

6 Hyperbolic 3-manifolds

6.1 Kleinian groups and hyperbolic manifolds

A *hyperbolic manifold* M^n is a connected, complete Riemannian manifold of constant sectional curvature -1 .

There is a unique simply-connected hyperbolic manifold \mathbb{H}^n of dimension n , up to isometry. Thus any hyperbolic manifold can be regarded as a quotient $M^n = \mathbb{H}^n/\Gamma$ where $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ is a discrete group.

Two explicit models for hyperbolic space are the *upper half-space model*,

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

with the metric $\rho = |dx|/x_n$; and the *Poincaré ball model*

$$\mathbb{H}^n = \mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

with $\rho = 2|dx|/(1 - |x|^2)$. Hyperbolic space has a natural *sphere at infinity* S_∞^{n-1} , corresponding to $\mathbb{R}^{n-1} \cup \{\infty\}$ in the upper half-space model and to $\partial\mathbb{B}^n$ in the Poincaré ball model.

The points on S_∞^{n-1} can be naturally interpreted as endpoints of geodesics.

Theorem 6.1 *For $n > 1$, every hyperbolic isometry extends continuously to a conformal automorphism of the sphere at infinity, establishing an isomorphism*

$$\text{Isom}(\mathbb{H}^n) \cong \text{Aut}(S_\infty^{n-1}).$$

Proof. First note that reflection through a hyperplane $P^{n-1} \subset \mathbb{H}^n$ extends continuously to (conformal) reflection through the sphere $S_\infty^{n-1} = \partial P^{n-1} \subset S_\infty^{n-1}$. Conversely, reflection through a sphere extends to reflection through a hyperplane in hyperbolic space. Since reflections generate both groups, we see the boundary values of any hyperbolic isometry are conformal, and any conformal map extends to an isometry.

Finally any isometry inducing the identity on S_∞^{n-1} must be the identity, since it stabilizes every geodesic. ■

A *Kleinian group* is a discrete subgroup $\Gamma \subset \text{Isom}(\mathbb{H}^n)$. The *limit set* $\Lambda \subset S_\infty^{n-1}$ of Γ is defined by $\Lambda = \overline{\Gamma x} \cap S_\infty^{n-1}$ for any $x \in \mathbb{H}^n$. The complement $\Omega = S_\infty^{n-1} - \Lambda$ is the *domain of discontinuity* for Γ .

Theorem 6.2 *The action of a Kleinian group on its domain of discontinuity is properly discontinuous. That is, for any compact set $K \subset \Omega$, the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite.*

A Kleinian group is *elementary* if it contains an abelian subgroup of finite index; equivalently, if $|\Lambda| \leq 2$.

Theorem 6.3 *If Γ is nonelementary, then Λ is the smallest nonempty closed Γ -invariant subset of S_∞^{n-1} .*

A *baseframe* ω for a hyperbolic manifold M is simply a point in the frame bundle of M . There is a natural bijection:

$$\begin{aligned} & \{\text{Baseframed hyperbolic manifolds } (M^n, \omega)\} \\ & \quad \leftrightarrow \\ & \{\text{Torsion-free Kleinian groups } \Gamma \subset \text{Isom}(\mathbb{H}^n)\}. \end{aligned}$$

General Kleinian groups correspond to hyperbolic orbifolds. Forgetting the baseframe amounts to only knowing Γ up to conjugacy in $\text{Isom}(\mathbb{H}^n)$.

6.2 Ergodicity of the geodesic flow

Theorem 6.4 *The geodesic flow on $M = \mathbb{H}^n/\Gamma$ is ergodic if and only if Γ acts ergodically on $S_\infty^{n-1} \times S_\infty^{n-1}$.*

Theorem 6.5 *The geodesic flow is ergodic on any finite-volume hyperbolic manifold M .*

Proof. (Hopf) Let $f \in C_0(T_1M)$ be a compactly supported continuous function on the unit tangent bundle. Let g_t denote the geodesic flow, and $I \subset L^2(T_1M)$ the subspace of functions invariant under g_t . To prove ergodicity we need to show I consists of the constant functions.

By the ergodic theorem,

$$f_+(v) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(g_t(v)) dt$$

exists for almost every $v \in T_1M$, and converges to the L^2 -projection F of f to I . On the other hand, the negative time average $f_-(v)$ converges to the same thing, so $F(v) = f_+(v) = f_-(v)$ for almost every v .

Now if v and w are vectors converging to the same point on S_∞^{n-1} in positive time, then the geodesic rays through v and w are asymptotic, so $f_+(v) = f_+(w)$ by uniform continuity of f . In other words, $F(v)$ is constant along the $(n -$

1)-spheres of the positive horocycle foliation of $T_1(M)$. Applying the same argument to $f_-(v)$, we see F is also constant along the negative horocycle foliation. Finally $F(v)$ is invariant under the geodesic flow. By Fubini's theorem, we conclude that $F(v)$ is constant.

Since $C_0(T_1M)$ is dense in $L^2(T_1M)$, we have shown I consists only of the constant functions, and thus the geodesic flow is ergodic. ■

6.3 Quasi-isometry

Let X and Y be complete metric spaces. A map $f : X \rightarrow Y$ is a K -quasi-isometry if for some $R > 0$ we have

$$R + Kd(x, x') \geq d(f(x), f(x')) \geq \frac{d(x, x')}{K} - R$$

for all $x, x' \in X$. In other words, on a large scale, f gives a bi-Lipschitz map to its image.

We say $f : X \rightarrow X$ is *close to the identity* if $\sup_X d(x, f(x)) < \infty$.

We say $f : X \rightarrow Y$ is a quasi-isometric *isomorphism* if there is a quasi-isometry $g : Y \rightarrow X$ such that $f \circ g$ and $g \circ f$ are close to the identity. If $f : X \rightarrow Y$ is a quasi-isometry and $B(f(X), R) = Y$ for some R , then in fact f is an isomorphism.

Example. The inclusion $f : \mathbb{Z}^n \rightarrow \mathbb{R}^n$ is a quasi-isometric isomorphism. An inverse is the map $g : \mathbb{R} \rightarrow \mathbb{Z}$ defined by taking the integer part, $g(x) = [x]$.

Groups. Let G be a finitely-generated group. Choosing a finite set of generators $\langle g_i \rangle$, we can construct the *Cayley graph* $C(G)$ by taking G as the vertices and connecting g and h by an edge if $g = g_i h$ for some g_i . Taking the edges to be of unit length, we obtain a metric d on G .

Alternatively, one can define $d(\text{id}, g) = n$ where n is the length of a minimal word expressing g in terms of $\langle g_i^{\pm 1} \rangle$, and then extend d to $G \times G$ so it is right-invariant.

Another choice of generators $\langle g'_i \rangle$ determines another metric d' on G . By expressing each g_i as a word in $\langle g'_i \rangle$ and vice-versa, one easily sees that (G, d) and (G, d') are quasi-isometric.

Theorem 6.6 *For any compact Riemannian manifold M , the universal cover \widetilde{M} and the group $\pi_1(M)$ are quasi-isometric.*

Proof. Realize $\pi_1(M, *) = G$ as a group of deck transformations acting on $(\widetilde{M}, *)$, and define $f : G \rightarrow \widetilde{M}$ by $f(g) = g*$. The map f can be extended to the Cayley graph by mapping each edge to a geodesic segment. Edges corresponding to the same generator of G map to segments of the same length, so f is Lipschitz.

To see f is a quasi-isometry, note that by compactness of M there is an $R > 0$ such that every point of the universal cover is within distance R of the orbit $G*$. Consider a geodesic segment $\gamma \subset \widetilde{M}$ of length $L = d(*, g*)$ joining $*$ to $g*$. Cut

γ into L segments of about unit length, and assign to each a point h_i^* within distance R . Then $d(h_i^*, h_{i+1}^*) \leq 2R + 1$. Now by discreteness of G^* , there are only finitely many h such that $d(*, h^*) \leq 2R + 1$; letting $C = \max d(\text{id}, g)$ over such g , we have $d(h_i, h_{i+1}) \leq C$, and thus $d(\text{id}, g) \leq CL = Cd(*, g^*)$. Thus f is a quasi-isometry. ■

Corollary 6.7 (Milnor, Švarc) *If M is a closed manifold with a metric of negative curvature, then $\pi_1(M)$ has exponential growth.*

Quasi-geodesics. A *quasi-geodesic* in a metric space X is a quasi-isometric map

$$\gamma : [a, b] \rightarrow X.$$

Often we will take $[a, b] = \mathbb{R}$.

Examples.

1. If $\gamma : [a, b] \rightarrow X$ is a geodesic and $f : X \rightarrow Y$ is a quasi-isometry, then $f \circ \gamma$ is a quasi-geodesic.
2. Let $\gamma : \mathbb{R} \rightarrow C(G)$ be a geodesic in the Cayley graph of a group $G = \pi_1(M)$, and let

$$f : (C(G), \text{id}) \rightarrow (\widetilde{M}, *)$$

be defined by $f(g) = g^*$ and by sending edges to geodesic segments. Then $f \circ \gamma$ is a quasi-geodesic.

3. Efficient taxis take quasi-geodesics along the grid of ‘streets’ connecting $\mathbb{Z}^2 \subset \mathbb{R}^2$. These *Manhattan geodesics* are far from unique; e.g. there are $\binom{2n}{n}$ geodesics from $(0, 0)$ to (n, n) . This example comes from the universal cover of a 2-torus.
4. If $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ is a C^2 curve parameterized by arclength, with geodesic curvature $k(s) < k_0 < 1$ at every point, then $\gamma(s)$ is a quasigeodesic. Indeed, the normal planes $P(s)$ through $\gamma(s)$ advance at a uniform pace C , so

$$d(\gamma(s), \gamma(t)) \geq d(P(s), P(t)) \geq C(k_0)|s - t|.$$

The borderline case is a horocycle, which is *not* a quasi-geodesic (in fact $d(\gamma(s), \gamma(0))$ grows like $\log s$). On the other hand, a curve at constant distance D from a geodesic in \mathbb{H}^2 has curvature $k_0(D) < 1$ (and is obviously a quasi-geodesic).

5. The loxodromic spiral $\gamma : [0, \infty) \rightarrow \mathbb{C}$ given by $\gamma(s) = s^{1+i}$ is a quasi-geodesic ray with no definite direction; that is, $\arg \gamma(s) = \log(s)$ moves around the circle an infinite number of times as $s \rightarrow \infty$.

This γ is not within a bounded distance of any Euclidean geodesic.

Lemma 6.8 For any closed convex set $K \subset \mathbb{H}^n$, nearest-point projection $\pi : \mathbb{H}^n \rightarrow K$ contracts by a factor of at least $\cosh(r)$ at distance r from K .

Proof. First calculate $\|d\pi(z)\|$ in the case where K is a geodesic in \mathbb{H}^2 . We can normalize coordinates so K is the imaginary axis and z lies on the unit circle; then $\pi(z) = i$. As a hyperbolic geodesic, the arclength parameterization of the unit circle is given by $(\tanh s, \operatorname{sech} s)$, so $\operatorname{Im} z = \operatorname{sech} r$ where $r = d(z, K)$. Now the geodesics normal to K are Euclidean circles, so $\|d\pi\| = 1$ in the Euclidean metric. Thus the hyperbolic contraction is by a factor of $\operatorname{Im}(\pi(z))/\operatorname{Im}(z) = \cosh r$.

The general case reduces to this one by considering a supporting hyperplane to K . ■

Theorem 6.9 Let $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ be a quasi-geodesic. Then γ is within a bounded distance of a unique hyperbolic geodesic.

Proof. For convenience, assume γ is continuous. For $T \gg 0$, let δ_T be the complete geodesic passing through $\gamma(-T)$ and $\gamma(T)$. Consider the cylinder $B(\delta_T, r)$ of radius r about δ_T , and suppose (a, b) is a maximal interval for which $\gamma(a, b)$ is outside the cylinder. Then $\gamma(a), \gamma(b)$ lie on the boundary of the cylinder, and the nearest point projection of $\gamma(a, b)$ to δ_T is contracting by a factor of $\cosh(r)$. Thus, if $|a - b| > R$, we have

$$\frac{|a - b|}{K} \leq d(\gamma(a), \gamma(b)) \leq 2r + \frac{d(\gamma(a), \gamma(b))}{\cosh(r)} \leq 2r + \frac{K|a - b|}{\cosh(r)}.$$

If we take r large enough that $\cosh(r) \gg K^2$, then the inequality above implies $|a - b|$ is not too large, and hence $\gamma[a, b]$ stays close to δ_T .

In other words, there is an absolute constant D such that $\gamma(-T, T) \subset B(\delta_T, D)$. Since the space of geodesics within distance D of $\gamma(0)$ is compact, we can pass to a convergent subsequence and obtain a geodesic δ with $\gamma \subset B(\delta, D)$.

In hyperbolic space, distinct geodesics diverge, so δ is unique. ■

6.4 Quasiconformal maps

We now revisit the notion of a quasiconformal map from a geometric point of view.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a homeomorphism (often required to preserve orientation). For each sphere $S = S(x, r)$, let

$$K(S) = \frac{\max\{d(f(y), f(x)) : y \in S\}}{\min\{d(f(y), f(x)) : y \in S\}}.$$

We say f is *geometrically quasiconformal* if $\sup K(S)$ is finite. A homeomorphism $f : S^n \rightarrow S^n$ is geometrically quasiconformal if $\sup K(S)$ is bounded in the spherical metric.

Theorem 6.10 *Let f be a quasiconformal homeomorphism of \mathbb{R}^n or S^n with $n \geq 2$. Then:*

- *f is analytically quasiconformal (f has derivatives in L^2 satisfying $\|Df\|^n \leq K|\det Df|$ almost everywhere);*
- *f is absolutely continuous ($f(E)$ has measure zero iff E has measure zero);*
- *f is differentiable almost everywhere; and*
- *if Df is conformal a.e., then f is a Möbius transformation.*

For proofs, see [LV]. To convey the spirit of the passage between the geometric and analytic definitions, we will prove:

Theorem 6.11 *A quasiconformal map $f : \mathbb{C} \rightarrow \mathbb{C}$ is absolutely continuous on lines (ACL).*

This means that for any line $L \subset \mathbb{C}$, the real and imaginary parts of f are absolutely continuous functions on $L + t$ for almost every $t \in \mathbb{C}$.

Proof. By making a linear change of coordinates in the domain of f , it suffices to show that $\operatorname{Re} f(x + iy)$ is an absolutely continuous function of $x \in [0, 1]$ for almost every y .

Consider the function $A(y) = \operatorname{area} f([0, 1] \times [0, y])$. Since $A(y)$ is monotone increasing, it has a finite derivative a.e. Choose y such that $A'(y)$ exists; we will show $F(x) = \operatorname{Re} f(x + iy)$ is absolutely continuous for $x \in [0, 1]$.

Consider a collection of disjoint intervals I_1, \dots, I_n in $[0, 1]$, with $\sum |I_i| < \epsilon$. We must show that $\sum |F(I_i)| < \delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

By subdividing the intervals, we can assume $|I_i| = h \ll \epsilon$ for all i , so $nh = \epsilon$. Let S_i be the $h \times h$ square resting on I_i . Since f is quasiconformal, we have $\operatorname{area}(f(S_i)) \asymp |J_i|^2$. Thus we have:

$$\begin{aligned} \left(\sum |J_i|\right)^2 &\leq n \sum |J_i|^2 \asymp n \sum \operatorname{area}(f(S_i)) \\ &\leq n \operatorname{area}(f([0, 1] \times [y, y + h])) \approx nhA'(y). \end{aligned}$$

Therefore $\sum |J_i| = O(\sqrt{nh}) = O(\epsilon^{1/2})$. ■

Measuring quasiconformality. The natural measure of distortion for a homeomorphism $f : \mathbb{C} \rightarrow \mathbb{C}$ is the *dilatation* $K(f)$.

If f is \mathbb{R} -linear, it maps circles to ellipses with major and minor axes M and m , and we define $K(f) = M/m$. For a general quasiconformal map, we define $K(f)$ to be the least constant such that $K(Df) \leq K(f)$ almost everywhere.

Perhaps surprisingly, $K(f)$ is not the same as $\sup K(S)$ over all spheres. For example, if f is given in polar coordinates by $f(r, \theta) = (r^\alpha, \theta)$, with $\alpha > 1$, then $K(f) = \alpha$ even though $K(S) = 2^\alpha$ for the sphere $S = \{z : |z - 1| = 1\}$.

The proper geometric definition of the dilatation is that $K(f)$ is the least constant such that $\limsup_{r \rightarrow 0} K(S(x, r)) \leq K(f)$ almost everywhere.

The 1-dimensional case. The geometric definition of quasiconformality makes sense even in dimension one, and yields the useful family of k -*quasisymmetric* homeomorphisms $f : \mathbb{R} \rightarrow \mathbb{R}$, satisfying $\sup K(S) \leq k$ for *every* sphere S . Of course a quasisymmetric map f is differentiable a.e. (since it is a monotone function), but f need *not* be absolutely continuous.

6.5 Quasi-isometries become quasiconformal at infinity

Let $f : \mathbb{H}^{n+1} \rightarrow \mathbb{H}^{n+1}$ be a quasi-isometry.

The *extension* $F : S_\infty^n \rightarrow S_\infty^n$ of f is defined as follows. Given $x \in S_\infty^n$, take a geodesic ray γ landing at x , and let δ be a geodesic ray that shadows the quasi-geodesic $f \circ \gamma$; then $F(x)$ is the endpoint of δ .

It is easy to see:

If f is close to the identity, then F is the identity.

Thus if f is an isomorphism with quasi-inverse g , the extensions of f and g satisfy $F \circ G = G \circ F = \text{id}$, so F is bijective. In fact we have:

Theorem 6.12 *The extension $F : S_\infty^n \rightarrow S_\infty^n$ of a quasi-isometry $f : \mathbb{H}^{n+1} \rightarrow \mathbb{H}^{n+1}$ is a homeomorphism.*

Proof. It suffices to show F is injective and continuous. Let us work in the Poincaré unit ball model, with $\mathbb{H}^{n+1} = \mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$. Composing with an isometry, we can assume $f(0) = 0$.

Consider $x, y \in S_\infty^n$ with $|x - y| = \epsilon > 0$. Let γ be the geodesic joining x and y , and let δ be the geodesic shadowing $f(\gamma)$. Then the endpoints are δ and $F(x)$ and $F(y)$. Since the endpoints are distinct, F is injective.

Moreover, $d(0, \gamma) = |\log \epsilon| + O(1)$ in the hyperbolic metric, so we have $d(0, \delta) \geq |\log \epsilon|/K + O(1)$. Therefore $|F(x) - F(y)| = O(\epsilon^{1/K})$, showing F is even Hölder continuous. ■

Corollary 6.13 *Any quasi-isometry $f : \mathbb{H}^{n+1} \rightarrow \mathbb{H}^{n+1}$ is an isomorphism.*

Proof. Let F be the extension of f . Given $z \in \mathbb{H}^{n+1}$, choose a geodesic δ through z with endpoints $F(x), F(y)$, and let γ be the geodesic from x to y . Then $f(\gamma)$ comes within a bounded distance of z , so f is essentially surjective and therefore it admits a quasi-inverse. ■

Theorem 6.14 *The extension of a quasi-isometry f on \mathbb{H}^{n+1} is a quasiconformal homeomorphism F on S_∞^n .*

Proof. To see F is quasi-conformal, we will show $F(S(a, r))$ has a bounded ratio of inradius to outradius for any small sphere $S(a, r) \subset S_\infty^n$. For convenience we normalize so $a = 0$ and F fixes 0 and ∞ . Let $|b| = |c| = r$ maximize the ratio

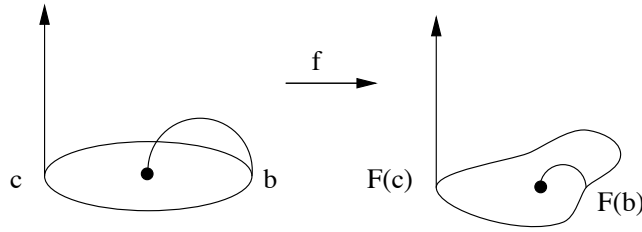


Figure 16. Geodesics, inradius and outradius.

$K = |F(c)/F(b)|$. Then the geodesics $[0, b]$ and $[c, \infty]$ are at distance $O(1)$, while the geodesics $[0, F(b)]$ and $[F(c), \infty]$ are at distance about $\log K$ (see Figure 16). Since f is a quasi-isometry, $f([0, b])$ and $f([\infty, c])$ lie at a bounded distance from $[0, F(b)]$ and $[\infty, F(c)]$, and from each other. Thus $\log K = O(1)$ and F is quasiconformal. ■

6.6 Mostow rigidity

In this section we prove that a compact hyperbolic manifold of dimension 3 or more can be reconstructed from its fundamental group.

Lemma 6.15 *Let $f : M \rightarrow N$ be a homotopy equivalence between compact Riemannian manifolds. Then the lift*

$$\tilde{f} : \tilde{M} \rightarrow \tilde{N}$$

of f gives a quasi-isometric isomorphism between the universal covers of M and N .

Proof. Let $g : N \rightarrow M$ be a homotopy inverse to f , and let $\tilde{g} : \tilde{N} \rightarrow \tilde{M}$ be a lift of g compatible with \tilde{f} . Then the homotopy h_t of $g \circ f$ to the identity lifts to a homotopy \tilde{h}_t of $\tilde{g} \circ \tilde{f}$ to the identity.

We can assume that f and g are smooth, so by compactness of M and N their lifts are K -Lipschitz for some K . Similarly, since \tilde{h}_t is a lift of a homotopy on M , there is a $D > 0$ such that

$$d(x, \tilde{g} \circ \tilde{f}(x)) \leq \text{diam } \tilde{h}_{0,1}(x) \leq D \quad (6.1)$$

for all $x \in \tilde{M}$. Therefore $\tilde{g} \circ \tilde{f}$ is close to the identity. Similarly, $\tilde{f} \circ \tilde{g}$ is close to the identity. It follows that \tilde{f} is a quasi-isometry. Indeed, we have the upper bound

$$d(\tilde{f}(x), \tilde{f}(y)) \leq Kd(x, y)$$

since \tilde{f} is Lipschitz, and the lower bound

$$d(x, y) \leq d(\tilde{g} \circ \tilde{f}(x), \tilde{g} \circ \tilde{f}(y)) - D \leq Kd(\tilde{f}(x), \tilde{f}(y)) - D$$

by (6.1). Similarly, \tilde{g} is a quasi-inverse for \tilde{f} . ■

Remark. One can also argue that \widetilde{M} and \widetilde{N} are quasi-isometry to $\pi_1(M)$ and $\pi_1(N)$ in such a way that \widetilde{f} is close to the quasi-isometric isomorphism $f_* : \pi_1(M) \rightarrow \pi_1(N)$.

Theorem 6.16 (Mostow) *Let M^n and N^n be compact hyperbolic n -manifolds, $n \geq 3$, and let*

$$\iota : \pi_1(M^n) \rightarrow \pi_1(N^n)$$

be an isomorphism. Then there is an isometry $I : M^n \rightarrow N^n$ such that $\iota = I_$.*

Proof. The manifolds M and N are $K(\pi, 1)$'s, so the isomorphism ι between their fundamental groups can be realized by a homotopy equivalence $f : M \rightarrow N$. By the preceding lemma, the lift $\widetilde{f} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a quasi-isometry, so its extension is a quasiconformal map $F : S_\infty^{n-1} \rightarrow S_\infty^{n-1}$ conjugating the action of $\pi_1(M)$ to that of $\pi_1(N)$. The map F is differentiable almost everywhere, by fundamental results on quasiconformal mappings.

If DF is conformal almost everywhere then, since $n > 2$, F is a Möbius transformation. (This step fails when $n = 2$). Then F extends to an isometry $\widetilde{I} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ which descends to the desired isometry $I : M \rightarrow N$.

Otherwise, DF fails to be conformal on a set of positive measure in S_∞^{n-1} . By ergodicity, DF is nonconformal almost everywhere.

Now for concreteness suppose $n = 3$. Then the conformal distortion of $DF(x)$ defines an ellipse in the tangent space $T_x S_\infty^2$ for almost every x . Let $L_x \subset T_x S_\infty^2$ be the line through the major axis of this ellipse.

Define $\theta : S_\infty^2 \times S_\infty^2 \rightarrow S^1$ as follows: given x, y on the sphere, use parallel transport along the geodesic joining x to y to identify $T_x S_\infty^2$ with $T_y S_\infty^2$, and let θ be the angle between the lines L_x and L_y .

Then θ is invariant under the action of $\pi_1(M)$, so by ergodicity of the geodesic flow it is a constant a.e. This means that if we choose coordinates on S_∞^2 so $x = \infty$, then the lines L_y have constant slope for $y \in \mathbb{R}_\infty^2$. But almost any point can play the role of x , while it is clearly impossible to arrange the linefield L_y to have constant slope in more than one affine chart on the sphere.

The proof for $n > 2$ is similar. ■

Note. Mostow rigidity also holds for finite volume manifolds.

6.7 Rigidity in dimension two

Here is a version of Mostow rigidity that works for hyperbolic surfaces.

Theorem 6.17 *Let $f : S_\infty^1 \rightarrow S_\infty^1$ be an orientation-preserving homeomorphism conjugating Γ to Γ' , where $X = \mathbb{H}/\Gamma$ and $X' = \mathbb{H}/\Gamma'$ are finite-volume hyperbolic surfaces. Then f is either singular or absolutely continuous. In the absolutely continuous case, f must be a Möbius transformation.*

Proof. For convenience we treat the case where X and X' are compact. By hypothesis, there is an isomorphism $\iota : \Gamma \rightarrow \Gamma'$ induced by f . We first observe that if $\Gamma = \Gamma'$ and ι is the identity, then f is the identity. This is because f must fix the attracting fixed-point of every $g \in \Gamma$, and such fixed points are dense on S_∞^1 .

By ergodicity of the action of Γ on the circle, f is either absolutely continuous or singular. We will show that in the former case, Γ is conjugate to Γ' inside $G = \text{Isom } \mathbb{H}$; in other words, $X \cong X'$. To this end, we identify S_∞^1 with $\widehat{\mathbb{R}}$ and choose coordinates so that $f(\infty) = \infty$. Then $f : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism; in particular, f' exists a.e.

Our hypothesis of absolute continuity implies that $f = \int f'$; in particular, $f'(x) \neq 0$ for some x . We can assume $x = 0$ and $f'(0) = 1$. Let $A_n(x) = nx$. Then $f_n = A_n f A_n^{-1}$ converges to the identity map $f_\infty(x) = x$ uniformly on compact sets. Moreover f_n intertwines the actions of certain conjugates Γ_n and Γ'_n of our original Fuchsian groups. Since G/Γ and G/Γ' are compact, we can pass to a subsequence so these conjugates converge, say to Γ_∞ and Γ'_∞ . Since $f_\infty(x) = x$, we have $\Gamma_\infty = \Gamma'_\infty$ and hence $X \cong X'$.

The same proof shows the isomorphism $\iota : \Gamma \rightarrow \Gamma'$ induced by the original map f is given by conjugation by an element $g \in G$. Thus our initial remarks show $f = g$. ■

6.8 Geometric limits

For another viewpoint on Mostow rigidity, we introduce in this section the *geometric topology* on baseframed hyperbolic manifolds.

For any topological space X , let $\text{Cl}(X)$ be the set of all closed subsets of X . When X is a compact Hausdorff space, we introduce a topology on $\text{Cl}(X)$ by defining $F_\alpha \rightarrow F$ iff

- for any open set $U \supset F$, we have $F_\alpha \subset U$ for all $\alpha \gg 0$; and
- for any open set with $U \cap F \neq \emptyset$, we have $F_\alpha \cap U \neq \emptyset$ for all $\alpha \gg 0$.

Theorem 6.18 *If X is a compact Hausdorff space, then so is $\text{Cl}(X)$.*

Next suppose X is only locally compact and Hausdorff. Then the one-point compactification $X' = X \cup \{\infty\}$ is compact, and there is a natural map $\text{Cl}(X) \rightarrow \text{Cl}(X')$ by sending F to $F \cup \{\infty\}$. Under this inclusion, $\text{Cl}(X)$ is closed, so it becomes a compact Hausdorff space with the induced topology.

Example. In $\text{Cl}(\mathbb{R})$, the intervals $I_n = [n, \infty)$ converge to the empty set as $n \rightarrow \infty$.

Now suppose G is a Lie group. The set of all closed subgroups of G forms a compact subset of $\text{Cl}(G)$. If H, H_n are subgroups, we say $H_n \rightarrow H$ *geometrically* if we have convergence in the Hausdorff topology.

Finally let $G = \text{Isom}(\mathbb{H}^n)$. Recall that every baseframed hyperbolic manifold (M, ω) determines a torsion-free discrete group $\Gamma \subset G$, and vice-versa. We say

(M_n, ω_n) converges *geometrically* to (M, ω) if the corresponding Kleinian groups satisfy $\Gamma_n \rightarrow \Gamma$ in $\text{Cl}(G)$.

Theorem 6.19 *The set \mathcal{H}_r^n of baseframed hyperbolic manifold (M, ω) with injectivity radius $\geq r$ at the basepoint is compact in the geometric topology.*

If we fix a hyperbolic manifold M , then we have a natural map $FM \rightarrow \text{Cl}(G)$ sending each ω in the frame bundle FM to the subgroup $\Gamma(M, \omega)$. This map is continuous. In particular, if M is compact, then the set of baseframed manifolds (M, ω) that can be formed from M is compact in the geometric topology.

We can now offer a proof of the last step of Mostow rigidity that does not make reference to ergodic theory of the geodesic flow.

Theorem 6.20 *Let $M = \mathbb{H}^3/\Gamma$ be a compact hyperbolic 3-manifold, and let $\mu \in M(\widehat{\mathbb{C}})$ be an L^∞ Γ -invariant Beltrami differential. Then $\mu = 0$.*

Proof. Suppose $\mu \neq 0$. Then, by a variant of the Lebesgue density theorem, there exists a $p \in \mathbb{C}$ such that $\mu(p) \neq 0$ and μ is almost continuous at p . That is, if we write $\mu = \mu(z) d\bar{z}/dz$, then for each $\epsilon > 0$ we have

$$\lim_{r \rightarrow 0} \frac{m\{x \in B(p, r) : |\mu(x) - \mu(p)| > \epsilon\}}{m(B(p, r))} = 0.$$

By a change of coordinates, we can assume that $p = 0$. Let $\mu(0) = a$. Then for $g_t(z) = tz$, we have the weak* limit

$$g_t^* \mu = \mu(tz) \frac{d\bar{z}}{dz} \rightarrow \nu = a \frac{d\bar{z}}{dz}$$

as $t \rightarrow 0$. Concretely, this means

$$\int_{\mathbb{C}} (g_t^* \mu) \phi = \int \mu(tz) \phi(z) |dz|^2 \rightarrow \int a \cdot \phi(z) |dz|^2$$

for every L^1 measurable quadratic differential $\phi = \phi(z) dz^2$.

Since μ is Γ -invariant, $g_t^*(\mu)$ is invariant under

$$\Gamma_t = g_t^*(\Gamma) = g_t^{-1} \Gamma g_t.$$

These conjugates Γ_t correspond to the baseframed manifolds (M, ω_t) as ω_t moves along a geodesic. Since M is compact, we can pass to a subsequence such that $\Gamma_t \rightarrow \Gamma'$, where Γ' is a conjugate of Γ . Indeed, Γ_t only depends on the value of $[g_t]$ in the compact space $\Gamma \backslash G$.

Then ν is invariant under Γ' . This implies Γ' fixes the point $z = \infty$, since ∞ is the only point at which ν is discontinuous. Therefore Γ' is an elementary group — which is impossible, since in fact every orbit of Γ' on $\widehat{\mathbb{C}}$ is dense. ■

Pushing this argument further, one can show:

Theorem 6.21 *Let $M^3 = \mathbb{H}^3/\Gamma$ be a hyperbolic manifold whose injectivity radius is bounded above. Then M^3 is quasiconformally rigid: the only measurable Γ -invariant Beltrami differential $\mu \in M(\widehat{\mathbb{C}})$ is $\mu = 0$.*

See [Mc5, Thm. 2.9].

6.9 Promotion

The basic mechanism of Mostow rigidity is *promotion*: we can use the expanding dynamics of a cocompact group to promote a point of measurable continuity to a point of topological continuity.

Here are two simpler results with the same promotion principle at work.

Theorem 6.22 *Let $M = \mathbb{H}^n/\Gamma$ be a compact hyperbolic manifold. Then the action of Γ on S_∞^{n-1} is ergodic.*

Proof 1. Let $A \subset S_\infty^{n-1}$ be a Γ -invariant set of positive measure. Then A has a point of Lebesgue density $p \in \mathbb{R}_\infty^{n-1}$; that is,

$$\lim_{r \rightarrow 0} \frac{m(B(p, r) \cap A)}{m(B(p, r))} = 1.$$

We may assume $p = 0$. Let $g_n(x) = x/n$; then $g_n \in G = \text{Isom}(\mathbb{H}^n)$. Since 0 is a point of density, we have $g_n^*(\chi_A) \rightarrow 1$ in the weak* topology on $L^\infty(S_\infty^{n-1})$. Since G/Γ is compact, we can write $g_n = \gamma_n h_n$ with $\gamma_n \in \Gamma$ and h_n in a compact subset of G . Then passing to a subsequence, we have $h_n \rightarrow h$, and therefore

$$g_n^*(\chi_A) = h_n^*(\gamma_n^* \chi_A) = h_n^*(\chi_A) \rightarrow h^*(\chi_A).$$

Therefore $h^*(\chi_A) = 1$, which shows $\chi_A = 1$ and A has full measure. Thus Γ acts ergodically. ■

Proof 2 (Ahlfors). Let $A \subset S_\infty^{n-1}$ be a Γ -invariant set of positive measure. Then the harmonic extension of χ_A to \mathbb{H}^n descends to a harmonic function $u : M \rightarrow \mathbb{R}$. By the maximum principle, u is constant, and thus $A = S_\infty^{n-1}$. ■

Theorem 6.23 *Let $f : X \rightarrow Y$ be a homotopy equivalence between a pair of compact hyperbolic surfaces, and let*

$$F : S_\infty^1 \rightarrow S_\infty^1$$

be the boundary values of \tilde{f} . Then either $F' = 0$ almost everywhere, or f is homotopic to an isometry.

Proof. Write $X = \mathbb{H}/\Gamma$ and $Y = \mathbb{H}/\Gamma'$. Suppose $F'(p) \neq 0$. By a change of coordinates, we can assume $p = F(p) = 0 \in \mathbb{R}_\infty^1$. Let $a = F'(0)$ and $g_n(x) = x/n$. Then we have

$$F_n(x) = g_n^{-1} \circ F \circ g_n(x) = nF(x/n) \rightarrow F_\infty(x) = ax$$

uniformly on compact sets. That is, the blowups of F yield in the limit the boundary values F_∞ of an isometry of \mathbb{H}^2 .

Now F_n conjugates $\Gamma_n = g_n^{-1}\Gamma g_n$ to $\Gamma'_n = g_n^{-1}\Gamma' g_n$. By compactness of X and Y , we can pass to a subsequence such that Γ_n and Γ'_n converge geometrically to groups Γ_∞ and Γ'_∞ that are conjugates of Γ and Γ' . Then F_∞ conjugates Γ_∞ to Γ'_∞ , so X and Y are isometric. With more care one can check that the isometry is in the homotopy class of f . ■

Notes. Bowen used this fact to prove that the limit set of a quasifuchsian group is either a round circle or a Jordan curve of Hausdorff dimension $d > 1$ [Bo].

6.10 Ahlfors' finiteness theorem

We now turn to results which control much more general hyperbolic 3-manifolds, with the constraint that $M^3 = \mathbb{H}^3/\Gamma$ is compact replaced by the assumption that $\pi_1(M^3)$ is finitely-generated.

Recall that the action of a Kleinian group on its domain of discontinuity Ω is properly discontinuous.

Theorem 6.24 (Ahlfors' finiteness theorem) *Let $\Gamma \subset \text{Isom}^+(\mathbb{H}^3)$ be a finitely-generated torsion-free Kleinian group. Then the quotient complex 1-manifold $X = \Omega/\Gamma$ is isomorphic to the complement of a finite set in a finite union of compact Riemann surfaces.*

Corollary 6.25 *The components of Ω fall into finitely many orbits under the action of Γ .*

Corollary 6.26 *The domain of discontinuity has no wandering domain; in fact, the stabilizer Γ^U of any component U is infinite.*

If Γ is elementary, then Ω/Γ is isomorphic to $\widehat{\mathbb{C}}$, \mathbb{C} , \mathbb{C}^* or a torus \mathbb{C}/L , so the conclusion of the finiteness theorem holds.

Let us assume from now on that $\Gamma \subset \text{Aut}(\widehat{\mathbb{C}})$ is an N -generator nonelementary Kleinian group. This condition implies that the centralizer of Γ in $\text{Aut}(\widehat{\mathbb{C}})$ is finite, and some of the results below will hold under that slightly weaker hypothesis.

We will start by proving the slightly weaker statement established in Ahlfors' original paper [Ah1]:

Theorem 6.27 *If Γ has N generators, then $\dim \text{Teich}(\Omega/\Gamma) \leq 3N - 3$.*

Remarks. The bound is sharp: an N -generator Schottky group has quotient surface X of genus $g = N$, satisfying $\dim \text{Teich}(X) = 3g - 3$.

While it is true that any *connected* Riemann surface of infinite hyperbolic area has an infinite-dimensional Teichmüller space, the result above does not exclude the possibility that X contains infinitely many components isomorphic to the triply-punctured sphere $Y = \widehat{\mathbb{C}} - \{0, 1, \infty\}$ (since $\dim \text{Teich}(Y) = 0$).

The proof has two main ingredients — the cohomology of deformations, and an estimate for quasiconformal vector fields.

Cocycles and group cohomology. Let G be a group, and A a G -module. A *1-cocycle* is a map $\xi : G \rightarrow A$ such that

$$\xi(gh) = \xi(g) + g \cdot \xi(h).$$

Such a cocycle is also called a *crossed homomorphism*, since it coincides with a homomorphism when G acts trivially on A . A *1-coboundary* is a cocycle of the form

$$\xi(g) = g \cdot a - a$$

for some $a \in A$. The quotient space, (cocycles)/(coboundaries), gives the group cohomology $H^1(G, A)$.

One can view the group $H^1(G, A)$ as classifying affine actions of G on A of the form $g(a) = g \cdot a + \xi(a)$, up to conjugacy by translation. This explains the cocycle rule: we need to have

$$(gh)(a) = gh \cdot a + \xi(gh) = g \cdot (h \cdot a + \xi(h)) + \xi(g) = gh \cdot a + g \cdot \xi(h) + \xi(g).$$

The action $g(a) = g \cdot a$ is considered trivial. It is conjugate to $g(a + b) - b = g \cdot a + (g \cdot b - b)$, so the coboundary $\xi(g) = g \cdot b - b$ is also considered trivial.

Holomorphic vector fields and deformations. Let $G = \text{PSL}_2(\mathbb{C})$, and let $A = \mathfrak{sl}_2(\mathbb{C})$ be the Lie algebra of holomorphic vector fields on the sphere. We have $\dim \mathfrak{sl}_2(\mathbb{C}) = \dim G = 3$. The adjoint action makes A into a G -module. We can regard an element $X \in \mathfrak{sl}_2(\mathbb{C})$ as a matrix or as a vector field, in terms of which the action can be written

$$g \cdot X = gXg^{-1} = g_*(X).$$

Lemma 6.28 *If Γ is a nonelementary N -generator Kleinian group, then*

$$\dim H^1(\Gamma, \mathfrak{sl}_2(\mathbb{C})) \leq 3N - 3.$$

Proof. The space of cocycles is at most $3N$ dimensional since a cocycle is determined by its values on generators. Since the centralizer of Γ is trivial, the space of coboundaries is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, hence 3-dimensional. The difference gives the bound $3N - 3$. ■

The group $H^1(\Gamma, \mathfrak{sl}_2(\mathbb{C}))$ can be interpreted as the tangent space to the variety of homomorphisms from Γ into G , modulo conjugacy, at the inclusion. That is, if $\rho_t : \Gamma \rightarrow G$ is a 1-parameter family of representations, with $\rho_0 = \text{id}$, and we set

$$\xi(g) = \frac{d}{dt} \rho_t(g) g^{-1},$$

then $\xi(g)$ gives a cocycle with values in $\mathfrak{sl}_2(\mathbb{C})$. This cocycle is a coboundary iff to first order we have $\rho_t(g) = \gamma_t g \gamma_t^{-1}$, i.e. if the deformation is by conjugacy.

From Beltrami differentials to cocycles. Now let $M(X)$ be the space of L^∞ Beltrami differentials on X , or equivalently the space of Γ -invariant μ on $\widehat{\mathbb{C}}$ supported on Ω . We now define a natural map

$$\delta : M(X) \rightarrow H^1(\Gamma, \mathfrak{sl}_2(\mathbb{C})).$$

Namely, we solve the equation $\bar{\partial}v = \mu$ and set $\delta\mu = \xi$ where

$$\xi(g) = g_*(v) - v.$$

Since μ is Γ -invariant, we have $\bar{\partial}\xi(g) = 0$, so $\xi(g)$ is indeed a holomorphic vector field and therefore an element of $sl_2(\mathbb{C})$. Note that the solution to $\bar{\partial}v = \mu$ is only well-defined modulo $sl_2(\mathbb{C})$, but changes v by an element of $sl_2(\mathbb{C})$ only changes ξ by a coboundary.

The idea of the construction is that $\mu \in M(X)$ should correspond to Riemann surface X_μ complex structure deformed infinitesimally in the direction μ , plus a quasiconformal map $f : X \rightarrow X_\mu$ with complex dilatation an infinitesimal multiple of μ . Since f is infinitely close to the identity, it should be represented by a vector field v . But the vector field does not quite live on X , because the target of f is X_μ rather than X . On the universal cover, v is really well-defined, and its failure to live on X is measured by the cocycle $\xi(g)$.

Quadratic differentials. Let $Q(X)$ be the Banach space of holomorphic quadratic differentials on X with finite L^1 -norm: $\|\phi\| = \int_X |\phi|$. There is a natural pairing $M(X) \times Q(X) \rightarrow \mathbb{C}$ given by

$$\langle \phi, \mu \rangle = \int \phi \mu.$$

Since $\langle \phi, \bar{\phi}/|\phi| \rangle = \int |\phi|$, the pairing descends to a perfect pairing on $M(X)/Q(X)^\perp \times Q(X)$. The quotient pairing is exactly that between the tangent space $T_X \text{Teich}(X)$ and the cotangent space $Q(X) = T_X^* \text{Teich}(X)$.

Let us say $\mu \in M(X)$ is *trivial* if $\mu \in Q(X)^\perp$; that is, if $\int \mu \phi = 0$ for all $\phi \in Q(X)$, or equivalently if $[\mu]$ represents the zero tangent vector to Teichmüller space.

Let $V(X)$ be the space of quasiconformal vector fields on X , and let $\|v\|_X = \sup \rho_X(z)|v(z)|$ denote the supremum of the hyperbolic length of v .

Lemma 6.29 *If $\delta\mu = 0$, then $\bar{\partial}v = \mu$ has a Γ -invariant solution vanishing on the limit set.*

Proof. If $\delta\mu = 0$, then we can modify any solution v by a holomorphic vector field to obtain $\xi(g) = 0$ for all g ; then v is Γ -invariant. Now if $z \in \Lambda$ is a hyperbolic fixed-point of an element $g \in \Gamma$, then $g'(z) \neq 1$, so the condition $g_*(v) = v$ implies $v(z) = 0$. Such points are dense in the limit set, so $v|_\Lambda = 0$. ■

Lemma 6.30 *Let v be a quasiconformal vector field on $Z = \widehat{\mathbb{C}} - \{0, 1, \infty\}$, vanishing at 0, 1 and ∞ . Then $\|v\|_Z$, the maximum speed of v in the hyperbolic metric on Z , is finite.*

Proof 1. The vector field v has an $|x \log x|$ modulus of continuity, which exactly balances the $1/|x \log x|$ singularity of the hyperbolic metric at the punctures 0, 1, ∞ .

Proof 2. The Teichmüller space of the 4-times punctured sphere is isometric to Z by the cross-ratio map. Thus the hyperbolic length of $v(z)$ is the same

as the Teichmüller length of the deformation of $\widehat{\mathbb{C}} - \{0, 1, \infty, z\}$ defined by $\bar{\partial}v$, which is controlled by $\|\bar{\partial}v\|_\infty$.

Proof 3. Let $\mu = \bar{\partial}v$. By linearity we can assume $\|\mu\|_\infty = 1$. Let $\phi_t : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be the unique holomorphic motion fixing 0, 1 and ∞ and with $\mu(\phi_t) = t\mu$. Then $f(t) = \phi_t(z)$ gives a holomorphic map $f : \Delta \rightarrow Z$. By the Schwarz lemma, with $w = d/dz$, we have

$$\|v(z)\|_X = \|Df(w)\|_X \leq \|w\|_\Delta = 2. \quad \blacksquare$$

Corollary 6.31 *If v is a quasiconformal vector field on a domain $\Omega \subset \widehat{\mathbb{C}}$, and $v|\partial\Omega = 0$, then $\|v\|_\Omega$ is finite.*

Proof. At any point $z \in \Omega$ we can find three points $\{a, b, c\}$ in $\partial\Omega$ such that the hyperbolic metric on Ω at z is comparable to the hyperbolic metric on $\widehat{\mathbb{C}} - \{a, b, c\}$. But the length of $v(z)$ in the hyperbolic metric on $\widehat{\mathbb{C}} - \{a, b, c\}$ is bounded. \blacksquare

Lemma 6.32 *If $v \in V(X)$ has $\|v\|_X < \infty$ then $\mu = \bar{\partial}v$ is trivial. In other words, if v has bounded hyperbolic speed, then $\int \mu\phi = 0$ for all $\phi \in Q(X)$.*

Proof. Let ρ be the hyperbolic metric on X , and fix $\phi \in Q(X)$. Let X_0 be a component of X , and consider a large ball $B(p, r) \subset X_0$. Then we have

$$\int_{X_0} |\phi| = \int_0^\infty dr \int_{\partial B(p, r)} |\phi|/\rho < \infty.$$

Integrating by parts and using the fact that $\bar{\partial}\phi = 0$, we find:

$$\int_{B(p, r)} \phi(\bar{\partial}v) = \int_{\partial B(p, r)} \phi v.$$

Now the sup-norm of ρv is bounded, while the L^1 -norm of $|\phi|/\rho$ tends to zero as $r \rightarrow \infty$, so we can conclude that $\int_{X_0} \phi \bar{\partial}v = 0$. Since X_0 and ϕ were arbitrary, we find $\mu = \bar{\partial}v \in M(X)$ is trivial. \blacksquare

Proof of Theorem 6.27. By the preceding lemmas, we have

$$\text{Ker } \delta \subset Q(X)^\perp.$$

Indeed, if $\delta\mu = 0$, then $\mu = \bar{\partial}v$ for a Γ -invariant v vanishing on $\partial\Omega = \Lambda$. Then v descends to a quasiconformal vector field on X with boundary hyperbolic speed, and hence $\mu = \bar{\partial}v$ is trivial, i.e. μ belongs to $Q(X)^\perp$.

Since the pairing between $Q(X)$ and $M(X)/Q(X)^\perp$ is perfect, we have:

$$\begin{aligned} \dim Q(X) &= \dim M(X)/Q(X)^\perp \\ &\leq \dim M(X)/\text{Ker } \delta \\ &\leq \dim H^1(\Gamma, sl_2(\mathbb{C})) \leq 3N - 3. \end{aligned}$$

■

Notes. Ahlfors' finiteness theorem *fails* for hyperbolic 4-manifolds; see [KP].

For another viewpoint on boundedness of $\|v\|_\Omega$ when $v|_{\partial\Omega} = 0$, one can use the fact that a quasiconformal vector field has modulus of continuity $x|\log x|$, while the hyperbolic metric near a puncture is at worst like $|dz|/|z|\log z|$.

6.11 Bers' area theorem

Theorem 6.33 (Bers) *Let Γ be a nonelementary N -generator Kleinian group. Then the hyperbolic area of $X = \Omega(\Gamma)/\Gamma$ is finite; in fact we have*

$$\text{area}(X) \leq 4\pi(N - 1).$$

Remark. By Gauss-Bonnet, once we know X has finite hyperbolic area, we have $\text{area}(X) = 2\pi|\chi(X)|$, and thus $|\chi(X)| \leq 2N - 2$. Again this inequality is sharp for a handlebody of genus g : we have $N = g$ and $\chi(X) = 2 - 2g$.

Proof. We follow the same lines as Ahlfors' finiteness theorem, but replace the space $sl_2(\mathbb{C}) = H^0(\widehat{\mathbb{C}}, \mathcal{O}(2))$ of holomorphic vector fields with the space $V_d = H^0(\widehat{\mathbb{C}}, \mathcal{O}(2d))$ of holomorphic sections of the d th power of the tangent bundle to $\widehat{\mathbb{C}}$.

Then a typical element of V_d has the form $v = v(z)(\partial/\partial z)^d$, and we have $\dim V_d = 2d + 1$. As before, the space of 1-coboundaries is isomorphic to V_d , so we have

$$\dim H^1(\Gamma, \mathcal{O}(2d)) \leq (2d + 1)(N - 1).$$

Then $\bar{\partial}v = \mu$ is a $(-d, 1)$ -form.

For the analytical part, let $M_d(X)$ denote the measurable $(-d, 1)$ -forms on X which are in L^∞ with respect to the hyperbolic metric. Similarly let $Q_d(X)$ denote the L^1 holomorphic sections $\phi(z) dz^{d+1}$ of the $(d + 1)$ st power of the canonical bundle on X . Then there is a natural pairing between $M_d(X)$ and $Q_d(X)$ as before.

We define

$$\delta_d : M_d(X) \rightarrow H^1(\Gamma, V_d)$$

as before: by solving $\bar{\partial}v = \mu$ and taking the resulting cocycle. Note that the lift of μ to Ω satisfies

$$\sup \rho^{d-1}(z)|\mu(z)| < \infty$$

where the hyperbolic metric $\rho(z)|dz|$ satisfies $\rho(z) \rightarrow \infty$ near $\partial\Omega$. Thus μ is locally in L^∞ and the $\bar{\partial}$ -equation is solvable. (For example, any $\mu \in M_d(\mathbb{H})$ satisfies $\mu(z) = O(y^{d-1})$.)

We have $\delta\mu = 0$ iff there is a Γ -invariant solution to $\bar{\partial}v = \mu$. In this case, $v = 0$ on Λ and one can again show v is bounded in the hyperbolic metric. Then integration by parts can be justified, showing $\mu \in Q_d(X)^\perp$. In conclusion, we find $\dim Q_d(X) \leq (2d+1)(N-1)$.

Now let us examine the integrability condition on $\phi = \phi(z) dz^{d+1}$. On the punctured disk, the hyperbolic metric is given by $\rho = |dz|/|z \log z|$. So if we have

$$\int_U \rho^{1-d} |\phi| < \infty$$

for some neighborhood U of $z = 0$, then ϕ has at worst a pole of order d at $z = 0$.

Thus if $X = \bar{X} - P$, where \bar{X} is a compact surface of genus g and $|P| = n$, then $Q_d(X) = H^0(\bar{X}, \mathcal{O}((d+1)K + dP))$, where K is a canonical divisor. We have $\deg(K + P) = 2g - 2 + n = |\chi(X)|$. By Riemann-Roch, the dimension of $Q_d(X)$ agrees with the degree of the divisor $d(K + P)$, up to an additive constant, so we have

$$\dim Q_d(X) = d|\chi(X)| + O(1).$$

Comparing with the dimension of the group cohomology, we find

$$d|\chi(X)| \leq (2d+1)(N-1) + O(1).$$

Dividing by d and letting $d \rightarrow \infty$ we get the area theorem. ■

Notes. Bers' theorem is proved in [Bers1]. The case of a Schottky group again shows the bound is sharp.

It is desirable to have a geometric interpretation of the cohomology $H^1(\Gamma, \mathcal{O}(d))$, generalizing the case $H^1(\Gamma, \mathcal{O}(2))$ which measures deformations of Γ inside $\mathrm{PSL}_2(\mathbb{C})$ and which appears in Ahlfors' finiteness theorem. Such an interpretation has been developed by Anderson. The idea is to use the embedding $\mathbb{P}^1 \rightarrow \mathbb{P}^n$ as a rational normal curve to extend the action of Γ to \mathbb{P}^n , and then investigate its deformations inside $PGL_{n+1}(\mathbb{C})$. See [And].

For example, $\mathrm{SL}_2(\mathbb{Z})$ acts on both \mathbb{RP}^1 and $\mathbb{RP}^2 = \mathbb{P}\mathbb{R}^{2+1}$. The latter action comes from the Klein and Minkowski models for hyperbolic space. The action of \mathbb{P}^1 is rigid but the action on \mathbb{P}^2 admits deformations; see [Sch].

6.12 No invariant linefields

Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ be a finitely-generated Kleinian group. In this section we will show:

Theorem 6.34 (Sullivan) *The limit set $\Lambda(\Gamma)$ supports no measurable, Γ -invariant field of tangent lines.*

Equivalently, if $\mu \in M(\widehat{\mathbb{C}})$ is a Γ -invariant Beltrami differential, and if $\mu = 0$ outside Λ , then $\mu = 0$ a.e.

Let $Jg(x) = |\gamma'(x)|_\sigma^2$ denote the Jacobian determinant of g at x for the spherical metric. If $Jg(x) = 1$ on a set of positive measure, then g is an isometry in the spherical metric and g has a fixed-point in \mathbb{H}^3 . Since Γ is torsion-free, we conclude that for almost every x , the map

$$g \mapsto Jg(x)$$

gives an injection of Γ into $(0, \infty)$.

We begin with a general analysis of the action of Γ on an invariant set $E \subset \widehat{\mathbb{C}}$ of positive measure. The measurable dynamical system (Γ, E) is *conservative* if for any $A \subset E$ of positive measure, we have $m(A \cap gA) > 0$ for infinitely many $g \in \Gamma$. At the other extreme, (Γ, E) is *dissipative* if it has a ‘fundamental domain’ $F \subset E$, meaning $E = \bigcup_{\Gamma} gF$ and $m(F \cap gF) = 0$ for all $g \neq e$.

Example: $f(z, t) = (e^{2\pi i \theta} z, t)$ gives a conservative action of \mathbb{Z} on $E = S^1 \times [0, 1]$. Note that f is far from ergodic.

Lemma 6.35 *Let $C = \{x \in E : \sum_{\Gamma} Jg(x) = \infty\}$ and let $D = E - C$. Then (Γ, C) is conservative and (Γ, D) is dissipative.*

Proof. Suppose $A \subset C$ has positive measure but $m(A \cap gA) = 0$ for all g outside a finite set $G_0 \subset G$. Then any point of C belongs to at most $n = |G_0|$ translates of A . It follows that

$$\int_A \sum_{\Gamma} Jg(x) dm \leq n \cdot m(C) < \infty,$$

contrary to the definition of C . Thus (Γ, C) is conservative.

To show (Γ, D) is dissipative, let $F \subset D$ be the set of y such that $Jg(y) < 1$ for all $g \in \Gamma$. Clearly gF is disjoint from F for $g \neq e$, because $Jg^{-1}(g(y)) = 1/Jg(y) > 1$ for all $g(y) \in gF$.

Now for almost any $x \in D$, the values $\{Jg(x) : g \in \Gamma\} \subset \mathbb{R}$ are discrete (by summability) and correspond bijectively to the elements of Γ . Thus there is a unique g maximizing $Jg(x)$, and therefore $y = g(x)$ belongs to F . Thus $\bigcup_{\Gamma} gF = D$ and (Γ, D) is dissipative. ■

Lemma 6.36 *Let $\Gamma \subset \text{Isom}(\mathbb{H}^3)$ be a finitely-generated Kleinian group with limit set Λ . Then (Γ, Λ) is conservative.*

Proof. Suppose not. Then there is a set $D \subset \Lambda$ of positive measure such that (Γ, D) is dissipative. Let $F \subset D$ be a measurable fundamental domain for Γ , and let $M(F) \subset M(\widehat{\mathbb{C}})$ denote the space of L^∞ Beltrami differentials supported on F . Each $\mu \in F$ can be freely translated by Γ to give a Γ -invariant Beltrami differential μ' supported on $D \subset \Lambda$. Thus the space $M(\Lambda)^\Gamma$ of Γ -invariant differentials supported on the limit set is infinite-dimensional.

On the other hand, the natural map

$$\delta : M(\Lambda)^\Gamma \rightarrow H^1(\Gamma, sl_2(\mathbb{C}))$$

is injective. Indeed, if $\delta\mu = 0$, then the equation $\bar{\partial}v = \mu$ has a solution vanishing on Λ , which implies $\mu = 0$ on Λ . Since Γ is finitely-generated, the cohomology group on the right is finite-dimensional, so we obtain a contradiction. ■

Lemma 6.37 *Let (Γ, E) be conservative and fix $\epsilon > 0$. Then there is a $g \in \Gamma - \{e\}$ such that $|Jg(x) - 1| < \epsilon$ on a set of positive measure.*

Proof. Suppose not. Then we can find $\epsilon > 0$ such that $|Jg(x) - 1| < \epsilon \implies g = e$ a.e. Therefore the image of $g \mapsto Jg(x)$ is discrete a.e.: since if $Jg(x)$ and $Jh(x)$ are close, then $J(hg^{-1})(y)$ is close to one at $y = g(x)$.

Let $E_+ = \{x : Jg(x) > 1 \ \forall g \in \Gamma\}$. Then $g(E_+) \cap E_+ = \emptyset$ for any $g \neq e$, since $J(g^{-1})(g(x)) = 1/Jg(x) < 1$ for $x \in E_+$. By conservativity, we have $m(E_+) = 0$. By similar reasoning, we conclude that for a.e. x there are unique elements $g_-, g_+ \in \Gamma$ (depending on x) such that

$$Jg_-(x) < 1 < Jg_+(x)$$

and no elements have Jacobians closer to 1 at x . Then the maps $F_\pm : E \rightarrow E$ defined by $F_\pm(x) = g_\pm(x)$ are inverses of one another, so both are injective. But F_+ is uniformly expanding, which is impossible since the total measure of E is finite. ■

Inducing. The preceding result is obvious if $E = S_\infty^2$. Indeed, for any $g \in \text{Isom } \mathbb{H}^3$, the Jacobian $Jg(x)$ is continuous and $\int_{S_\infty^2} Jg = 1$, so the Jacobian is close to one on a set of positive measure.

For applications it is useful to have the following stronger statement.

Lemma 6.38 *Let (Γ, E) be conservative and fix $\epsilon > 0$ and a set $F \subset E$ of positive measure. Then there is a $g \in \Gamma - \{e\}$ and a set of positive measure $A \subset F$ such that $g(A) \subset F$ and $|Jg(x) - 1| < \epsilon$ for all $x \in A$.*

The most conceptual formulation of this result is via *inducing*. First we must generalize our setting. A *partial automorphism* $g : E \dashrightarrow E$ is an invertible measurable map whose domain and range are in E . A collection of partial automorphisms G forms a *pseudo-group* if whenever the composition $g \circ h$ of $g, h \in G$ is defined on a set of positive measure A , we have $g \circ h = k|A$ for some $k \in G$. We say (G, E) is *conservative* if whenever $m(A) > 0$, we have $m(A \cap g(A)) > 0$ for infinitely many $g \in G$.

Now suppose Γ is a group acting on an invariant set E , and suppose $F \subset E$ has positive measure. Let

$$G = \Gamma|F = \{g|F \cap g^{-1}(F) : g \in \Gamma\}.$$

Then G is a pseudo-group acting on F . If (Γ, E) is conservative, so is (G, F) .

It is then easy to see that Lemma 6.37 applies just as well to pseudo-group actions. Applying the Lemma to $\Gamma|F$ yields Lemma 6.38.

Theorem 6.39 *Let Γ be any Kleinian group, and suppose (Γ, E) is conservative, $E \subset \widehat{\mathbb{C}}$. Then E supports no Γ -invariant linefield.*

Remark. For this Theorem we do not assume Γ is finitely generated. This very general statement shows that the Γ -invariant complex structure is unique on the conservative part of the dynamics; any variation must be supported on the dissipative part, where it can be freely specified in a measurable fundamental domain.

Proof. Suppose to the contrary that E supports a Γ -invariant line field, specified by a Beltrami differential with $|\mu| = 1$. Fix a small $\epsilon > 0$. Choose coordinates so that $z = \infty$ is not fixed by any $g \neq e$ in Γ . Writing $\mu = \mu(z)d\bar{z}/dz$, we can find a compact set of positive measure $K \subset \mathbb{C}$ on which the linefield has nearly constant slope. Rotating the linefield, we can assume the slope is nearly one, i.e. $|\mu(z) - 1| < \epsilon$ for all $z \in K$.

Almost every point of K is a point of Lebesgue density. So shrinking K , we can assume there exists an $r_0 > 0$ such that for any $z \in K$ and $r < r_0$, we have $|\mu(w) - 1| < \epsilon$ for 99% of the points $w \in B(z, r)$

By the measure-theory Lemma above, for any $\delta > 0$ there is a $z \in K$ and $g \neq e$ in Γ such that $g(z) \in K$ and $|Jg(z) - 1| < \delta$. (In fact there is a positive measure set of such z , but only one is necessary to get a contradiction.) Since $g(\infty) \neq \infty$, we can write $g = I \circ R$ where $I : \mathbb{C} \rightarrow \mathbb{C}$ is a Euclidean isometry and $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a reflection through a circle $S = S(c, s)$.

Now the Möbius transformation g is determined by its 2-jet, $g(z)$, $g'(z)$ and $g''(z)$. The first two terms range in a compact set. By taking K small, we can exclude g from any finite subset of Γ , so $g''(z)$ must be large. But this means the radius s of S is small. So with a suitable choice of δ , we can assume $s \ll r$. Since $Jg(z) = JR(z) = s^2|z - c|^{-2}$ is close to one, the point z lies in the annulus $A = \{w : s/2 < |w - c| < 2s\}$ about S , which satisfies $R(A) = A$.

But it is clear that R sends the parallel linefield on A to a linefield that is very far from parallel. On the other hand, $g(A) = I(R(A)) = I(A)$ is isometric to A , so it still has diameter $2s \ll r$ and it contains $g(z) \in K$. Thus $\mu|_A$ and $\mu|_{g(A)}$ are both nearly 1, which is impossible. ■

Notes.

1. Sullivan's proof appears in [Sul2]; see also [Ot, Ch. 7]. One of its first applications, as explained in [Ot], was in the endgame of Thurston's construction of hyperbolic structures on 3-manifolds that fiber over the circle. For a survey of the ergodic theory of Kleinian groups, see [Sul3].
2. Attached to any measurable dynamical system (Γ, E) there is a von Neumann algebra A . To construct A , one first forms a bundle of Hilbert spaces $H \rightarrow E/\Gamma$ whose fiber over Γx is $\ell^2(\Gamma x)$. Then A is the space of L^∞ sections of the bundle of operator algebras $B(H)$.

One can also describe A as a space of matrices of the form $T = (T_{xy})$ with $x, y \in E$, such that $T_{xy} = 0$ unless x and y lie in the same orbit. Composition is defined by $(ST)_{xz} = \sum_y S_{xy}T_{yz}$.

The space $L^\infty(E)$ forms the commutative subalgebra of diagonal operators in A . That is, for each $x \in E$ and $f \in L^\infty(E)$, we have a bounded operator

$$F_x : \ell^2(\Gamma x) \rightarrow \ell^2(\Gamma x)$$

defined by $F_x(a_{\gamma x}) = f(\gamma x)a_{\gamma x}$, and these fit together to give a section of $B(H)$. The group Γ also embeds in A , acting by a unitary shift of each Hilbert space $\ell^2(\Gamma x)$.

The center of A corresponds to $L^\infty(E/\Gamma)$, so A is a *factor* (the center is trivial) iff (Γ, E) is ergodic. It can also be shown that A admits a trace (a linear functional satisfying $\text{tr}(ab) = \text{tr}(ba)$ and $\text{tr}(1) = 1$) iff (Γ, E) admits an invariant measure. In this case, (Γ, E) is said to be a factor of type II_1 .

The proof that Λ supports no Γ -invariant line field (when Γ is finitely generated) also shows that (Γ, Λ) is an algebra of type III_1 — it admits no invariant measure, and the ‘ratio set’ of Radon-Nikodym derivatives is dense in \mathbb{R}^* for measure equivalent to Lebesgue.

3. A nice example of an infinitely-generated group Γ which is quasiconformally rigid is the group generated by reflections in a hexagonal packing of circles in \mathbb{C} . Here the quotient Riemann surface Ω/Γ is a countable union of triply-punctured spheres (corresponding to the interstices in the packing), so its Teichmüller space is trivial.

A fundamental domain F for Γ is the region above the hemispheres resting on the circles in the hexagonal packing. The set $E = \overline{F} \cap S_\infty^2$ is just the closure of the union of the interstices, and it serves as a fundamental domain for the dissipative part of the action on S_∞^2 . Since $m(E \cap \Lambda) = 0$, the action on the limit set is conservative, so $M(\Lambda)^\Gamma = 0$. Therefore Γ is rigid.

Using the fact, one can show that circle packings furnish an algorithm for the construction of Riemann mappings [RS]. This algorithm has been used to apply conformal mappings to the human brain.

6.13 Sullivan’s bound on cusps

Theorem 6.40 (Sullivan) *Let Γ be a nonelementary N generator Kleinian group. Then the number of cusps of $M = \mathbb{H}^3/\Gamma$ is at most $5N - 5$.*

The idea of the proof is similar to the proof of Bers’ area theorem, but with an analytical twist: we allow *distributional* Beltrami coefficients.

More precisely, let $C \subset \Lambda$ be the countable set of *cusps* of Γ , i.e. fixed-points of parabolic elements. Let $M_d(C)$ be the vector space of Γ -invariant $(-d, 1)$ -forms on C with the quality of *measures*. That is, $\mu \in M_d(C)$ is locally of the form $\mu(z)d\bar{z}/(dz)^d$, where $\mu(z)$ is a measure supported on C .

Theorem 6.41 *For $d \geq 2$, we have $\dim M_d(C) = |C/\Gamma|$.*

Proof. We claim that for any cusp $c \in \widehat{\mathbb{C}}$,

$$\sum_{\Gamma/\Gamma_c} \|\gamma'(c)\|^3 < \infty.$$

Here Γ_c is the stabilizer of c . Note that $\gamma'(c)$ is well-defined since $\gamma'(c) = 1$ for $\gamma \in \Gamma_c$. Then $\|\gamma'(c)\|$ is bounded above by the Euclidean volume of the horoball (in the ball model for \mathbb{H}^3) resting on $\gamma(c)$. Since these are disjoint we get convergence.

Now consider, for $c \in C$, the measure-differential $\mu_c = \delta(z - c)d\bar{z}/(dz)^d$. Then $|\mu_c|$ transforms by $|\gamma'(c)|^{2+d}$ (the 2 comes from the action on area densities the sphere), so

$$\mu = \sum_{\Gamma_c} \sum_{\Gamma_c} \gamma^* \mu_c$$

converges to an element of $M_d(C)$ supported on Γ_c . Since μ 's with different supports are linearly independent, we obtain the bound on $\dim M_d(C)$. ■

Proof of Theorem 6.40. Solving the $\bar{\partial}$ -equation, we have as usual a cocycle map $\delta : M_d(C) \rightarrow H^1(\Gamma, V_d)$. We claim δ is injective. Indeed, if $\delta\mu = 0$, then $\bar{\partial}v = \mu$ for some Γ -invariant d -field $v = v(z)(d/dz)^d$. Since the fundamental solution $1/z$ to the $\bar{\partial}$ -equation μ is in L^1_{loc} , so is v . Now v is holomorphic on Ω , so it vanishes there — the Riemann surface Ω/Γ admits no holomorphic vector fields or d -fields. On the other hand, v also vanishes on the limit set by a variant of the no-invariant linefields theorem. Thus $v = 0$.

Taking $d = 2$, we find

$$|C/\Gamma| = \dim M_2(C) \leq \dim H^1(\Gamma, V_2) \leq 5N - 5.$$

■

Note. Sullivan's bound on the number of cusps appears in [Sul1]. (In this reference, the bound of $5N - 5$ is misstated as $5N - 4$.)

6.14 The Teichmüller space of a 3-manifold

Let M be a compact 3-manifold. A *convex* hyperbolic structure on M is a Riemann metric g of constant curvature -1 such that ∂M is locally convex. Equivalently, for any homotopy class of path γ between two points in $\text{int}(M)$, the shortest representative of γ also lies in $\text{int}(M)$.

Given such a metric, the developing map $\widetilde{M} \rightarrow \mathbb{H}^n$ is injective and its image is convex. Thus (M, g) can be extended to a complete hyperbolic manifold N in a unique way. The manifold $N = \mathbb{H}^n/\Gamma$ is *geometrically finite* and indeed *convex cocompact*: that is, the convex core of N , given by

$$\text{core}(N) = \text{hull}(\Lambda)/\Gamma \subset N,$$

is compact.

Conversely, for any convex cocompact hyperbolic manifold N , a unit neighborhood M of its convex core carries a (strictly) convex hyperbolic structure.

We can complete N to a *Kleinian manifold*

$$\overline{N} = (\mathbb{H}^n \cup \Omega)/\Gamma,$$

with a complete hyperbolic metric on its interior and a conformal structure on its boundary. The manifolds M and \overline{N} are homeomorphic.

In analogy with the Teichmüller space of a Riemann surface, given a compact oriented 3-manifold M , we can consider the space $GF(M)$ of all geometrically finite hyperbolic manifolds *marked* by M .

A point $(f, \overline{N}) \in GF(M)$ is represented by a homeomorphism

$$f : M \rightarrow \overline{N},$$

where \overline{N} is an oriented Kleinian manifold, and f respects orientations. As usual, two pairs $(f_1, \overline{N}_1), (f_2, \overline{N}_2)$ are *equivalent* if there is an orientation-preserving isometry $\iota : \overline{N}_1 \rightarrow \overline{N}_2$ and an $h : M \rightarrow M$ homotopic to the identity such that

$$\begin{array}{ccc} M & \xrightarrow{f_1} & \overline{N}_1 \\ h \downarrow & & \downarrow \iota \\ M & \xrightarrow{f_2} & \overline{N}_2 \end{array}$$

commutes.

Theorem 6.42 *Let M be a compact 3-manifold admitting at least one convex hyperbolic structure. Then there is a naturally isomorphism*

$$GF(M) \cong \text{Teich}(\partial M)$$

given by

$$(N, f) \mapsto (\partial N, f|_{\partial M}).$$

6.15 Hyperbolic volume

Let $\pi(\theta) = -\int_0^\theta \log |2 \sin t| dt$ be the Lobachevsky function. An ideal tetrahedron $T(\alpha, \beta, \gamma)$ is determined by its dihedral angles, which sum to π .

Theorem 6.43 *The hyperbolic volume of $T(\alpha, \beta, \gamma)$ is given by $\pi(\alpha) + \pi(\beta) + \pi(\gamma)$.*

Idea of the proof.

Corollary 6.44 *The regular tetrahedron has maximum volume.*

Proof. By Lagrange multipliers, at the maximum of $\pi(\alpha) + \pi(\beta) + \pi(\gamma)$ subject to $\alpha + \beta + \gamma = \pi$ we have $\pi'(\alpha) = \pi'(\beta) = \pi'(\gamma)$. Since $\pi'(\theta)$ determines $|\sin \theta|$, this means $\sin(\alpha) = \sin(\beta) = \sin(\gamma)$. By the law of sines, this means the associated triangle is equilateral. ■

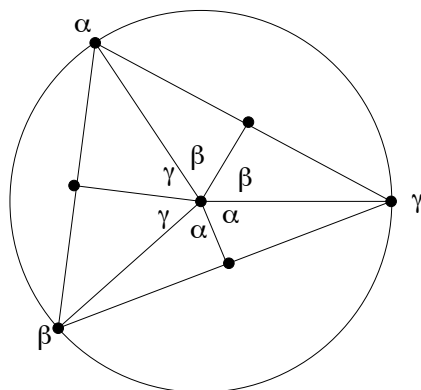


Figure 17. Dissection of an ideal tetrahedron into 6 pieces.

Ideal octahedron, Whitehead link, Apollonian gasket, Scott's theorem.

Theorem 6.45 *Surface groups are LERF.*

This means every finitely generated subgroup of $\pi_1(\Sigma_g)$ comes from a subsurface of a finite-sheeted covering space.

Corollary 6.46 *Every closed geodesic on a closed surface is covered by a simple geodesic on a finite cover.*

The proof is based in part on:

Theorem 6.47 \mathbb{H} *is exhausted by convex regions tiled by right pentagons.*

We note that in fact the layers of pentagons surrounding a given one always form convex regions. Indeed, the total number of edges e_n and right angles a_n around the n th layer satisfies

$$\begin{pmatrix} a_{n+1} \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a_n \\ e_n \end{pmatrix}.$$

By Gauss-Bonnet the total number of pentagons p_n out to generation n is $a_n - 4$. Thus one can easily compute $p_1, p_2, \dots = 1, 11, 51, 201, 761, 2851, 10651, \dots$ with $p_i \sim C\lambda^i$, $\lambda = 2 + \sqrt{3}$.

Three dimensions. It is also known that the Bianchi groups $SL_2(\mathcal{O}_D)$ are LERF, and in particular that the figure eight and Whitehead link complements are LERF. However the proof is much more difficult in 3-dimensions, because a finitely-generated subgroup of a geometrically finite group need not be geometrically finite. In particular, the proof for the Whitehead link complement does

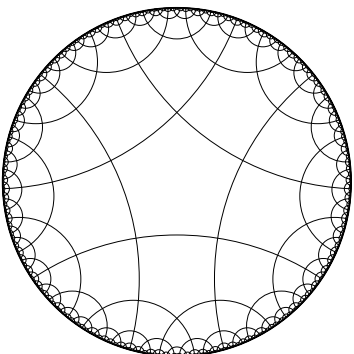


Figure 18. Tiling by right pentagons.

not just follow from commensurability with a reflection group, as indicated in [Sc].

There are also now examples of 3-manifold groups which are *not* LERF; however these examples are torus-reducible.

Question. Is the fundamental group of every compact hyperbolic 3-manifold LERF?

Question. Is every surface subgroup of a compact hyperbolic 3-manifold represented by a virtually embedded surface?

Remark: Apollonian gasket. The submanifold of M obtained by cutting along a totally geodesic triply-punctured sphere (the disk spanning one component of W) has, as its limit sets, the Apollonian gasket.

Remark: Vol3 and volume coincidences. There is a *closed* hyperbolic 3-manifold whose volume is a rational multiple of the volume of the figure eight knot complement [JR]. In general, for an arithmetic hyperbolic manifold, $\text{vol}(M)$ is a rational multiple of $\zeta_k(2)$, where k is the invariant trace field of M ; but the commensurability class of M depends on the associated quaternion algebra over k .

In general a quaternion algebra over a quadratic imaginary field is uniquely determined by specifying the even number of places where it ramifies. It gives a cocompact lattice *unless* this set of places is empty, in which case $\mathcal{Q} \cong M_2(k)$.