

ChatGPT takes on Riemann surfaces homework

User: Let M_0 be the field of germs of complex analytic functions defined near $z = 0$, i.e. the field of Laurent series with positive radius of convergence. Does M_0 have a unique discrete valuation?

Assistant: No— M_0 does **not** have a unique discrete valuation (as a valuation on the field).

Assistant: Globally on the field, there are many other discrete valuations.

A standard way to see this:

- M_0 is uncountable, while the algebraic closure of $\mathbf{C}(z)$ inside M_0 is still “small” in comparison, so M_0 contains elements transcendental over $\mathbf{C}(z)$

With a bit more care one can arrange genuinely discrete extensions (value group \mathbf{Z}), ...

User: I’m not convinced the extension is discrete.

Assistant: You’re right to be skeptical: . . .

A reliable way to produce genuinely discrete valuations is to use **divisorial valuations** coming from function fields of higher transcendence degree.

Here is a clean route (non-constructive, but standard): . . .

This produces a discrete valuation on M_0 that is not ord_0 (for instance, it can be trivial on $\mathbf{C}(z)$, while ord_0 is not).

Assistant: Bottom line

- Your objection is correct: **an arbitrary valuation extension need not be discrete.**
- Nevertheless, M_0 **does not have a unique discrete valuation** as an abstract field: besides ord_0 , there exist many other discrete (rank-1) valuations, but producing them cleanly requires using a 2-variable rational subfield and a *divisorial* valuation (or an equivalent construction), not just “extend the t -adic valuation and hope”.

User: It seems to me that the same proof that works for \mathbf{Q}_p shows in fact there is only 1 discrete valuation on M_0 . This is because every element of the form $1 + \sum_1^\infty a_n z^n$ in M_0 has an n th root of M_0 , so any discrete valuation must be one on these elements.

Assistant: You’re basically right: . . . Let me write it cleanly....

Assistant: **Conclusion** As an abstract field, M_0 has a **unique nontrivial discrete valuation up to scaling**, namely ord_0 .

So your instinct is correct: my earlier “many valuations” intuition conflated *valuations* with *discrete valuations*; in this 1-dimensional henselian-ish setting, discreteness forces the valuation to be the obvious one.

User: You did write above, “Then uniqueness fails, and there are many other discrete valuations on M_0 —” Was that wrong? If so, where was your argument wrong?

Assistant: Yes — that earlier statement was wrong **for** M_0 .

Where the argument went off the rails:

Assistant: **What I said then (the flawed idea)** I sketched: ...

Two separate gaps were hiding there: So any attempted extension of a “new” discrete valuation from some subfield must, upon extension to all of M_0 , **either**

- collapse to a multiple of ord_0 , **or**
- become **non-discrete** (value group no longer \mathbf{Z}).

Assistant: **Exactly what was wrong in the earlier argument.** The incorrect step was the implicit leap:

“Extend a discrete valuation from a subfield to M_0 ” \Rightarrow “the extension is still discrete.”

That implication is false in general, and for M_0 it’s *especially* false....

Assistant: So: **I was wrong** to suggest “many discrete valuations” on M_0 as an abstract field. The correct statement is:

M_0 has a **unique nontrivial discrete valuation up to scaling**, namely ord_0 .