Teichmüller Theory
Problems and Exercises
Math 275 — Harvard University — Spring 2005
C. McMullen

1. For $g \geq 2$, the period map $\mathcal{T}_g \to \mathfrak{H}_g$ from Teichmüller space to the Siegel upper halfspace is not injective. Why not? Which points in $\mathcal{T}_g$ map to the same point in $\mathfrak{H}_g$? Why is the map injective for $g = 1$?

2. The Jacobian of a surface of genus two with 5-fold symmetry can be described as $\mathrm{Jac}(X) = \mathbb{C}^2/L$, where $L = \mathbb{Z}[\zeta]$, $\zeta = \exp(2\pi i/5)$ is a fifth root of unity, and L is embedded in $\mathbb{C}^2$ by a ring homomorphism sending ζ to (ζ, ζ^2).

   What is a symplectic structure on L coming from its identification with $H_1(X,\mathbb{Z})$? Give your answer in terms of a matrix $\langle \zeta^i, \zeta^j \rangle$, or in terms of a bilinear form $B : L \times L \to \mathbb{Z}$ involving the trace $\mathrm{Tr} : L \to \mathbb{Z}$.

3. What is the most symmetric Riemann surface of genus two? Determine its Jacobian as a principally polarized Abelian variety, and give its period matrix.

4. Let $D \subset \Delta$ be a regular hyperbolic decagon that yields a smooth surface of genus two upon identifying opposite sides. Normalize so that D is centered at $z = 0$ and has a vertex on the imaginary axis. Give the coordinates for the centers and radii (c_i, r_i) of the ten circles whose arcs form the sides of D.

5. Let X be the triply-punctured sphere with its complete hyperbolic metric of finite volume. What is the shortest closed geodesic on X? Why is it not simple? What is its length?

6. What is the hyperbolic metric on the punctured disk Δ^*? On the annulus $A(R) = \{z : 1 < |z| < R\}$? What is the length of the unique closed geodesic on $A(R)$?

7. Describe all the pairs-of-pants decomposition of a surface of genus three by enumerating trivalent graphs.

8. Show that for any pair of elliptic curves $X, Y \in \mathcal{M}_1$, there is a sequence of finite covers X_n of X such that $X_n \to Y$ in $\mathcal{M}_1$.

9. Let $\Gamma(n) = \{A \in \mathrm{SL}_2(\mathbb{Z}) \, : \, A \equiv I \bmod n\}$, and let $X(n) = \mathbb{H}/\Gamma(n)$.

   (a) Show the automorphism group of $X(n)$ is $\mathrm{SL}_2(\mathbb{Z}/n)$.

   (b) Describe $X(n)$ concretely as a copy of $\mathbb{P}^1$ with a finite set E_n removed, for $n = 2, 3, 4, 5$. What is the action of $\mathrm{SL}_2(\mathbb{Z}/n)$ on E_n?

   (c) Describe the compactification of $X(n)$ as an algebraic curve, for $n = 6$ and 7.

10. Calculate the boundary of a bigon $B \subset \Delta^*$ meeting $\partial\Delta$ at ± 1. What is Euclidean radius of the maximal inscribed ball $B(0, r) \subset B$?

11. Let f be a measurable function on $\mathbb{R}^2$. Suppose that f is constant on almost every vertical line and on almost every horizontal line. Show that f is constant almost everywhere.

12. Give an explicit example of a pair of points $x, y \in \mathbb{R}$ such that the projection to $\mathbb{H}/\,\mathrm{SL}_2(\mathbb{Z})$ of the geodesic joining x to y is neither closed nor dense.

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13. Determine the subset of $\mathrm{Aut}(\mathbb{H})$ represented by $N^t AN$. Explain your answer geometrically, in terms of the horocycle and geodesic flows on $\mathrm{T}_1(\mathbb{H})$.

14. Show there is no discrete subgroup $\Gamma \subset AN$ such that $(AN)/\Gamma$ is compact; and that AN is not unimodular.

15. Let $f : X \to X$, $X = \mathbb{R}^2/\mathbb{Z}^2$ be a linear map given by a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $|a+d| > 2$. Show that $f(x,y)$ is ergodic and mixing with respect to the area measure $dx\,dy$ on X. (Hint: use the Fourier transform to replace $L^2(X)$ with $\ell^2(\mathbb{Z}^2)$. Or, use the pair of linear foliations preserved by f.)

16. The *limit set* of a discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H})$ is given by $\Lambda = \overline{\Gamma x} \cap S^1_\infty$ for any $x \in \mathbb{H}$. Prove that the following are equivalent: (i) $X = \mathbb{H}/\Gamma$ has a dense geodesic; (ii) $\Lambda = S^1_\infty$; (iii) every AN orbit in G/Γ is dense.

17. Let $f : \mathbb{H}^2 \to M = \mathbb{H}^3/\Gamma$ be a totally geodesic immersion of a hyperbolic plane into a compact hyperbolic 3-manifold. Using Ratner's theorem, show that $N = \overline{f(\mathbb{H}^2)}$ is either M itself or an immersed surface of finite area.

18. Let $\gamma : [0,T] \to X$ be a geodesic path on a compact hyperbolic surface, parameterized by arclength. Show that there exists a sequence of closed geodesics $\gamma_n : [0,T_n] \to X$, $T_n \to \infty$, such that $\gamma_n \to \gamma$ uniformly on $[0,T]$.

19. Let $\mathcal{G} = \mathbb{RP}^2 - \overline{\Delta}$ be the space of unoriented geodesics in the Klein model.

    (i) Show that the measure $\mu = dx\,dy/(x^2+y^2-1)^{3/2}$ on G is invariant under $\mathrm{SO}(2,1)$.

    (ii) Show there is a constant C such that for any geodesic segment $\alpha \subset \Delta$, if we let $I(\alpha) \subset G$ be the set of geodesics meeting α, then $\ell(\alpha) = C\mu(I(\alpha))$. Find the value of C.

20. Let L_X be the Liouville current on $T_1(X)$. Show that there exists a sequence of weighted closed geodesics such that $C_n\gamma_n \to L_X$ in the space of geodesic currents.

21. Let $X = \mathbb{H}/\Gamma$ be compact, and let $x \in \mathcal{G}$ be a point whose orbit Γx is closed in $\mathcal{G}$. (i) Show that x corresponds to a closed geodesic *loop* on X. (ii) What happens if X is noncompact, but of finite volume?

22. Let $X = \mathbb{H}/\Gamma$ be compact. Show that (although the action of Γ on $\mathcal{G}$ is not properly discontinuous) there exists a compact 'fundamental domain' $K \subset \mathcal{G}$, such that $\mathcal{G} = \bigcup \Gamma \cdot K$.

23. Find a train track τ for the once-punctured torus T that carries two different simple closed curves. Determine the map from the space of weights on τ (satisfying the switching conditions) to $H_1(T, \mathbb{R})/(v \equiv -v)$.

24. Let α be a geodesic current on X. Show that α binds X iff $i(\alpha, \beta) > 0$ for all $\beta \in \mathcal{C}(X)$. (Recall that α *binds* X if every geodesic γ crosses a geodesic α_0 in the support of α.)

    Is it enough to require that $i(\alpha, \gamma) > 0$ for all closed geodesics γ on X?

25. (Bonus) Let $\alpha, \beta_n \in \mathcal{C}(X)$ be closed curves, such that $C_n\beta_n \to \alpha$ for some $C_n > 0$. Show directly that $C_n i(\alpha, \beta_n) \to 0$.

26. (Bonus) Let $(\ell_i, \tau_i)_1^3$ be Fenchel-Nielsen coordinates for $\mathcal{T}_2$, coming from a pair of pants decomposition with 3-fold symmetry. Let $X_L \in \mathcal{T}_2$ denote the Riemann surface with $\ell_i = L$ and $\tau_i = 0$. What is the limit of X_L in $\mathbb{P}\mathcal{ML}_2$ as $L \to \infty$?

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27. For $t > 0$ let $X_t \in \mathcal{M}_2$ denote the hyperelliptic curve given by $y^2 = x(x-t^3)(x-t^2)(x-t)(x-1)$. Find approximate Fenchel-Nielsen coordinates for X_t when t is small.

28. Let α and β denote the Euclidean distances between the a-sides and b-sides of a quadrilateral $Q \subset \mathbb{C}$. Show that $\text{area}(Q) \geq \alpha\beta$.

29. Prove that if $f : A(R_1) \to A(R_2)$ is a holomorphic covering map, where $A(R) = \{z : 1 < |z| < R\}$, then $f(z) = az^n$ for some $n \neq 0$. Conclude that $R_2 = R_1^{|n|}$ and $R_1 = R_2$ if f is an isomorphism.

30. Let $Q = [0,1] \times [0,1]$ with b-sides $0 \times [0,1]$ and $1 \times [0,y]$. Show that there exist $C_1, C_2 > 0$ such that for all $0 < y < 1/2$ we have

$$C_1 \log(1/y) < \text{mod}(Q) < C_2 \log(1/y).$$

31. Show that for all $\epsilon > 0$ there exists a pair of Riemann surfaces $X_1, X_2 \in \mathcal{T}_g$ such that $d_T(X_1, X_2) > 1$ even though

$$(1-\epsilon) < \frac{L_\gamma(X_1)}{L_\gamma(X_2)} < (1+\epsilon)$$

for all closed geodesics γ.

(Hint: let $X_1 = \tau^n(X_2)$, where X_1 is a punctured torus with a geodesic δ of length $1/n$, and τ is a Dehn twist about δ. See (Lixin, 1999) for references around this problem.)

32. Let q be a meromorphic quadratic differential on a compact Riemann surface X.

(i) Show that if q vanishes to of order $i \geq -1$ at p, then there is a local chart such that $q = z^i\, dz^2$. (The case $i = -1$ corresponds to a simple pole.)

(ii) Show this result fails for $i = -2$, and formulate a version that it correct.

(iii) Show that $\int_X |q| < \infty$ iff q has at worst simple poles.

33. Prove the Koch snowflake curve has bounded turning.

34. Let $f : X \to Y$ be a quasiconformal map. Suppose $\mu(f) = tq/|q|$ for some $q \in Q(X)$ and $t > 0$. Show that there exists a quadratic differential r on Y such that $f : (X, q) \to (Y, r)$ is affine.

35. Let $f : X \to X$ be a pseudo-Anosov mapping relative to a holomorphic quadratic differential q on X. Show that f is ergodic with respect to the invariant measure $\mu(E) = \int_E |q|$.

36. Give a concrete model for the quadratic differential (X, q) and the group $\mathrm{PSL}(X, q)$ associated to the Coxeter graph A_3 (written O–O–O).

37. (Bonus) Suppose $\phi \in \mathrm{Mod}_g$ has infinite order and the degree $2g$ polynomial $p(t) = \det(tI - \phi|H_1(Z_g, \mathbb{Z}))$ is irreducible over $\mathbb{Q}$.

38. (Bonus) Give an explicit example of a pair of (connected) simple closed curves α, β on a surface of genus two that bind the surface. Show that a binding pair exists on a surface of any genus.