Rigid/Flexible Dynamics

Problems and Exercises

Math 275 — Harvard University — Fall 2006

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1. Give necessary and sufficent conditions on $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ such that T is in $U(1,1)$, i.e. such that T preserves $|z|^2 = |z_1|^2 - |z_2|^2$. Then show every element of $\mathrm{PU}(1,1) = U(1,1)/S^1$ is represented by a matrix of the form $\left(\begin{smallmatrix} a & b \\ \overline{b} & \overline{a} \end{smallmatrix}\right)$, $|a|^2 - |b|^2 = 1$.

2. Can every hyperbolic surface of genus two be tiled by 8 right-angled pentagons?

3. Give explicit coordinates for the vertices of a square in the hyperbolic plane with internal angles of 45°, say centered at $z = 0$ in the Poincaré disk model.

4. Let $X(2) = \mathbb{H}/\Gamma(2)$ be the triply-punctured sphere, and let $X(1) = \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. (i) Show that $\mathrm{Aut}(X(2)) \cong S_3$, and $X(2)/S_3 = X(1)$. (ii) Let $Y = X(2)/\tau$, where $\tau \in \mathrm{Aut}(X)$ has order two. Show $\mathrm{Aut}(Y) = \mathbb{Z}/2 = \langle\sigma\rangle$. Describe the orbifolds Y and $Z = Y/\langle\sigma\rangle$. (iii) Show that although $X(2)$ covers $X(1)$ and Z there is no orbifold W covered by both Z and $X(1)$. (iv) Conclude by giving an explicit example of a group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ such that $\Gamma \cap \mathrm{SL}_2(\mathbb{Z})$ has finite index in both Γ and $\mathrm{SL}_2(\mathbb{Z})$, but Γ and $\mathrm{SL}_2(\mathbb{Z})$ together generate an indiscrete group.

5. Consider the ideal hyperbolic octahedron F in the upper halfspace model, with vertices at $(0, \infty, \pm 1 \pm i)$. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ be a Kleinian group with fundamental domain F, such that $M = \mathbb{H}^3/\Gamma$ is the Whitehead link complement $S^3 - W$.

   (i) Find explicit generators for Γ. (ii) Identify the stabilizer in Γ of the point at infinity. (iii) Find two distinct, totally geodesic triply-punctured spheres in M. (iv) Show what one of these surfaces looks like in S^3.

6. Show that the complement of the Borromean rings double covers the Whitehead link complement, and is hence also a hyperbolic manifold.

7. Let $f : \mathbb{H}^3 \to \mathbb{H}^3$ be a quasi-isometry. Suppose that for every (totally geodesic) plane $P_1 \cong \mathbb{H}^2 \subset \mathbb{H}^3$, there is a hyperbolic plane P_2 such that

$f(P_1)$ is within a bounded distance of P_2. Show that f is a bounded distance from an isometry.

8. Let $X = S^1 \times S^1$ be the 2-torus, and let $C = \sum a_i f_i \in C_2(X, \mathbb{R})$ be a 2-cycle that is homologous to zero. Show there is a universal $K > 0$ such that $C = \partial D$ where D is a 3-chain satisfying $\|D\| \leq K\|C\| = K \sum |a_i|$.

9. What is the simplicial volume of the 3-manifold $T_1\Sigma_g$ formed by the unit tangent bundle to a surface of genus g? (Hint: use the preceding result and the fact that the unit tangent bundle of Σ_g − (a disk) is trivial.)

10. Recall that the limit set of a discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ is defined by $\Lambda = S^{n-1}_\infty \cap \overline{\Gamma x}$ for any $x \in \mathbb{H}^n$.

    (i) Show that if Λ contains at least three points, then Λ is the smallest, non-empty, closed, Γ-invariant subset of S^{n-1}_∞; in particular, $\Lambda = \overline{\Gamma p}$ for any $p \in \Lambda$.

    (ii) Show that if $|\Lambda| \leq 2$, then Γ is virtually abelian.

11. Let $K \subset \mathbb{H}^3$ be the convex hull of a closed set $F \subset S^2_\infty$. Show there is a universal constant R such that every $x \in K$ lies within distance R of a geodesic with endpoints in F.

12. Let $M^3 \to S^1$ be a torus bundle over the circle. Show that there is a self-map $f : M^3 \to M^3$ of degree $d > 1$.

13. Construct a map $\phi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathbb{R}$ satisfying $\phi(xy) = \phi(x)+\phi(y)+O(1)$ such that the restriction of ϕ to $\widetilde{\mathrm{SO}_2(\mathbb{R})}$ is an isomorphism.

14. Show $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ is quasi-isometric to $\mathbb{R} \times \mathbb{H}$.

15. Let $\Gamma_n \subset \mathrm{PSL}_2(\mathbb{R})$ be the (n, n, n)-triangle group, $n \geq 7$, so $X_n = \mathbb{H}/\Gamma_n$ is the hyperbolic orbifold of genus zero with three cone points of type $\mathbb{Z}/n$. (a) Show that X_7 is covered by a surface of genus 3. (b) Show more generally that for n odd, there is an isometric action of $\mathbb{Z}/n$ on a hyperbolic surface Y_n yielding X_n as quotient. Compute the genus of Y_n.

    (Hint: find a surjective map

    $$\pi_1(X_n) = \langle a, b, c \,:\, a^n = b^n = c^n = abc = e\rangle \to \mathbb{Z}/n$$

    sending each of a, b and c to a generator of $\mathbb{Z}/n$. Alternatively, construct Y_n from a regular $2n$-gon.)

16. Given an example of a closed, orientable, flat 3-manifold that is not homeomorphic to the torus $\mathbb{R}^3/\mathbb{Z}^3$.

17. In each of the following examples, describe the geometric structure on the manifold $M^3 - \gamma$.

    (a) $M^3 = \mathbb{R}^3/\mathbb{Z}^3$ and γ is a closed geodesic;

    (b) M^3 is the torus bundle over the circle with monodromy $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, and γ is the zero section;

    (c) $M^3 = \Sigma_g \times S^1$ and $\gamma = \gamma_0 \times \{1\}$, where γ_0 is an essential simple loop on the surface Σ_g of genus $g \geq 2$. (Hint: this case needs to be cut along a torus.)

18. A geometry (G, X) has *no moduli* if whenever $M_i = X/\Gamma_i$, $i = 1, 2$ are compact, homeomorphic manifolds modeled on (G, X), the groups Γ_1 and Γ_2 are conjugate in G. Which of the 8 three-dimensional geometries have no moduli?

19. Let X_t denote the foliated torus obtained by suspending the rotation $x \mapsto x + t$ on $S^1 = \mathbb{R}/\mathbb{Z}$. Show there is a diffeomorphism from X_t to X_s, respecting the foliations, if and only if $t \sim s$ under the action of $\mathrm{GL}_2(\mathbb{Z})$ by Möbius transformations.

20. Let $f_n(x) = nx \bmod 1$ as an endomorphism of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Construct a Cantor set $K \subset S^1$ such that $f_3|K$ admits an monotone semiconjugacy to f_2. That is, there should be a continuous, monotone map $h : S^1 \to S^1$ such that $h(K) = S^1$ and $f_2(h(x)) = h(f_3(x))$.

    What is the Hausdorff dimension of K?

21. Let $A : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be a toral automorphism or endomorphism by a matrix $A \in \mathrm{M}_2(\mathbb{Z})$. Show that its entropy is given by $h(A) = \log \rho(A)$, where $\rho(A)$ is the spectral radius of A.

22. Let $X = \mathbb{H}/\Gamma$ be a hyperbolic surface of genus two. (a) Show that the number $N(L)$ of simple closed geodesics on X of length $\leq L$ grows at most like a polynomial in L. (b) Let $S \subset S^1 \times S^1$ be the set of endpoint–pairs for all simple (not necessarily closed) geodesics on X. Show that S is closed. (c) Show that $\mathrm{H.dim}(S) = 0$.

23. Let $N = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : t \in \mathbb{R}\} \subset \mathrm{SL}_2(\mathbb{R})$. Given $L \in \mathcal{L}_2$, what rings can arise as $\mathrm{End}_N(L) = \mathrm{End}(L) \cap \mathbb{R} \cdot N$? When is $\mathrm{End}_N(L)$ bigger than $\mathbb{Z}$? What is the orbit $N \cdot L$ in this case?

24. Let $L \subset \mathbb{R}^n$ be a lattice spanned by n linearly independent vectors of length one. Show that the covering radius of L is at most $\sqrt{n}/2$: that is, every point of $\mathbb{R}^n$ is within distance $\sqrt{n}/2$ of a point of L.

25. Let $L \subset \mathbb{R}^n$ be a well-rounded lattice of dimension $n \leq 4$. Show that the shortest vectors in L generate L. Show this statement fails for $n = 5$.

26. Let $A = \left(\begin{smallmatrix} 5 & 4 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. (i) Find an explicit lattice $L \in \mathcal{L}_2$ on the closed geodesic determined by A. (ii) What is the discriminant of the quadratic ring $\mathrm{End}_A(L) = \mathrm{End}(L) \cap A$?

27. Let p be a prime and let $K = \mathbb{Q}(X)$ be the cubic field generated by $X^3 = p$. Consider the ideal

$$I = \mathbb{Z}(p^5) + \mathbb{Z}(p^2 X) + \mathbb{Z}(p^2 X^2)$$

for the order

$$A = \mathbb{Z} + \mathbb{Z}(p^2 X) + \mathbb{Z}(p^2 X^2) \subset K.$$

(i) Show that I is a proper ideal (no larger order leaves I invariant). (ii) Show I is not invertible (there is no fractional ideal J such that $IJ = A$). (iii) Show that $[A : I]$ is different from $\gcd(N(x) \,:\, x \in I)$.

28. Show there is no connected Lie group lying strictly between $\mathrm{SO}(2,1,\mathbb{R})^0$ and $\mathrm{SL}_3(\mathbb{R})$.

29. Show the statement:

For all $\alpha, \beta \in \mathbb{R}$ we have $\inf_{n>0} n^{\delta} \|n\alpha\| \cdot \|n\beta\| = 0$

is true for $\delta < 1$. Find the best value $\delta > 1$ you can for which it is false.

30. *Show there is an $r > 0$ such that for all $x \in S^1$, the balls $B(nx, r/n)$, $r = 1, 2, 3 \ldots$ do not cover S^1. Equivalently, for all x we have:

$$\sup_y \inf_{n>0} n\|nx - y\| \geq r > 0.$$

(See e.g. [Ca, Ch. V].)

31. Using the preceding result, show that any unimodular lattice $L \subset \mathbb{R}^2$ satisfies

$$\mathrm{cov}_N(L) = \sup_{x \in \mathbb{R}^2} \inf_{y \in L} N(x - y) > c_2 > 0.$$

It is not known if there is a corresponding lower bound $c_n > 0$ for $n \geq 3$ [GL, p.585].

References

[Ca] J. W. S. Cassels. *An Introduction to Diophantine Approximation.* Cambridge University Press, 1957.

[GL] P. M. Gruber and C. G. Lekkerkerker. *Geometry of Numbers.* Elsevier, 1987.