Math 213A F23 Homework 7 Solutions

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Given $R > 1$, let $A_R = \{z : 1 < |z| < R\}$. Let $f : A_R \to A_S$ be a bijective holomorphic map. Prove that $R = S$. (Hint: apply Schwarz reflection to extend f to $\mathbb{C}^*$.)

We'll use the following notation: for $0 < p < q < \infty$, we let $A(p, q) := \{z : p < |z| < q\}$ and $S^1(p) = \{z : |z| = p\}$, so that $\partial A(p, q) = S^1(q) - S^1(p)$ and $A_R = A(1, R)$.

Proof. Note that f is a biholomorphism, and hence in particular a homeomorphism and hence proper. For any σ such that $1 < \sigma < S$, the preimage $f^{-1}(S^1(\sigma))$ is compact, so there is a ρ such that $1 < \rho < R$ such that $A_\rho \subset A_R \setminus f^{-1}(S^1(\sigma))$. Since A_ρ is connected, it must map to a connected component of $A_S \setminus S^1(\sigma)$, and so either $|f(z)| < \sigma$ for all $z \in A_\rho$ or $|f(z)| > \sigma$ for all $z \in A_\rho$. In the first case, properness implies that $|f(z)| \to 1$ as $|z| \to 1$ in A_R, whereas in the second case, properness implies $|f(z)| \to S$ as $|z| \to 1$ in A_R. In the second case, we may replace f by the function $z \mapsto S/f(z)$ to reduce to the first case, and hence we may assume without loss of generality that $|f(z)| \to 1$ as $|z| \to 1$ in A_R. By the same reasoning, either $|f(z)| \to 1$ as $|z| \to R$ in A_R, or $|f(z)| \to S$ as $|z| \to R$ in A_R, but in fact the second must hold: indeed, from $|f(z)| \to 1$ as $|z| \to 1$ in A_R, it follows that $|f^{-1}(w)| \to 1$ as $|w| \to 1$ in A_S, and so if $|f(z)| \to 1$ as $|z| \to R$ in A_R, then taking $w = f(z)$ gives $|z| = |f^{-1}(w)| \to 1$ as $|z| \to R$ in A_R, which is absurd.

To reduce clutter, we introduce some terminology.

Definition 0.0.1. Let $p, q, u, v \in (0, \infty)$ be such that $p < q$ and $u < v$. A biholomorphism $f : A(p, q) \to A(u, v)$ is said to be acceptable if $|f(z)| \to u$ as $|z| \to p$ in $A(p, q)$, and $|f(z)| \to v$ as $|z| \to q$ in $A(p, q)$.

Now we use the Schwarz reflection principle in the following form.

Lemma 0.0.2. Let $p, q, u, v \in (0, \infty)$ be such that $p < q$ and $u < v$. Define $s := q/p$ and $t := v/u$, and note that $s, t \in (1, \infty)$. If $f : A(p, q) \to A(u, v)$ is an acceptable biholomorphism, then for any $n \geq 0$, the function f extends to an acceptable biholomorphism

$$f : A\left(p \cdot s^{-(3^n-1)/2}, q \cdot s^{(3^n-1)/2}\right) \to A\left(u \cdot t^{-(3^n-1)/2}, v \cdot t^{(3^n-1)/2}\right).$$

Proof. The statement is set up in such a way that, by induction, it suffices to check $n = 1$ (verify!), where it is just the usual Schwarz reflection principle in the inner and outer boundary circles (check!), and the extension is defined by

$$f(z) = u^2 \overline{f(p^2 \overline{z}^{-1})}^{-1} = v^2 \overline{f(q^2 \overline{z}^{-1})}^{-1}.$$

These formulae also guarantee that the extended f is still a biholomorphism (why?). ■

Let's get back to our original problem. The first paragraph shows that $f : A_R \to A_S$ is an acceptable biholomorphism. Applying the lemma to $p = u = 1$ and $q = R, v = S$, we get that for any $n \geq 0$, the function f extends to an acceptable biholomorphism

$$f : A(R^{(-3^n+1)/2}, R^{(3^n+1)/2}) \to A(S^{(-3^n+1)/2}, S^{(3^n+1)/2}).$$

Since

$$\bigcup_{n \geq 0} A(R^{(-3^n+1)/2}, R^{(3^n+1)/2}) = \bigcup_{n \geq 0} A(S^{(-3^n+1)/2}, S^{(3^n+1)/2}) = \mathbb{C}^*,$$

it follows that f extends to a biholomorphism $f : \mathbb{C}^* \to \mathbb{C}^*$ with further the property that $|f(z)| \to 0$ as $|z| \to 0$ and $|f(z)| \to \infty$ as $|z| \to \infty$. Therefore, defining $f(0) = 0$ and $f(\infty) = \infty$, the function f extends (by say Riemann's Removable Singularities Theorem) to a biholomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that fixes 0 and ∞. But we know all the biholomorphisms of $\hat{\mathbb{C}}$: these are all Möbius transformations, and it is easy to check that all Möbius transformations that fix 0 and ∞ are of the form $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}^*$, and thus f itself must be of this form. But then $|f(z)| \to 1$ as $|z| \to 1$ implies that $|\lambda| = 1$, so that $|f(z)| \to S$ as $|z| \to R$ implies that $R = S$ as needed.

■

Aliter. (See [1, p. 334]) By the same initial considerations as in the previous solutions, f is an acceptable biholomorphism. Consider the function $u : A_R \to \mathbb{R}$ defined by

$$u(z) := \log|f(z)| - \lambda \log|z|,$$

where $\lambda := \log_R S > 0$. This is a harmonic function, and by the acceptability of f extends continuously to a function $\overline{A_R} \to \mathbb{R}$, with $u|_{\partial A_R} = 0$. It follows by the Maximum Principle for harmonic functions that $u \equiv 0$. Taking the derivative $\partial/\partial z$ of both sides of $u \equiv 0$, we conclude that in A_R, we have

$$\frac{f'(z)}{f(z)} - \frac{\lambda}{z} = 0.$$

It follows that for any ρ with $1 < \rho < R$, we have

$$\lambda = \frac{1}{2\pi \mathrm{i}} \oint_{S^1(\rho)} \frac{f'(z)}{f(z)} \mathrm{d}z = W(f(S^1(\rho)), 0)$$

where latter quantity is the winding number of $f(S^1(\rho))$ around the origin, and hence in particular an integer. It follows that $\lambda \geq 1$, showing that $S \geq R$. The same result applied to f^{-1} proves that $R \geq S$, showing that $R = S$ as needed. ■

Remark 1. Showing that, as $|z_n| \to 1$ for some sequence $z_n \in A_R$, the $\lim_{n\to\infty} |f(z_n)|$ exists and belongs to $\{1, S\}$ (and similarly for $|z_n| \to R$) is a crucial part of the argument. I have taken points off from any argument to this effect that fails to be airtight. Quite a few submissions argued that, because f is a biholomorphism, it extends continuously to the closures; this is not true in general–we constructed examples in the class of domains with non locally connected boundaries such that the Riemann map from the disk to the domain does not extend continuously to the closure. Similarly, the boundedness of f is not sufficient to imply that it extends continuously to the boundary: for instance, $\exp(1/(z-1))$ is a bounded holomorphic function on Δ that does not extend continuously to the boundary.

Q2. Compute

$$\oint_{|z|=1/2} \frac{\mathrm{d}z}{z\sin^2(z)}.$$

Solution. The zeroes of $z\sin^2(z)$, and hence the poles of the integrand, are at $\pi\mathbb{Z}$, and of these none lies on the circle $S^1(1/2)$ and only $z = 0$ lies inside the disk of radius $1/2$. It follows from the Residue Theorem that

$$\oint_{|z|=1/2} \frac{\mathrm{d}z}{z\sin^2(z)} = 2\pi\mathrm{i} \operatorname*{Res}_{z=0} \frac{1}{z\sin^2(z)}.$$

Note that around $z = 0$, we have

$$\frac{\sin z}{z} = 1 - \frac{z^2}{6} + [z^4]$$

so that we have the Laurent expansion

$$\frac{1}{z\sin^2(z)} = \frac{1}{z^3}\left(1 - \frac{z^2}{6} + [z^4]\right)^{-2} = \frac{1}{z^3}\left(1 + \frac{z^2}{3} + [z^4]\right) = \frac{1}{z^3} + \frac{1}{3}\cdot\frac{1}{z} + [z],$$

so that the residue, which is the coefficient of $1/z$ is exactly $1/3$. By combining these results, we conclude that

$$\oint_{|z|=1/2} \frac{\mathrm{d}z}{z\sin^2(z)} = \frac{2\pi\mathrm{i}}{3}.$$

■

Q3. Find all proper holomorphic maps

(a) $\mathbb{C}^* \to \mathbb{C}$, and
(b) $\mathbb{C} \to \mathbb{C}^*$.

Solution.

(a) Let $f : \mathbb{C}^* \to \mathbb{C}$ be a proper holomorphic map. By properness, $f(z) \to \infty$ as $z \to 0, \infty$ in $\mathbb{C}^*$, so by Riemann's Removable Singularities Theorem (how?), f extends to a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with poles exactly at $z = 0$ and $z = \infty$. By subtracting away the principal parts at 0 and ∞, we obtain a bounded entire function, which is then a constant by Liouville's Theorem; it follows that any such rational map can be written as a finite Laurent polynomial of the form

$$f(z) = \sum_{j=-n}^{m} a_j z^j$$

for some $m, n \geq 1$ and $a_j \in \mathbb{C}$ with $a_{-n}, a_m \neq 0$, and the same reasoning shows that any such expression defines proper holomorphic map $\mathbb{C}^* \to \mathbb{C}$.

(b) There are no proper holomorphic maps $\mathbb{C} \to \mathbb{C}^*$.

Proof 1. Suppose that $f : \mathbb{C} \to \mathbb{C}^*$ is a proper holomorphic map. By lifting to the universal cover, we can write $f(z) = e^{g(z)}$ for some entire $g : \mathbb{C} \to \mathbb{C}$. We claim that g is proper; indeed, if $K \subset \mathbb{C}$ is compact, then so is $\exp(K) \subset \mathbb{C}^*$, and then $g^{-1}(K) \subset f^{-1}(\exp(K))$ along with the compactness of $f^{-1}(\exp(K))$ implies that $g^{-1}(K)$ is compact as well. In particular, g is surjective and for any $t \in \mathbb{C}^*$, we have $f^{-1}(t) = g^{-1}(\log t)$ where the set $\log t := \{w \in \mathbb{C} : e^w = t\}$ is infinite, so it follows that every fiber of f is infinite, and f can't be proper. This is a contradiction.

Proof 2. Suppose $f : \mathbb{C} \to \mathbb{C}^*$ is a proper holomorphic map. Note Since $f^{-1}(S^1)$ is compact, there is an $R \gg 1$ such that $f^{-1}(S^1) \subset \Delta(R)$. Then f maps $\mathbb{C} \setminus \Delta(R)$ into a connected component of $\mathbb{C}^* \setminus S^1$, i.e. into one of Δ^* and $\mathbb{C} \setminus \overline{\Delta}$. By replacing f by $1/f$ if needed, we may assume that f maps $\mathbb{C} \setminus \Delta(R)$ into Δ^*. This tells us that f is bounded (by $\max\{1, \sup_{|z| \leq R} |f(z)|\}$), and is hence constant, by Liouville's Theorem; but a constant map $f : \mathbb{C} \to \mathbb{C}^*$ cannot be proper.

Proof 3. Suppose $f : \mathbb{C} \to \mathbb{C}^*$ is any nonconstant holomorphic map. We claim that f has an essential singularity at ∞, i.e. that $f(1/z)$ has an essential singularity at zero–indeed, by Liouville's Theorem, f can't be bounded in a neighborhood of ∞, but then neither can $1/f$, so the singularity at ∞ can neither be removable nor a pole. It follows from the Casorati-Weierstrass Theorem that for any $R > 0$, the image of $\mathbb{C} \setminus \Delta(R)$ under f is dense in $\mathbb{C}^*$, and hence if $K \subset \mathbb{C}^*$ is any compact set with nonempty interior, then its preimage $f^{-1}(K)$ is unbounded, and hence, in particular, not compact. Therefore, f can't be proper. Since constant maps can't be proper either, we are done.

■

Remark 2. In fact, you can show using Picard's Little Theorem that all the fibers of any nonconstant holomorphic map $\mathbb{C} \to \mathbb{C}^*$ must be infinite, and this gives yet another way to see that there are no proper holomorphic maps $\mathbb{C} \to \mathbb{C}^*$. See Chapter 4, Exercise 32 in the version of the notes dated 11/02/23.

Remark 3. Quite a few solutions failed to correctly answer (b), "producing" proper holomorphic maps $\mathbb{C} \to \mathbb{C}^*$. A common mistake was to argue that $e^z \to \infty$ as $z \to \infty$ in $\mathbb{C}$, which is not true: e^z has an essential singularity at ∞ and does not limit to any fixed value as $|z| \to \infty$. Students who submitted such answers are encouraged to compare their answers against the various proofs above to see what went wrong. Another common mistake was to argue that a proper map $f : \mathbb{C} \to \mathbb{C}^*$ must be proper when considered as a map $i \circ f : \mathbb{C} \to \mathbb{C}$, but this is not correct, since $(i \circ f)^{-1}(\overline{\Delta}) = f^{-1}(\overline{\Delta} \setminus 0)$ has no reason to be compact, say.

Q4. Compute the hyperbolic metric on $\Delta^* = \Delta \setminus \{0\}$.

Solution. The universal cover of Δ^* is given by the map

$$\pi : \mathbb{H} \to \Delta^*, \quad q \mapsto \mathrm{e}^{2\pi \mathrm{i} q}$$

which has group of deck transformations $\mathbb{Z}$ acting by $n \cdot z = z + n$.[1] We claim that the metric defined by

$$\rho_{\Delta^*}(z)|\mathrm{d}z| = \frac{|\mathrm{d}z|}{-|z| \log |z|}$$

is the required hyperbolic metric on Δ^*. To show this, it suffices to check that the pullback of this metric under π is the usual hyperbolic metric of $\mathbb{H}$, and we compute this pullback as

$$\pi^*(\rho_{\Delta *}(z)|\mathrm{d}z|) = \rho_{\Delta^*}(\pi(q)) \cdot |\pi'(q)| \cdot |\mathrm{d}q| = \frac{1}{-|\mathrm{e}^{2\pi \mathrm{i} q}| \log |\mathrm{e}^{2\pi \mathrm{i} q}|} \cdot |2\pi \mathrm{i} \mathrm{e}^{2\pi \mathrm{i} q}| \cdot |\mathrm{d}q|.$$

Now $|\mathrm{e}^{2\pi \mathrm{i} q}| = \mathrm{e}^{-2\pi \operatorname{Im} q}$, and hence the above yields

$$\frac{2\pi}{-\log \mathrm{e}^{-2\pi \operatorname{Im} q}} |\mathrm{d}q| = \frac{|\mathrm{d}q|}{\operatorname{Im} q} = \rho_{\mathbb{H}}(q)|\mathrm{d}q|,$$

which is the hyperbolic metric on $\mathbb{H}$ as needed. ■

Remark 4. To come up with the above expression for the hyperbolic metric on Δ^*, you can pull back the hyperbolic metric on $\mathbb{H}$ under local sections of π. These are given by

$$\sigma(z) = \frac{1}{2\pi \mathrm{i}} \log z,$$

where $\log z$ denotes some locally defined branch of the logarithm, and you are welcome to check that locally $\sigma^*(\rho_{\mathbb{H}}(q)|\mathrm{d}q|)$ agrees with our expression above.

[1] As you can check, any map $\mathbb{H} \to \Delta^*$ of the form $q \mapsto \mathrm{e}^{\lambda \mathrm{i} q}$ for $\lambda > 0$ works, and is related to this one by scaling on the domain (which is an isometry, and hence doesn't change the induced metric).The choice of normalization is made so that the deck transformation group $\mathbb{Z}$ acts by translation as indicated.

Q5. Prove that for any nonzero polynomial $p(z)$, the function $e^z - p(z)$ has infinitely many zeroes.

Solution. Recall that the order of an entire function $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$\rho(f) := \inf\{\rho \in \mathbb{R}_{\geq 0} : \exists\, A, B : \forall z \in \mathbb{C}, \text{ we have } |f(z)| \leq A \exp(B|z|^\rho)\}.$$

This satisfies the following properties:

(a) If f is a polynomial function other than $f \equiv 0$, then $\rho(f) = 0$.[2]
(b) If $f = \exp(\alpha z^n)$ for some $n \geq 0$ and $\alpha \in \mathbb{C}^*$, then $\rho(f) = n$.
(c) If f and g are entire, then $\rho(f+g) \leq \max\{\rho(f), \rho(g)\}$, and if $\rho(f) \neq \rho(g)$, then equality holds.

Finally, we quote Hadamard's Factorization Theorem:

Theorem 0.0.3 (Hadamard)**.** Suppose f is an entire function of growth order $\rho < \infty$ that is not identically zero. Let $p := \lfloor \rho \rfloor$. If a_n are the nonzero roots of f counted with multiplicity, then

$$f(z) = z^m e^{Q(z)} \prod_{n=1}^{\infty} E_p(z/a_n),$$

where $m := \operatorname{ord}_0 f$ and $Q(z)$ is a polynomial of degree $\leq p$.

Proof. See [2, Thm. 5.1]. I explain the relationship of this version to Theorem 3.15 in the version of the notes dated 11/02/23 in Remark 5. ■

Now onto the main solution. By properties (a), (b), (c) it follows that $f(z) = e^z - p(z)$ has order 1. Suppose that f has only finitely many zeroes, say $a_1, \ldots, a_n$. Then by Hadamard's Theorem, we have $f(z) = q(z)e^{az}$ for some nonzero polynomial $q(z) \in \mathbb{C}[z]$ and $a \in \mathbb{C}$ (where we have absorbed any constants into $q(z)$), with $a \neq 0$ by $\rho(f) = 1$. It follows inductively that the n^{th} derivative $f^{(n)}(z)$ looks like

$$f^{(n)}(z) = q_n(z)e^{az}$$

for some polynomial $q_n(z) \in \mathbb{C}[z]$, where $q_0(z) = q(z)$ and the polynomials $q_n(z)$ satisfy the recurrence that for $n \geq 0$ we have

$$q_{n+1}(z) = aq_n(z) + q_n'(z).$$

Since $a \neq 0$, each $q_n(z)$ is nonzero and $\deg q_n = \deg q$ is constant. Now for $n \gg 1$ (taking $n > \deg p$ suffices), we have from the expression $f(z) = e^z - p(z)$ that

$$e^z = f^{(n)}(z) = q_n(z)e^{az}.$$

If $\deg q = \deg q_n \geq 1$, then the left hand side never vanishes, whereas the right hand side has some zero, which is a contradiction. Therefore, $q(z)$ is a constant, say $q \in \mathbb{C}$, and each $q_n(z) = a^n q$. Then we may write the above equation as

$$e^{(1-a)z} = a^n q,$$

so, since the right hand side is constant, we must have $a = 1$. Then

$$e^z - p(z) = q \cdot e^z,$$

so that

$$(1-q)e^z = p(z).$$

If $q \neq 1$, then by (b), the left side has order 1, whereas by (a), the right side has order 0, a contradiction. Therefore, $q = 1$, and then $p(z) = 0$, which is a contradiction to the hypothesis that $p(z)$ is a nonzero polynomial.[3] ■

[2] We define $\rho(0) := -\infty$. The converse is, by the way, not true. For instance, $\sum_1^\infty e^{-n^2} z^n$ has order 0.
[3] This is where the proof fails for $p(z) = 0$, and indeed e^z has no zeroes.

Q6. Give an example of a convergent canonical product $f(z) = \prod_1^\infty (1 - z/a_n)$ that has order 1.

Solution. Let (a_n) be a sequence in $\mathbb{C}$ such that $a_n \to \infty$ with critical exponent[4] $\alpha := \alpha(a_n) < \infty$. Let $p \geq 0$ be the unique integer such that $\sum_1^\infty 1/|a_n|^p = \infty$ but $\sum_1^\infty 1/|a_n|^{p+1} < \infty$, so that $p \leq \alpha \leq p+1$. In this case, the product

$$f(z) := \prod_{n=1}^{\infty} E_p(z/a_n)$$

is called the canonical product associated to a_n. Then we have:

Theorem 0.0.4. In the above set-up, the canonical product $f(z)$ converges to an entire function and of order $\rho(f) = \alpha(a_n)$.

Proof. This is Theorem 3.14 in the version of the notes dated 11/02/24. ∎

Therefore, it suffices to produce a sequence (a_n) such that $a_n \to \infty$, the critical exponent $\alpha(a_n) = 1$ and $\sum_1^\infty 1/|a_n| < \infty$, so that $p = 0$. One such sequence is given by $a_n = n(\log n)^2$ (and starting indexing at $n = 2$, say).

- Note that $\alpha(a_n) \leq 1$ is clear because for any $\varepsilon > 0$, the sequence $1/|a_n|^{1+\varepsilon}$ is majorized by $1/n^{1+\varepsilon}$, which converges. On the other hand, we can't have $\alpha(a_n) < 1$, because that would mean that $1/|a_n|^{1-\varepsilon}$ converges for some $\varepsilon \in (0,1)$, which is false: for $n \gg 1$, we have the inequality $(\log n)^{2(1-\varepsilon)} \leq (\log n)^2 \leq n^{\varepsilon/2}$ and hence for $n \gg 1$ we have $1/|a_n|^{1-\varepsilon} \geq 1/n^{1-\varepsilon/2}$, the sum of which cannot converge. Therefore, $\alpha(a_n) = 1$.
- It remains to check that $\sum_2^\infty 1/|a_n| < \infty$. There are many ways to do this: one could use the Integral Test to compare with

$$\int_2^\infty \frac{1}{x(\log x)^2} \mathrm{d}x = -\frac{1}{\log x}\bigg|_2^\infty = \frac{1}{\log 2} < \infty.$$

  Alternatively, one could use the Cauchy Condensation Test to compare with the series

$$\sum_2^\infty \frac{2^n}{|a_{2^n}|} = \sum_2^\infty \frac{2^n}{2^n \cdot (n \log 2)^2} = \frac{1}{(\log 2)^2}\left(\frac{\pi^2}{6} - 1\right) < \infty.$$

∎

Remark 5. The integer p is called the genus of the canonical product associated with the sequence a_n. If the critical exponent $\alpha \geq 0$ is not an integer, then the genus p and critical exponent α are related by $p = \lfloor \alpha \rfloor$. If $\alpha = 0$, then $p = 0$; else if $\alpha \in \mathbb{Z}_{\geq 1}$, then we know only that $p \in \{\alpha, \alpha - 1\}$, and both are possible: if $a_n = n$, then $p = \alpha = 1$, whereas if $a_n = n(\log n)^2$, then $\alpha = 1$ but $p = 0$; this is what the above problem is supposed to illustrate. Note also that this definition of p only works when there are infinitely many a_n; for finitely many a_n (and each nonzero), the above results for α and p don't work the same way. Suppose now that f is any entire function. If f has only finitely many zeroes, we may factor them out, and then Theorem 0.0.3 applies as written, whereas Theorem 3.15 in the notes doesn't. If f has infinitely many zeroes a_n, then they can be ordered so $0 < |a_1| \leq |a_2| \leq \cdots$ (ignoring the zero of finite order at $z = 0$) with $a_n \to \infty$. Then the critical exponent $\alpha(a_n) \leq \rho(f)$. In particular, if $\rho(f) < \infty$, then $p' := \lfloor \rho(f) \rfloor \geq \lfloor \alpha \rfloor$ is certainly at least as big as the genus p of the canonical product associated with a_n, and hence the infinite product $\prod_1^\infty E_{p'}(z/a_n)$ still converges, so Theorem 3.15, which is, so to speak, the sharpest version of the Hadamard Factorization Theorem, implies Theorem 0.0.3 above.

Remark 6. Some answers argued that the example $\sin(\pi z)/(\pi z)$ works, with

$$a_n := \begin{cases} (n+1)/2, & n \notin 2\mathbb{Z}, \\ n/2, & n \in 2\mathbb{Z}. \end{cases}.$$

You can now see from the above remark why this doesn't quite work: this sequence has critical exponent $\alpha = 1$, but genus $p = 1$ as well. Therefore, while the infinite product $\prod_1^\infty (1 - z/a_n)$ converges pointwise, it is not the canonical product associated with this sequence, which is $\prod_1^\infty E_1(z/a_n)$.

[4]Recall the definition: $\alpha(a_n) = \inf\{\alpha > 0 : \sum_1^\infty 1/|a_n|^\alpha < \infty\} \in [0, \infty]$, where the sum ignores the finitely many $a_n = 0$.

References

[1] Z. Nehari, *Conformal Mapping.* Dover Books on Advanced Mathematics, Dover Publications, Inc. New York, 1975.

[2] E. M. Stein and R. Shakarchi, *Complex Analysis.* No. 2 in Princeton Lectures in Analysis, Princeton University Press, 2007.