\gamma \in \Gamma$ such that H encloses γF and $d(\gamma F, H) > R$. Using this, show every horocycle is of the type considered in (iv).

75. Characterize, in terms of their dihedral angles, when three planes A, B, C in $\mathbb{H}^3$ have a common perpendicular plane D.

76. Let A, B, C be 3 planes in hyperbolic 3-space, passing through a single point p. Then the intersections AC and BC determine 2 lines in the plane C, meeting at p with angle θ. Give a formula for $\cos\theta$ in terms of the dihedral angles between the three planes A, B and C.

77. Using the solution to the preceding problem, show that the cosines of the interior angles of one of the triangular faces of the arithmetic hyperbolic tetrahedron T with Coxeter diagram:



are given by $\sqrt{1/2}$, $\sqrt{1/3}$ and $\sqrt{2/3}$. In particular, two of the angles are not rational multiples of π. How can this be consistent with the fact that this tetrahedron tiles $\mathbb{H}^3$?

References

[GL] P. M. Gruber and C. G. Lekkerkerker. *Geometry of Numbers*. Elsevier, 1987.