Math 213A F23 Homework 9 Solutions

Dhruv Goel

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Show that if $f, g \in S$ and $f(\Delta)$ contains $g(\Delta)$, then $f = g$.

Solution. If $g(\Delta) \subset f(\Delta)$, then $h := f^{-1} \circ g : \Delta \to \Delta$ is a holomorphic map with

$$h'(0) = (f^{-1})'(g(0)) \cdot g'(0) = \frac{1}{f'(0)} \cdot g'(0) = 1 \cdot 1 = 1,$$

so by the Schwarz Lemma, we must have $h = \mathrm{id}_\Delta$, and hence that $f = g$. ■

Q2. Show that if $f \in \Sigma$ and w is not in the image of f, then $|w| \leq 2$. (Hint: apply the bound $|b_1| \leq 1$ to the function $\sqrt{f(z^2) - w}$.)

Solution. Let $\mathcal{S}$ denote the class of schlicht functions defined in the lecture. We first have:

Lemma 0.0.1. If $p \in \mathcal{S}$ is any function, there is an odd $q \in \mathcal{S}$ such that $p(z^2) = q(z)^2$.

Proof. The function $g : \Delta \to \mathbb{C}^*$ defined by $g(z) = p(z^2)/z^2$ (and $g(0) = 1$) is an even holomorphic function, so, since Δ is simply connected, admits an unique holomorphic square root say $h(z)$ with $h(0) = 1$; then, we have

$$h(-z)^2 = g(-z) = g(z),$$

and so by uniqueness, we must have $h(z) = h(-z)$, i.e. h itself is even. Take $q(z) := zh(z)$. This is clearly odd, and satisfies $q(0) = 0$ and $q'(0) = h(0) = 1$ with $q(z)^2 = p(z^2)$. Finally, to check that $q(z)$ is univalent, note that if $z, w \in \Delta$ are such that $q(z) = q(w)$, then

$$p(z^2) = q(z)^2 = q(w)^2 = p(w^2),$$

so that from $p \in \mathcal{S}$ we have $z^2 = w^2$. If $z = w$, we are done; else, if $z = -w$, then since q is odd, we get that

$$q(z) = q(w) = -q(z)$$

so that $q(z) = 0$. Then,

$$p(z^2) = q(z)^2 = 0,$$

and again, from univalence of p, we conclude that $z^2 = 0$, so $z = w = 0$. ■

If $p(z) = z + \sum_2^\infty a_n z^n$, then

$$q(z) = z\left(1 + \sum_2^\infty a_n z^{2n-2}\right)^{1/2} = z + \frac{a_2}{2} z^3 + \mathcal{O}(z^5)$$

around $z = 0$. This lemma was used in the proof of Theorem 4.18 in the notes dated 11/15/23.

Now we prove the main result. Since $f(\infty) = \infty$, we have $w \in \mathbb{C}$. Apply the lemma to the function $p \in \mathcal{S}$ defined by

$$p(z) := \frac{1}{f(1/z) - w} = z + wz^2 + \mathcal{O}(z^3)$$

to get an odd $q \in S$ such that $p(z^2) = q(z)^2$; defining $\varphi(z) := 1/q(1/z)$, we get a function $\varphi \in \Sigma$ with the property that $\varphi(z)^2 = f(z^2) - w$. It follows that, irrespective of $f(z)$, we have around ∞ that

$$\varphi(z) = z - \frac{w}{2}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^3}\right).$$

Therefore, from the result $|b_1| \leq 1$ for functions in Σ, we conclude that $|w| \leq 2$ as needed. ■

Remark 1. We can also argue for the existence of $\sqrt{f(z^2) - w}$ more directly topologically. Namely, the map $g : \mathbb{C} \setminus \overline{\Delta} \to \mathbb{C}^*$ given by $g(z) = f(z) - w$ is a holomorphic map that behaves like $f(z) \approx z$ near $z = \infty$; therefore, for $R \gg 1$, the map $g|_{S^1(R)} : S^1(R) \to \mathbb{C}^*$ induces an isomorphism on fundamental groups, and hence so does g. Therefore, since the maps given by $z \mapsto z^2$ provide the two sheeted covers of $\mathbb{C} \setminus \overline{\Delta}$ and $\mathbb{C}^*$ respectively, it follows that the map $h : \mathbb{C} \setminus \overline{\Delta} \to \mathbb{C}^*$ given by $h(z) = g(z^2) = f(z^2) - w$ satisfies $h(\pi_1(\mathbb{C} \setminus \overline{\Delta})) \subset 2\pi_1(\mathbb{C}^*)$ and hence by covering space theory lifts to give a square root $\varphi : \mathbb{C} \setminus \Delta \to \mathbb{C}^*$. It then remains to check that plugging $\varphi(\infty) := \infty$ back in makes φ into an element of Σ.

Remark 2. In any case, careful justification must be given for the existence and univalence of $\sqrt{f(z^2) - w}$. I have taken points off from any answer that does not provide such a justification.

Q3. For $a > b > 0$ consider the ellipse $E \subset \mathbb{C}$ with major axis $[-a, a]$ and minor axis $[-\mathrm{i}b, \mathrm{i}b]$. Let $I \subset [-a, a]$ be the segment joining the foci of E. Finally, let B be the annular region between E and I. Find an explicit conformal map $f : A(R) \to B$ for some $R > 1$. (Hint: think about $z + 1/z$.)

Solution. The map $z \mapsto z + 1/z$ takes for $R > 1$ the circle $S^1(R)$ to the parametrized curve

$$Re^{\mathrm{i}\theta} \mapsto Re^{\mathrm{i}\theta} + \frac{1}{Re^{\mathrm{i}\theta}} = \left(R + \frac{1}{R}\right)\cos\theta + \mathrm{i}\left(R - \frac{1}{R}\right)\sin\theta,$$

which is an ellipse with major axis $[-(R+R^{-1}), R+R^{-1}]$ and minor axis $[-\mathrm{i}(R-R^{-1}), \mathrm{i}(R-R^{-1})]$, with focii at ± 2. It also takes $S^1 = S^1(1)$ to the interval $[-2, 2]$, and it follows easily that it takes the region $A(R)$ biholomorphically to the annular region between the boundary and the focal segment of this ellipse. Since the foci of the ellipse $E_{a,b}$ defined above are at $\pm\sqrt{a^2 - b^2}$, the map $f : A(R) \to B$ must be given explicitly by

$$f(z) = \frac{\sqrt{a^2 - b^2}}{2}\left(z + \frac{1}{z}\right),$$

and $R > 1$ has to satisfy

$$a = \frac{\sqrt{a^2 - b^2}}{2}\left(R + \frac{1}{R}\right),$$

which gives us

$$R = \sqrt{\frac{a+b}{a-b}} = \sqrt{\frac{1 + \sqrt{1 - e^2}}{1 - \sqrt{1 - e^2}}},$$

where $e := \sqrt{1 - (b/a)^2}$ is the eccentricity of the ellipse. (We knew already that R had to be a function of the eccentricity only–why?) ∎

Q4. Let $f : \Delta \to \mathbb{C}$ be analytic and suppose that $\int_\Delta |f'(z)||\mathrm{d}z|^2 < \infty$.[1] Prove that

$$F(z) = \lim_{r\to 1^-} f(rz)$$

exists and is finite for almost every $z \in S^1$.

Solution. For $r \in [0,1]$, define the function $\ell_r : S^1 \to [0,\infty]$ by

$$\ell_r(z) = \int_0^r |f'(\rho z)|\,\mathrm{d}\rho$$

which is the length of the radial segment $[0,r)z$ under f. For each $r \in [0,1)$, the function ℓ_r is $[0,\infty)$-valued and continuous; therefore, the function $\ell := \ell_1 = \sup_{r\in[0,1)} \ell_r$ is certainly measurable. Note that if for some $z \in S^1$, the value $\ell(z) < \infty$, then the Dominated Convergence Theorem applied to the collection of functions $\{\rho \mapsto f'(\rho z)\cdot \mathbb{1}_{[0,r)}(\rho)\}_{r\in[0,1)}$ on $[0,1)$ gives us that

$$F(z) = \lim_{r\to1^-} f(rz) = \lim_{r\to1^-}\left(f(0) + z\int_0^1 f'(\rho z)\cdot \mathbb{1}_{[0,r)}(\rho)\,\mathrm{d}\rho\right) = f(0) + z\int_0^1 f'(\rho z)\,\mathrm{d}\rho$$

exists and is finite. Therefore, it remains to show that $\ell(z) < \infty$ for almost all $z \in S^1$, for which by standard real analysis it suffices to show:

Lemma 0.0.2. The function ℓ is in $L^1(S^1)$, i.e.

$$\int_{S^1} \ell(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta) < \infty,$$

where μ is the Lebesgue measure on S^1.

Proof. This integral can be broken up as

$$\int_0^{2\pi}\int_0^1 |f'(\rho\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^{1/2} |f'(\rho\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{2\pi}\int_{1/2}^1 |f'(\rho\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\rho\,\mathrm{d}\theta,$$

where the first term on the right is clearly finite by the continuity of the integrand and compactness of $\overline{\Delta(1/2)}$. The second term is bounded above by

$$2\int_0^{2\pi}\int_{1/2}^1 |f'(\rho\mathrm{e}^{\mathrm{i}\theta})|\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = 2\int_{\Delta\setminus\Delta(1/2)} |f'(z)|\,|\mathrm{d}z|^2 < 2\int_\Delta |f'(z)|\,|\mathrm{d}z|^2 < \infty,$$

where the application of Fubini's Theorem in the first step is justified by the nonnegativity of the integrand. ■

Remark 3. This is essentially a property of the Hardy spaces H^p defined on an earlier solution set; c.f. [1, Thm. 17.11].

[1] There is a typo in the notes–when integrating against Δ, the integrand should have an area element $|\mathrm{d}z|^2$. If you interpreted the problem to have the hypothesis $\int_\Delta |f'(z)|^2|\mathrm{d}z|^2 < \infty$, that's fine: the Cauchy-Schwarz inequality implies in this case that

$$\left(\int_\Delta |f'(z)|\,|\mathrm{d}z|^2\right)^2 \le \pi\int_\Delta |f'(z)|^2\,|\mathrm{d}z|^2,$$

so we are again reduced to the hypothesis above. The hypothesis $\int_\Delta |f'(z)|^2|\mathrm{d}z|^2$ has a more geometric interpretation since this quantity (up to constants) is the area $\mathrm{Area}(f(\Delta))$ of the image of the Δ under f; this result can be then interpreted to say that if the image of Δ under an analytic map $f : \Delta \to \mathbb{C}$ has finite area, then the images of almost all the radial segments under f have finite length. This is part of what goes into the proof of the continuous extension of a Riemann map to the boundary, as discussed in class.

Q5. What are the conformal radii of the following pointed regions?

(a) $(\mathbb{H}, \mathrm{i})$,
(b) $(\{z : -1 < \operatorname{Re} z < 1\}, 0)$,
(c) $(\{z \in \mathbb{H} : -\pi < \operatorname{Re} z < \pi\}, \mathrm{i})$
(d) $(\hat{\mathbb{C}} \smallsetminus [-2,2], \infty)$, and
(e) (S_α, r) where $r > 0$ and $S_\alpha := \{z : \arg(z) \in (-\alpha, \alpha)\}$.

Solution. Here we collect some general remarks about conformal radii.

(i) Here's one definition of the conformal radius amenable to computation: if (U,p) is a simply connected proper open subset of the plane and if $f : (\Delta, 0) \to (U,p)$, is any biholomorphism, then the conformal radius of the pointed region (U,p) is defined to be

$$r(U,p) := |f'(0)|,$$

and this definition is independent of the choice of f by the Schwarz lemma. This definition is clearly equivalent to the one in the notes (why?).

(ii) If (U,p) and (V,q) are two such regions, and $g : (U,p) \to (V,q)$ a biholomorphism, then we have from (i) that

$$r(V,q) = |g'(p)| \cdot r(U,p).$$

(iii) If $\varphi \in \mathrm{PU}_2$ is any isometry of the Riemann sphere that takes U to another subset $\varphi(U)$ of $\mathbb{C}$, then we clearly have $r(U,p) = r(\varphi(U), \varphi(p))$, because $|\varphi'(z)| = 1$ for all $z \in \mathbb{C}$ with $\varphi(z) \in \mathbb{C}$. This observation allows us to extend the notion to subsets of $\hat{\mathbb{C}}$: if $(U,p) \subset \hat{\mathbb{C}}$ instead (still simply connected, and missing at least two points), then we can take U to a simply connected proper subset of $\mathbb{C}$ via an isometry $\varphi \in \mathrm{PU}_2$, and we can define $r(U,p) := r(\varphi(U), \varphi(p))$, with this radius being independent of the choice of φ by the above remark.

Now let's compute these radii.

(a) More generally, for $t \in (0, \infty)$, the map $f_t : (\Delta, 0) \to (\mathbb{H}, \mathrm{i}t)$ given by

$$f_t(z) = \mathrm{i}t \left(\frac{1+z}{1-z} \right)$$

is a biholomorphism, so that

$$r(\mathbb{H}, \mathrm{i}t) = |f_t'(0)| = 2t.$$

Therefore, by a horizontal translation and property (ii), we get that $r(\mathbb{H}, \alpha) = 2 \operatorname{Im} \alpha$ for any $\alpha \in \mathbb{H}$. In particular, $r(\mathbb{H}, \mathrm{i}) = 2$.

(b) The map $f : (\Delta, 0) \to (\{z : -1 < \operatorname{Re} z < 1\}, 0)$ given by

$$f(z) = \frac{2}{\mathrm{i}\pi} \log \left(\frac{1+z}{1-z} \right)$$

is a biholomorphism (where the branch of log is chosen so $f(0) = 0$)[2], and hence

$$r\left(\{z : -1 < \operatorname{Re} z < 1\}, 0\right) = |f'(0)| = \frac{4}{\pi}.$$

(c) Let's call this half-strip T; we showed in class that the map $g : T \to \mathbb{H}$ defined by $g(z) = \sin(z/2)$ is a biholomorphism. Therefore, by using the result from (ii) and (a) above, we have that

$$r(T, \mathrm{i}) = \frac{1}{|g'(i)|} r\left(\mathbb{H}, \sin\left(\frac{\mathrm{i}}{2}\right)\right) = \frac{2}{|\cos(i/2)|} \cdot 2 \operatorname{Im} \sin\left(\frac{\mathrm{i}}{2}\right) = \frac{2}{\cosh(1/2)} \cdot 2 \sinh\left(\frac{1}{2}\right) = 4 \tanh\left(\frac{1}{2}\right).$$

(d) Here, we are in the situation of (iii) above, and we choose $\varphi(z) := 1/z$. We discussed in class that the map $f : \Delta \to \hat{\mathbb{C}} \smallsetminus [-2,2]$ given by $f(z) = z + z^{-1}$ is a biholomorphism. The composite is given by $\varphi \circ f(z) = (z + z^{-1})^{-1} = z(1+z^2)^{-1}$ and hence from (i) and (iii) above, we have

$$r(\hat{\mathbb{C}} \smallsetminus [-2,2], \infty) = (\varphi \circ f)'(0) = 1.$$

[2] See the discussion in my solutions to questions 1 and 2 of Homework 5 if you need a reminder on the properties of f.

(e) The map $f : (\Delta, 0) \to (S_\alpha, r)$ given by

$$f(z) = r\left(\frac{1+z}{1-z}\right)^{2\alpha/\pi}$$

is a biholomorphism (where the branch is chosen to so that $f(0) = r$), and hence

$$r(S_\alpha, r) = |f'(0)| = r \cdot \frac{2\alpha}{\pi} \cdot 2 = \frac{4\alpha r}{\pi}.$$

■

Remark 4. Many submissions got the computation of $r(\hat{\mathbb{C}} \setminus [-2, 2], \infty)$ wrong. You cannot take derivatives around ∞ the same way you do in the finite plane–you need to choose coordinates to move things to the finite plane. If f is a function defined and holomorphic (in particular $\mathbb{C}$-valued) in a neighborhood of ∞, the correct definition of $f'(\infty)$ is not $\lim_{z\to\infty} f'(z)$ but rather $\lim_{z\to\infty} z(f(z) - f(\infty))$, which is the derivative of $f(1/z)$ around $z = 0$. (This is an important idea that shows up when you think about analytic capacities and extensions of holomorphic functions, etc.) In particular, if you take the definition of the conformal radius to be the following: if $f : (U, p) \to (\Delta, 0)$ is a biholomorphism, then $r(U, p) := 1/|f'(p)|$, then you can use this to compute $r(\hat{\mathbb{C}} \setminus [-2, 2], \infty)$ as well; indeed, we can take $f : \hat{\mathbb{C}} \setminus [-2, 2] \to (\Delta, 0)$ given by

$$f(z) := \frac{z - \sqrt{z^2 - 4}}{2} = \frac{2}{z + \sqrt{z^2 - 4}},$$

and then we have

$$f'(\infty) := \lim_{z\to\infty} zf(z) = 1.$$

Q6. For $t \geq 0$, let $U_t = \mathbb{H} \cup (-\mathbb{H}) \cup (-t, t)$. Let $f_t : (\mathbb{H}, \mathrm{i}) \to (U_t, \mathrm{i})$ be the Riemann mapping, normalized so that $f_t'(\mathrm{i}) > 0$.

(a) Find $f_t(z)$ explicitly for $t > 0$.
(b) Show that as $t \to 0$, [we have] $f_t(z) \to f_0(z) = z$ in $\mathcal{O}(\mathbb{H})$.
(c) Note that $-\mathbb{H} \subset f_t(\mathbb{H})$ for all $t > 0$, but $-\mathbb{H}$ is disjoint from $f_0(\mathbb{H})$. How is this possible if $f_t \to f_0$?

Solution.

(a) Note that the map $f : \mathbb{H} \to U_2$ given by $f(z) = z + z^{-1}$ is biholomorphic and has the property that $f'(\mathrm{i}t) > 0$ for all $t \in (0, \infty)$. It maps the exterior $\mathbb{H} \setminus \overline{\Delta}$ of the unit disc in $\mathbb{H}$ to $\mathbb{H}$, the semicircle $\mathbb{H} \cap S^1$ to $(-2, 2)$ and the interior $\mathbb{H} \cap \Delta$ to $-\mathbb{H}$. Further, it maps the upper half of the y-axis $\mathrm{i}(0, \infty)$ isomorphically to the y-axis, and has the property that $\mathrm{i}(1, \infty)$ maps to the upper half y-axis, and $\mathrm{i}(0, 1)$ maps to the lower half of the y-axis. See Figure (1).

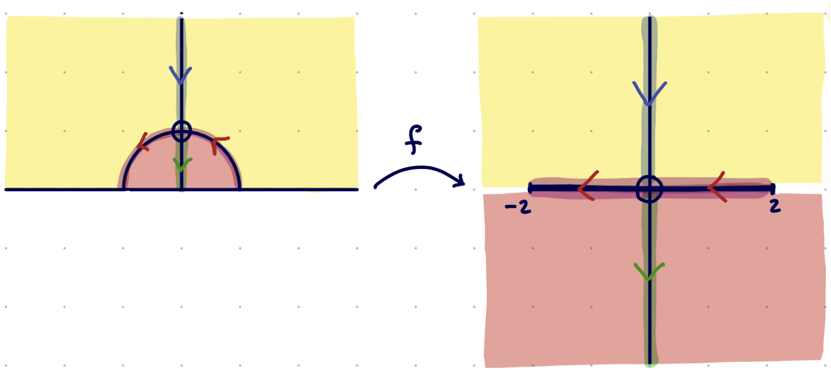

Figure 1: The mapping f.

Therefore, the map $(t/2) \cdot f : \mathbb{H} \to U_t$ is biholomorphic. To normalize this as needed, we note that, for $t > 0$, the preimage of $2t^{-1}\mathrm{i}$ under f is $f^{-1}(2t^{-1}\mathrm{i}) = \lambda_t \mathrm{i}$, where

$$\lambda_t := \frac{\sqrt{t^2+1} + t + 1}{\sqrt{t^2+1} + t - 1} > 1,$$

and hence the explicit formula is simply

$$f_t(z) = \frac{t}{2}\left(\lambda_t z + \frac{1}{\lambda_t z}\right).$$

(b) Note that as $t \to 0$, we have $\lambda_t \sim 2t^{-1}$, which is to say that $\lim_{t \to 0} t\lambda_t = 2$. Now if $K \subset \mathbb{H}$ is compact, find an $R \gg 1$ such that $K \cup (1/K) \subset \Delta(R)$[3], and then use that for any $t > 0$ we have

$$\sup_{z \in K} |f_t(z) - z| = \sup_{z \in K} \left| \left(\frac{1}{2} t\lambda_t - 1\right) z + \frac{t}{2\lambda_t} \frac{1}{z} \right| \leq \left(\left| \frac{1}{2} t\lambda_t - 1 \right| + \frac{t}{2\lambda_t} \right) R,$$

so that

$$\lim_{t \to 0} \sup_{z \in K} |f_t(z) - z| = 0.$$

Since this is true for every compact $K \subset \mathbb{H}$, we have shown that f_t converges uniformly on compact sets to $f_0(z)$ as $t \to 0$, i.e. that $f_t(z) \to f_0(z) = z$ in $\mathcal{O}(\mathbb{H})$.

(c) This is not a contradiction of any sort: we have

$$f_t^{-1}(-\mathbb{H}) = \mathbb{H} \cap \Delta(\lambda_t^{-1}),$$

which is getting smaller as $t \to 0^+$. As $t \to 0^+$, the Riemann map f_t is trying to funnel the smaller and smaller inner half-discs to go and occupy the lower half plane $-\mathbb{H}$, while stabilizing on most of the upper half-plane (uniformly on compact subsets of it) to the identity map. ■

[3]Here $1/K := \{1/z : z \in K\}$.

Q7. Is there a univalent map $f : \mathbb{C} \smallsetminus \overline{\Delta} \to \mathbb{C}$ of the form $f(z) = z + \sum_1^\infty b_n/z^n$ with $b_2 = 1/\sqrt{2}$?

Solution. No. Suppose f were such; then the extension $f : \hat{\mathbb{C}} \smallsetminus \overline{\Delta} \to \hat{\mathbb{C}}$ of f defined by $f(\infty) = \infty$ is in the class Σ.[4] By Corollary 4.16 in the notes dated 11/15/23, we have $\sum_1^\infty n|b_n|^2 \leq 1$, so that if $b_2 = 1/\sqrt{2}$, then $b_n = 0$ for all $n \neq 2$. It follows that f must be given by

$$f(z) = z + \frac{1}{\sqrt{2}z^2},$$

but this function is not univalent on $\mathbb{C} \smallsetminus \overline{\Delta}$; indeed, the derivative

$$f'(z) = 1 - \frac{\sqrt{2}}{z^3}$$

vanishes at say $z = \sqrt[6]{2} \in \mathbb{C} \smallsetminus \overline{\Delta}$, and hence f is not injective in any neighborhood of $\sqrt[6]{2}$ in $\mathbb{C} \smallsetminus \overline{\Delta}$. ■

[4] If it makes you more comfortable, you could analyze the structure of the isolated singularity at $z = 0$ of $f(1/z) = z^{-1} + \sum_1^\infty b_n z^n$; here, we must have $\overline{\lim}_{n\to\infty} |b_n|^{1/n} \leq 1$, and this function clearly has a simple pole at $z = 0$ with residue 1.

References

[1] W. Rudin, *Real and Complex Analysis.* McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Company, third ed., 1987.