Math 213A F23 Homework 5 Solutions

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Find a bounded harmonic function $u : \Delta \to \mathbb{R}$ such that $\lim_{r\to 1^-} u(re^{i\theta}) = 1$ when $\theta \in (0, \pi)$ and -1 when $\theta \in (\pi, 2\pi)$.

Solution. The function

$$u(z) = \frac{2}{\pi} \arg\left(\frac{1+z}{1-z}\right) = \frac{2}{\pi} \arctan\left(\frac{2y}{1-x^2-y^2}\right) = \frac{2}{\pi} \arctan\left(\frac{2r\sin\theta}{1-r^2}\right)$$

works, where $z = x + iy = re^{i\theta}$. Here,

$$\arg : \{z : \operatorname{Re} z > 0\} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

is the branch of argument defined by $\arg(u + iv) = \arctan(v/u)$ for $u, v \in \mathbb{R}$ with $u > 0$. This function is bounded because arctan is, it is harmonic because it is the imaginary part of a holomorphic function, namely that of an appropriate branch of

$$\frac{2}{\pi} \log\left(\frac{1+z}{1-z}\right),$$

and you can check directly that

$$\lim_{r\to 1^-} u(re^{i\theta}) = \begin{cases} 0, & \theta \in \{0, \pi\}, \\ 1, & \theta \in (0, \pi), \\ -1, & \theta \in (\pi, 2\pi). \end{cases}$$

∎

Remark 1. Geometrically, the map $z \mapsto (1+z)/(1-z)$ takes Δ biholomorphically to the right half plane $\{z : \operatorname{Re} z > 0\}$, and further takes the upper (resp. lower) boundary of Δ to the strict upper (resp. lower) half of the imaginary axis. Therefore, renormalizing the argument of this map appropriately suffices. Another way to obtain this function is to note that convolution against the Poisson kernel can be defined more generally for an L^1-function h, and that the resulting function has radial limit equal to h at any point of continuity of h; the following theorem gives a precise result.

Theorem 0.0.1. If $h \in L^1(S^1)$, then the function $P[h] : \Delta \to \mathbb{R}$ defined by

$$P[h](re^{i\theta}) := \int_{-\pi}^{\pi} P_r(\varphi)\, h(e^{i(\theta-\varphi)}) \frac{d\varphi}{2\pi}$$

is a harmonic function. Further, if $h \in L^\infty(S^1)$ and h is continuous at a point $e^{i\theta}$,[1] then we have that

$$\lim_{r\to 1^-} P[h](re^{i\theta}) = h(e^{i\theta}).$$

[1] Recall that functions in $L^p(X)$ are almost-everywhere equivalence classes; what this hypothesis is saying that there is some measurable function in the equivalence class represented by h that is continuous at $e^{i\theta}$, and we pick and work with this representative. As in any course on analysis, we blur the distinction between functions and their almost-everywhere equivalence classes.

Proof. For the first part, note that if h is real-valued, then

$$P[h](z) = \mathrm{Re} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\varphi} + z}{\mathrm{e}^{\mathrm{i}\varphi} - z} h(\mathrm{e}^{\mathrm{i}\varphi}) \frac{\mathrm{d}\varphi}{2\pi},$$

and this integral is clearly a holomorphic function of z.[2] For the second part, for a given $\varepsilon > 0$, if we pick a $\delta \in (0, \pi)$ so that $|h(\mathrm{e}^{\mathrm{i}(\theta-\varphi)}) - h(\mathrm{e}^{\mathrm{i}\theta})| \leq \varepsilon$ whenever $|\varphi| \leq \delta$, we have then that

$$h(r\mathrm{e}^{\mathrm{i}\theta}) - h(\mathrm{e}^{\mathrm{i}\theta}) = \int_{-\pi}^{\pi} P_r(\varphi)(h(\mathrm{e}^{\mathrm{i}(\theta-\varphi)}) - h(\mathrm{e}^{\mathrm{i}\theta})) \frac{\mathrm{d}\varphi}{2\pi} = \left(\int_{-\delta}^{\delta} + \int_{|\varphi| \in [\delta, \pi]} \right) P_r(\varphi)(h(\mathrm{e}^{\mathrm{i}(\theta-\varphi)}) - h(\mathrm{e}^{\mathrm{i}\theta})) \frac{\mathrm{d}\varphi}{2\pi}.$$

Then the first term is bound in absolute value by

$$\varepsilon \int_{-\delta}^{\delta} P_r(\varphi) \frac{\mathrm{d}\varphi}{2\pi} \leq \varepsilon,$$

and the second term is bound by

$$2 \left\| h \right\|_\infty \int_{|\varphi| \in [\delta, \pi]} P_r(\varphi) \frac{\mathrm{d}\varphi}{2\pi},$$

which for a fixed δ has limit 0 as $r \to 1^-$, as you can check (this is what makes the family $\{P_r(\varphi)\}_{r \in (0,1)}$ an approximate identity). It follows that in this case, we have that

$$\lim_{r \to 1^-} |h(r\mathrm{e}^{\mathrm{i}\theta}) - h(\mathrm{e}^{\mathrm{i}\theta})| \leq \varepsilon$$

for all $\varepsilon > 0$, which proves the result. ∎

At a point $\mathrm{e}^{\mathrm{i}\theta}$ of discontinuity of h, the radial limit $\lim_{r\to 1^-} P[h](r\mathrm{e}^{\mathrm{i}\theta})$ is in a certain sense the limit of the average values of h on decreasingly smaller neighborhoods of $\mathrm{e}^{\mathrm{i}\theta}$ in the circle; for instance, if h is piecewise continuous, then this radial limit of u is the average of two one-sided limits of h at this point. For much more on harmonic functions and Poisson integrals, see [1, Chapter X] or [2, Chapter 11]. For instance, we have the more general Theorems 11.16 and 11.23 of [2] on nontangential limits of functions $P[h]$ for $h \in L^1(S^1)$, and Exercise X.6.1 of [1] on piecewise continuous functions h.

Finally, note that, in our case, you can take h_0 to be the function defined on S^1 by $h_0(\mathrm{e}^{\mathrm{i}\theta}) = 1$ when $\theta \in (0, \pi)$ and -1 when $\theta \in (\pi, 2\pi)$ (and defined arbitrarily when $\theta \in \{0, \pi\}$), then the function $u = P[h_0]$ will be a solution to the above problem. In fact, one can make the evaluation of this integral explicit by using the indefinite integral

$$\int \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \frac{\mathrm{d}\varphi}{2\pi} = -\frac{1}{\pi} \arctan\left(\frac{1+r}{1-r} \tan\left(\frac{\theta - \varphi}{2} \right) \right) + C,$$

which can be derived easily using the standard half-angle substitution. This obtains again the function u defined above.

Remark 2. Yet another way to obtain u is to use the Fourier coefficients of the square wave. Recall our proof of Theorem 1.47 in the notes gives a constructive way of finding the unique harmonic extension of a continuous function $h : S^1 \to \mathbb{C}$ to the disk: expand $h(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ in its Fourier series, and then

$$u(z) := a_0 + \sum_{n \geq 1} a_n z^n + \sum_{n \leq -1} a_n \overline{z}^n$$

will converge uniformly on compact subsets of Δ and will be the required function. This procedure also works more generally for L^1 functions h. In our case, the function $h = h_0$ is the square wave, and this admits the Fourier expansion

$$h(\mathrm{e}^{\mathrm{i}\theta}) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin((2n+1)\theta)$$

[2]Incidentally, this argument can also be used to show that a continous $u : \overline{\Delta} \to \mathbb{R}$ with $u|_\Delta$ harmonic is the real part of a holomorphic function on Δ, namely the integral above applied to $h = u|_{S^1}$.

so that $a_0 = 0$ and for $n \in \mathbb{Z}$ nonzero

$$a_n = \frac{((-1)^n - 1)\mathrm{i}}{n\pi},$$

and then you can check that the $u(z)$ you get by the formula in Equation 2 is the same as the one obtained above. The justification that the limiting values are h can then proceed as above, or using, say Abel's Lemma directly.

Remark 3. It is no coincidence that all of these methods give us the same answer–and indeed, there is a unique such u satisfying the result of the theorem. Indeed, for $1 \leq p \leq \infty$ and any harmonic $u : \Delta \to \mathbb{R}$, define

$$\|u\|_p := \sup_{r\in[0,1)} \|u_r\|_p\,,$$

where $u_r : S^1 \to \mathbb{R}$ is the function defined by $u_r(\mathrm{e}^{\mathrm{i}\theta}) = u(r\mathrm{e}^{\mathrm{i}\theta})$, and $\|\cdot\|_p$ denotes the p-norm of a Borel function $S^1 \to \mathbb{R}$. Then the space

$$h^p(\Delta) := \{u : \Delta \to \mathbb{R} \text{ harmonic } : \|u\|_p < \infty\}$$

is called the p^{th} harmonic Hardy space. This is a separable normed linear space (actually a Banach space) equipped with the norm $\|\cdot\|_p$. Since the $\|\cdot\|_\infty = \operatorname{ess\,sup}$ is the essential supremum which agrees with the supremum for continuous functions, the space $h^\infty(\Delta)$ is the space of bounded harmonic functions on Δ. Now if $u \in h^\infty(\Delta)$, then the family $\{u_r\}_{r\in[0,1)}$ of functions on S^1 is a norm-bounded sequence in $L^\infty(S^1) = L^1(S^1)^*$, and so by the Banach-Alaoglu theorem contains a subsequence u_{r_n} for some sequence of radii $r_n \to 1$ that converges in the weak-$*$ topology to some function $h \in L^\infty(S^1)$. Now if u satisfies the property in the definition, then u_r converges pointwise almost everywhere to the function h_0 so, by boundedness of u, the Dominated Convergence Theorem tells us that the function h that u_r converges to in the weak-$*$ topology must be none other than this h_0. Finally, you can show, by using that the Poisson kernel is in $L^1(S^1)$, that u then must be given by $P[h_0]$, and is in particular unique.

In fact, we have:

Theorem 0.0.2. For $1 < p \leq \infty$, the map $h \mapsto P[h]$ is an isometry of $L^p(S^1)$ to $h^p(\Delta)$.

Such a uniqueness statement would not be true, if we asked for say a positive harmonic function $u : \Delta \to \mathbb{R}$ with $u_r \to 1 + h$ pointwise almost everywhere, because, for instance, you could add a positive multiple of the the Poisson kernel itself without changing this condition; the correct statement at $p = 1$ comes from thinking about Poisson integrals of measures. For much more on this fascinating aspect of harmonic function theory, see [3, Chapter 6] or [2, Chapter 17].

Remark 4. This function is usually presented as a "counterexample" to the statement that if $h : S^1 \to \mathbb{R}$ is any function, then the sequence $P_r[h] : S^1 \to \mathbb{R}$ defined by $P_r[h](\mathrm{e}^{\mathrm{i}\theta}) = P[h](r\mathrm{e}^{\mathrm{i}\theta})$ converges uniformly to h on S^1, which is true when h is continuous (see [4, Ex. 11A.12] and [2, §11.15]).

Q2. Give an example of a bounded harmonic function u on the unit disk whose harmonic conjugate is unbounded.

Solution. The same function $u(z)$ from the previous solution works, because the harmonic conjugate of u is the function v defined by

$$v(z) := -\frac{2}{\pi} \log \left| \frac{1+z}{1-z} \right|,$$

which is clearly unbounded as $z \to 1^-$.

$\blacksquare$

Q3. Let $f(z) = \sum_n a_n z^n$ be a power series with coefficients $a_n \in \mathbb{Z}$. Suppose that $f(z)$ defines a bounded analytic function on the unit disk. Prove that $f(z)$ is a polynomial.

Solution. Let $M := \|f\|_\infty = \sup_{z\in\Delta} |f(z)| \in [0, \infty)$. Note that for each $r \in (0, 1)$, Parseval's Theorem applied to $\overline{\Delta(r)}$ gives us that

$$\sum_n |a_n|^2 r^{2n} = \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \leq M^2.$$

Therefore, since the left-hand side is a monotonically increasing nonnegative function of $r \in (0, 1)$, we have that [3]

$$\sum_n |a_n|^2 = \sum_n \lim_{r\to 1^-} |a_n|^2 r^{2n} = \lim_{r\to 1^-} \sum_n |a_n|^2 r^{2n} \leq M^2 < \infty.$$

It follows that if that $a_n \in \mathbb{Z}$, then only finitely many of the a_n can be nonzero, i.e. $f(z)$ has to be a polynomial.

■

Remark 5. For $1 \leq p \leq \infty$, define the p^{th} Hardy space

$$H^p(\Delta) := h^p(\Delta; \mathbb{C}) \cap \mathcal{O}(\Delta)$$

to be the space of holomorphic functions with finite p-norms. Then, our $f \in H^\infty(\Delta) \subset H^2(\Delta)$, and we have that a function $f(z) = \sum_n a_n z^n \in H^2(\Delta)$ iff $\sum_n |a_n|^2 < \infty$; this is essentially what we have shown in the above proof. See [2, Chapter 17], particularly Theorem 17.12.

Remark 6. Some solutions "proved" that a power series around 0 that is defined on, and remains bounded in, Δ, must have a radius of convergence $R > 1$. This statement is false, and an example is furnished by the series

$$f(z) := \sum_{n\geq 1} \frac{z^{2^n}}{n^2}.$$

It is easy to see from the Cauchy-Hadamard formula that this sequence has a radius of convergence of exactly $R = 1$, and it is clear that

$$\sup_{z\in\Delta} |f(z)| = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6} < \infty,$$

so that $f(z)$ is in particular a bounded function. However, $f(z)$ has a natural boundary at S^1: there is no connected open set $U \supset \Delta$ properly containing Δ that admits a holomorphic extension of f.[4] The reason for this mistake is often a wrong interpretation of the following statement: "a power series must have a singularity on its circle of convergence". All this is saying is that if $f(z) = \sum_n a_n z^n$ is a power series with radius of convergence $0 < R < \infty$, then there is some point $z_0 \in \mathbb{C}$ with $|z_0| = R$ such that there is no open subset $U \subset \mathbb{C}$ containing z_0 and holomorphic function $g : U \to \mathbb{C}$ such that $g(z) = f(z)$ for all $z \in U \cap \Delta(0, R)$.[5] This doesn't necessarily imply that $f(z)$ has to be unbounded near z_0.

Remark 7. Some solutions used the statement that a bounded analytic function on the disk extends continuously to the boundary. This is false; here's a counterexample using infinite Blaschke products. We have the following result:

Theorem 0.0.3. If $\{\alpha_n\}$ is a sequence of points in $\Delta \setminus \{0\}$ such that

$$\sum_{n=1}^\infty (1 - |\alpha_n|) < \infty,$$

[3] One could invoke the Dominated Convergence Theorem, although this is, in essence, much more elementary and follows simply from the monotonicity and limiting properties of summation–a variant of what is usually called the Monotone Convergence Theorem. Alternatively, one could argue that if $\sum_n |a_n|^2 = \infty$, then we could choose a suitable $0 < r < 1$ such that $\sum_n |a_n|^2 r^{2n} > M^2$, which is a contradiction.

[4] One can prove that $\lim_{r\to 1^-} f'(r\omega)$ blows up for a dense subset of ω in S^1. One can also apply one of various "Gap Theorems" (see [5, Chapter 6].

[5] What's the correct statement for when $R = 0$ or $R = \infty$?

for any $k \geq 0$ the infinite product

$$B(z) := z^k \prod_{n=1}^{\infty} \frac{z - \alpha_n}{1 - \overline{\alpha}_n z} \cdot \left(-\frac{|\alpha_n|}{\alpha_n} \right)$$

converges uniformly on compact sets to define a holomorphic function on Δ that is bounded in absolute value by 1 and has zeroes exactly at the points α_n.

Proof. Taken from [2, Theorem 15.21]. The result follows from noting that if $\rho \in (0,1)$ and $|z| \leq \rho$, then for each $n \geq 1$ we have

$$\left| 1 - \frac{z - \alpha_n}{1 - \overline{\alpha}_n z} \left(-\frac{|\alpha_n|}{\alpha_n} \right) \right| = \left| \frac{1 + (|\alpha_n|/\alpha_n) z}{1 - \overline{\alpha}_n z} \right| (1 - |\alpha_n|) \leq \frac{1+\rho}{1-\rho}(1 - |\alpha_n|).$$

■

Suppose that we choose such a convergent infinite Blaschke product $B(z)$. By the discussion preceding Theorem 0.0.2 (see also [2, Theorem 15.24]), the radial (even nontangential) limit $\lim_{r \to 1^-} B(re^{i\theta})$ exists and has absolute value 1 almost everywhere on S^1. If B extended continuously to $\overline{\Delta}$, then B would have absolute value 1 (not just almost, but literally) everywhere on S^1 by continuity. On the other hand, by the compactness of $\overline{\Delta}$, the sequence α_n has a convergent subsequence α_{n_k} that converges to a point $e^{i\theta} \in S^1$, and then we would have by continuity

$$0 = \lim_{k \to \infty} B(\alpha_{n_k}) = B(\lim_{k \to \infty} \alpha_{n_k}) = B(e^{i\theta}),$$

which is a contradiction to $|B(e^{i\theta})| = 1$.

Q4. Prove that there are no proper holomorphic map $f : \Delta \to \mathbb{C}$.

Solution. We present two proofs.

Proof 1. Suppose f is such a map. Then f can't be constant, since Δ is noncompact (why?). Next, any nonconstant proper holomorphic map of (nonempty) Riemann surfaces is surjective, so that f is in particular surjective. Finally, note that every point in $\mathbb{C}$ has only finitely many f-preimages (why?). In particular, f has only finitely many zeroes; let's call them $a_1, \ldots, a_n$, repeated according to their multiplicities. Now let $g(z) := \prod_{i=1}^{n}(z - a_i)$, and consider the function $f/g : \Delta \smallsetminus \{a_i\} \to \mathbb{C}$. This has removable singularities at the a_i and hence extends to a holomorphic function on all of Δ, which we again call f/g. By construction, f/g has no zeroes. The proof will be over if we show that f/g is proper, because then it would have to be surjective by the first argument, but we just showed that it doesn't have any zeroes–this would be the required contradiction.

To show that f/g is proper, suppose that $\{z_n\}$ is a sequence in Δ that escapes every compact set. Since f is proper, the sequence $\{f(z_n)\}$ escapes every compact set, i.e. $\lim_{n\to\infty} |f(z_n)| = \infty$. Since g is a polynomial, there is an M with $0 < M < \infty$ such that $\sup_{z\in\Delta} |g(z)| < M$. It follows that (ignoring the finitely many a_i if they happen to show up in the $\{z_n\}$), we have that

$$\lim_{n\to\infty} \frac{|f(z_n)|}{|g(z_n)|} \geq \lim_{n\to\infty} \frac{|f(z_n)|}{M} = \infty,$$

which proves that f/g is proper as well.

Alternatively, at this point, you could consider the function $g/f : \Delta \to \mathbb{C}$ which extends continuously to $\overline{\Delta}$ by defining it to be zero on $\overline{\Delta} = S^1$. Then the Maximum Principle tells us that $g/f = 0$ everywhere, which is not possible. ■

Proof 2. Since $\mathbb{H}$ and Δ are biholomorphic via a Möbius transformation, it suffices to show that there are no proper holomorphic maps $f : \mathbb{H} \to \mathbb{C}$; suppose f is such a map. Since f has finitely many zeroes (as in the previous proof), there is some $x \in \mathbb{R}$ and $r > 0$ such that if $\Omega := \Delta(x, r)$ and $\Omega^+ : \Delta(x, r) \cap \mathbb{H}$, then $f|_{\Omega^+}$ has no zeroes. Define the function $g : \Omega^+ \to \mathbb{C}$ by

$$g(z) := \frac{1}{f(z)}.$$

This is a holomorphic function in Ω^+ with the property that $\operatorname{Im} g(z) \to 0$ when $\operatorname{Im} z \to 0^+$ (since in fact $f(z) \to \infty \Leftrightarrow g(z) \to 0$ as $\operatorname{Im} z \to 0^+$ by the properness of f), and hence by the Schwarz reflection principle extends to a holomorphic function on $\Delta(x, r)$. Since such an extension is continuous, we must then have that $g \equiv 0$ on $\Delta(x, r) \cap \mathbb{R}$, but then the identity principle tells us that $g \equiv 0$ on $\Delta(x, r)$, which is a contradiction since $g(z) \neq 0$ for $z \in \Omega^+$.

■

Q5. Let $\Gamma \subset \mathrm{Aut}(\hat{\mathbb{C}})$ be a finite group. A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quotient map for Γ if [for all$x, y \in \hat{\mathbb{C}}$, we have] $f(x) = f(y)$ iff $\gamma(x) = y$ for some $\gamma \in \Gamma$.

(a) Find a quotient map for the dihedral group D_{2m} generated by $z \mapsto \zeta_m z$ and $z \mapsto 1/z$, where $\zeta_m = \exp(2\pi \mathrm{i}/m)$.

(b) Let $\Gamma \subset \mathrm{Aut}(\hat{\mathbb{C}})$ be the subgroup leaving the set $\{0, 1, \infty\}$ invariant. Give generators for Γ and find a quotient map for Γ, normalized so that $f(\infty) = \infty$, $f(\zeta_6) = 0$ and $f(-1) = 1$.

Solution.

(a) The function

$$f(z) = z^m + \frac{1}{z^m}$$

suffices. This is clearly stable under the generators $z \mapsto \zeta_m z$ and under $z \mapsto 1/z$, and hence under all of D_{2m}. Note that $f(z) = \infty$ iff $z \in \hat{\mathbb{C}} \smallsetminus \{0, \infty\}$. If $x, y \in \hat{\mathbb{C}}$ are such that $f(x) = f(y)$, then either this quantity is ∞ in which case we are done, or we have $x, y \in \hat{\mathbb{C}} \smallsetminus \{0, \infty\}$ and then

$$f(x) = f(y) \Rightarrow (x^m y^m - 1)(x^m - y^m) = 0$$

so that $x^m = y^{\pm m}$, which tells us that $x = \zeta_m^j y^{\pm 1}$ for some $0 \le j \le m-1$ and some choice of sign as needed.

(b) Since $\mathrm{Aut}(\hat{\mathbb{C}})$ acts uniquely triply transitively on $\hat{\mathbb{C}}$, the map $\Gamma \to S_3$ given by sending an automorphism in Γ to its induced permutation of $\{0, 1, \infty\}$ is an isomorphism. In particular, $|\Gamma| = 6$ and in fact we may use the above correspondence to write down all elements of Γ explicitly as given by rational maps

$$z, \quad \frac{1}{z}, \quad 1 - z, \quad \frac{1}{1-z}, \quad \frac{z-1}{z}, \text{ and } \frac{z}{z-1}.$$

Any two distinct transpositions in generate S_3, so, for instance, we may take the maps $z \mapsto 1/z$ and $z \mapsto 1 - z$ as generators of Γ. The rational map $J : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$J(z) := \frac{4}{27} \frac{(1 - z(1-z))^3}{z^2(1-z)^2}$$

is the required quotient map, as you can check. One way to do this is to note that it is clearly invariant under the generators $z \mapsto 1/z$ and $z \mapsto 1 - z$ of Γ, and hence under all of Γ. Next, we have the identity

$$J(z) - J(w) = \frac{4(z-w)(z+w-1)(zw-1)(zw-z+1)(zw-w+1)(zw-z-w)}{27z^2(1-z)^2w^2(1-w)^2}$$

and we may argue as before (also taking care of the special values of z, namely $z \in \{0, 1, \infty\}$, where $J(z) = \infty$ separately). Alternatively, once you show that J is invariant under Γ, then all you need to do is argue that it has degree exactly six. Indeed, since J has degree six, $J(z) = J(w)$ implies that z and w are in the same Γ-orbit as soon as we know that z (or equivalently w) has trivial stabilizer in Γ. The only points of $\hat{\mathbb{C}}$ with nontrivial stabilizers in Γ are the eight points $0, \infty, \pm 1, 2^{\pm 1}, \zeta_6^{\pm 1}$ for which $J(z)$ is one of $0, 1, \infty$,[6] and these finitely many cases we can do by hand. Specifically, we have

$$\begin{aligned} J^{-1}(0) &= \{\zeta_6, \zeta_6^{-1}\} \\ J^{-1}(1) &= \{-1, 1/2, 2\}, \\ J^{-1}(\infty) &= \{0, 1, \infty\}, \end{aligned}$$

and it is easy to check that Γ acts transitively on these subsets as well.[7]

■

[6] This is the reason, in some ways, why elliptic curves with j-invariant 0 and 1728 are special. See Remark 10 for a comment on normalization, where $j(z) = 1728 \cdot J(z)$.

[7] See Theorem 0.0.4(b) for a more general result.

Remark 8. Quite a few solutions argued after showing that a proposed quotient map f is invariant under Γ that $\deg f = |\Gamma|$ suffices to conclude that f is a quotient map. This is correct but needs careful justification. This is because the fibers $f^{-1}(w)$ for the critical values $w \in \hat{\mathbb{C}}$ of f will be "nonreduced", i.e. will have cardinality strictly less than $|\Gamma|$; this happens iff some point in such a fiber has a nontrivial stabilizer in Γ. For instance, this is true for $w \in \{\pm 2, \infty\}$ in (a) and $w \in \{0, 1, \infty\}$ in (b). One has to therefore be slightly careful with these exceptional fibers, as shown above. Here is a general precise statement using the theory of Riemann surfaces:

Theorem 0.0.4. Let X be a compact Riemann surface, and $\Gamma \subset \mathrm{Aut}(X)$ be a finite group.

(a) The quotient X/Γ exists as a Riemann surface, i.e. there is a (necessarily compact) Riemann surface Y and a holomorphic map $\pi : X \to Y$ with the property that given any other Riemann surface Z and $f : X \to Z$ a Γ-invariant holomorphic map, i.e. a holomorphic map such that $f(\gamma x) = f(x)$ for all $x \in X$ and $\gamma \in \Gamma$, there is a unique holomorphic map $\tilde{f} : Y \to Z$ such that the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

commutes. Further, Y is unique up to unique isomorphism commuting with the quotient maps.

(b) If Z is any Riemann surface and $f : X \to Z$ is a Γ-equivariant map of degree $|\Gamma|$, then f is a quotient map for Γ, i.e. the map $\tilde{f}$ above is an isomorphism.

Proof. For a gentle exposition of (a), see, for instance, [6, III.3]; clearly, if such a Y exists then it is unique up to unique isomorphism commuting with the quotient maps. The statement in (b) is an immediate consequence of (a) and the multiplicatively of degree: in this case, we have

$$|\Gamma| = \deg f = \deg(\tilde{f} \circ \pi) = (\deg \tilde{f})(\deg \pi) = (\deg \tilde{f}) \cdot |\Gamma|$$

so that $\deg \tilde{f} = 1$, telling us that $\tilde{f}$ is an isomorphism. ∎

This is not the most general result in this direction; Γ can be taken to be infinite but acting properly discontinuously (i.e. a discrete Γ acting properly), as in the quotient of $\mathbb{C}$ by a lattice Λ. This theorem can be used to reduce some of the computations above: for instance, in (a), the map $f(z)$ is clearly invariant under D_{2m} and of degree $|D_{2m}| = 2m$, and is hence the required quotient map. Similarly, in (b), once you have checked that $J(z)$ is invariant under $z \mapsto 1/z$ and $z \mapsto 1 - z$ and has degree six, it must be the required quotient map.

Remark 9. The above solution for (b) makes it seem like we pulled the function $J(z)$ out of a hat, but there are in fact ways to obtain it "from first principles". One way to do this is to note that the subgroup leaving the set $\{1, \omega, \omega^2\}$ invariant is exactly the D_6 from the first problem, i.e. our Γ and D_6 from (a) are conjugate under a Möbius transformation taking $0, 1, \infty$ to $1, \omega, \omega^2$, say

$$g(z) := \frac{\omega^2 z + \omega}{z + \omega}$$

where $\omega := \zeta_3 = \exp(2\pi \mathrm{i}/3)$. Therefore, the rational function

$$h(z) = g(z)^3 + \frac{1}{g(z)^3}$$

is a quotient map for Γ. Normalizing as needed, we find the required quotient map to be

$$J(z) = \frac{-4}{h(z) - 2},$$

which you can check agrees with the answer above. Another approach, that you can find in more detail in [7, §19.9], involves trying to use the symmetric polynomials in the six rational maps of Γ. Clearly, the first and sixth symmetric polynomials (i.e. the sum and the product) don't work, so we can try the second symmetric polynomial or alternatively the second power sum, i.e. the sum of the squares, of these rational functions. This approach works, and gives you another way to get the function $J(z)$, after appropriately normalizing.

Remark 10. This choice of normalization for the quotient map in (b) is not arbitrary: the resulting quotient map $J(z)$ obtained above (or more precisely the function $j(z) := 1728 \cdot J(z)$) is the modular j-invariant[8], which is a very beautiful function that shows up all over the place in math, e.g. in connections with modular forms and the moduli space of elliptic curves, transcendental number theory, and representations of the Monster group to name a few. We will talk more about this function in class later on, but you can look at this webpage or this one to whet your appetite now.

[8]The normalization $J(z)$ is classical, whereas the normalization $j(z)$ is chosen because, to name one of many reasons, the Fourier series of $j(\lambda(\tau))$, where $\lambda(\tau)$ is the modular λ-function has integral coefficients when expanded in $q = e^{2\pi i\tau}$.

Q6.

(a) Verify that the spherical metric $2(1+|z|^2)^{-1}|\mathrm{d}z|$ is invariant under $h(z) = (z-1)/(z+1)$.
(b) Show that the rotations $g_\theta(z) = \mathrm{e}^{\mathrm{i}\theta}z$, together with h, generate the full group $\mathrm{SU}_2/\{\pm I\} \subset \mathrm{PSL}_2\,\mathbb{C}$.

The group $\mathrm{SU}_2/\{\pm I\}$ is usually called PSU_2 or PU_2.

Proof.

(a) Let $\rho(z) := 2(1+|z|^2)^{-1}$ so the metric is $\rho(z)|\mathrm{d}z|$. We compute

$$h^*(\rho(z)|\mathrm{d}z|) = \rho(h(z))|h'(z)||\mathrm{d}z| = \frac{2}{1+\left|\dfrac{z-1}{z+1}\right|^2}\cdot\left|\frac{2}{(z+1)^2}\right||\mathrm{d}z| = \frac{4}{|z+1|^2+|z-1|^2}|\mathrm{d}z|.$$

Now

$$|z+1|^2+|z-1|^2 = (|z|^2+1+2\operatorname{Re} z)+(|z|^2+1-2\operatorname{Re} z) = 2(1+|z|^2),$$

and hence

$$h^*(\rho(z)|\mathrm{d}z|) = \frac{2}{1+|z|^2}|\mathrm{d}z| = \rho(z)|\mathrm{d}z|$$

as needed. Technically, this proof only works on $\mathbb{C}\smallsetminus\{-1\}$. One can then either work with charts at ∞ to do this at (i.e. in a neighborhood of) ∞ and -1, or argue by using that $z\mapsto 1/z$ is also an isometry for this spherical metric; this I leave to the reader.

(b) Let $\Gamma := \langle g_\theta, h\rangle$ is the subgroup of automorphisms of $\hat{\mathbb{C}}$ generated by all the g_θ and h. We have to show that $\Gamma = \mathrm{PU}_2$, where note that $\mathrm{PU}_2 = \{f\in\mathrm{PSL}_2\,\mathbb{C} : f^*\rho = \rho\}$. We do this in three steps:

(i) First, we show that Γ acts transitively on $\hat{\mathbb{C}}$, for which it suffices to show that if $z\in\hat{\mathbb{C}}$ is any element, then there is an $g\in\Gamma$ such that $g(z)=0$. Note that h takes $0\mapsto -1\mapsto\infty\mapsto 1\mapsto 0$, so we are done when $z\in\{0,\pm1,\infty\}$. When $|z|=1$, we may rotate by some g_θ to get $z=1$ and we are done. If $|z|=r$ with $r\in(0,\infty)\smallsetminus\{1\}$, then $|h(r)|<1$ but $|h(-r)|>1$ (check!), so that the image of the circle $S^1(0,r)$ under h (which is another circle[9]) must intersect $S^1 = S^1(0,1)$ somewhere by continuity; we send z via a rotation g_θ to the point in $S^1(0,r)$ that is sent via h to S^1, reducing us to the already shown case $r=1$.

(ii) Next, we know that rotations $g_\theta(z) = \mathrm{e}^{\mathrm{i}\theta}z$ are isometries, and we showed in (a) that $h(z)$ is an isometry, so that we have $\Gamma\subset\mathrm{PU}_2$.

(iii) Finally, we show that if $h\in\mathrm{PU}_2$ satisfies $h(0)=0$, then $h=g_\theta$ for some θ. There are many ways to do this; here are two. Note that any $f\in\mathrm{PU}_2$ looks like $f(z) = (\alpha z+\beta)/(-\overline{\beta}z+\overline{\alpha})$ for some $\alpha,\beta\in\mathbb{C}$ with $|\alpha|^2+|\beta|^2=1$. Then $f(0)=\beta/\overline{\alpha}$, so that

$$f(0)=0 \Rightarrow \beta = 0 \Rightarrow |\alpha| = 1$$

and $f(z) = \alpha^2 z$, which is of the form g_θ for some θ. Alternatively, one could argue that if $h\in\mathrm{PU}_2$ satisfies $h(0)=0$, then h preserves

$$\Delta = \{z\in\mathbb{C} : |z|<1\} = \{z\in\hat{\mathbb{C}} : d_\rho(0,z) < d_\rho(0,1) = \pi/2\},$$

and so we may apply the Schwarz lemma to conclude that h is a rotation.

Now we are ready to prove that $\Gamma = \mathrm{PU}_2$; for this, let $f\in\mathrm{PU}_2$ be arbitrary. By (i) applied to $z=f(0)$, there is a $g\in\Gamma$ such that $g(f(0))=0$. By (ii), $g\in\mathrm{PU}_2$ and hence $g\circ f\in\mathrm{PU}_2$ takes zero to zero, so by (iii) we have $g\circ f = g_\theta$ for some θ. Since $g_\theta\in\Gamma$, it follows that $f = g^{-1}\circ g_\theta\in\Gamma$ as well.

■

[9] Here "circle" means "generalized circle", i.e. circle or straight line.

Q7. The quaternions $\mathbb{H}$ are the algebra $\mathbb{R}[\mathbf{i},\mathbf{j},\mathbf{k}]$ subject to the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \text{ and } \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

As a vector space, we have $\mathbb{H} \cong \mathbb{R}^4$, and the unit norm quaternions U are a multiplicative group isomorphic to S^3. Prove that SU_2 is isomorphic, as a group, [to] the unit norm quaternions U. (Hint: find an embedding $\mathbb{H} \hookrightarrow \mathrm{M}_2(\mathbb{C})$ as an $\mathbb{R}$-algebra, by choosing of matrices $I, J, K \in \mathrm{SU}_2$ that satisfy the same relations as $\mathbf{i},\mathbf{j},\mathbf{k}$.)[10]

Solution. Take

$$I = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}.$$

It is straightforward to check that these satisfy the same relations as the $\mathbf{i},\mathbf{j},\mathbf{k}$, and hence the map

$$\mathbb{H} \to \mathrm{M}_2(\mathbb{C}), \quad a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \mapsto a\,\mathrm{id}_2 + bI + cJ + dK = \begin{bmatrix} a + b\,\mathrm{i} & c + d\,\mathrm{i} \\ -c + d\,\mathrm{i} & a - b\,\mathrm{i} \end{bmatrix}$$

is an embedding of finite dimensional $\mathbb{R}$-algebras. The image consists of exactly the matrices of the form

$$\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}$$

for $\alpha, \beta \in \mathbb{C}$, with the correspondence given by $\alpha = a + b\,\mathrm{i}$ and $\beta = c + d\,\mathrm{i}$. We further have that the norm in $\mathbb{H}$ agrees with the determinant in $\mathrm{M}_2(\mathbb{C})$, i.e. if $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ and $\alpha = a + b\,\mathrm{i}$ and $\beta = c + d\,\mathrm{i}$, then

$$\|q\|^2 = a^2 + b^2 + c^2 + d^2 = |\alpha|^2 + |\beta|^2 = \det \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}.$$

The group of unit norm quaternions

$$U = \{q \in \mathbb{H} : \|q\|^2 = 1\} \cong S^3$$

therefore maps isomorphically under this embedding to the group

$$\left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1 \right\} = \mathrm{SU}_2,$$

and this is a (Lie) group isomorphism because multiplication of quaternions on the left corresponds to matrix multiplication on the right, since $\mathbb{H} \hookrightarrow \mathrm{M}_2(\mathbb{C})$ is an embedding of algebras. ∎

Remark 11. These matrices I, J, K can be found using the same general principles that allow us to construct $\mathbb{C}$ from $\mathbb{R}$ by finding a $\mathbb{R}$-subalgebra of $\mathrm{M}_2(\mathbb{R})$ generated by the matrix J with satisfies $J^2 = -1$. This is a general construction that is used to find matrix representations of real and complex Clifford algebras, with the above being the cases $\mathrm{Cl}_0(\mathbb{R}) \cong \mathbb{R}$, $\mathrm{Cl}_1(\mathbb{R}) \cong \mathbb{C}$ and $\mathrm{Cl}_2(\mathbb{R}) \cong \mathbb{H}$ respectively.

[10] It is an unfortunate fact that the blackboard bold $\mathbb{H}$ is overloaded notation for both the upper half plane and the quaternions. Hopefully, this will not cause any confusion.

References

[1] T. W. Gamelin, *Complex Analysis.* Springer, 2001.

[2] W. Rudin, *Real and Complex Analysis.* McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Company, third ed., 1987.

[3] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory.* No. 137 in Graduate Texts in Mathematics, Springer, second ed., 1992.

[4] S. Axler, *Measure, Integration & Real Analysis.* No. 282 in Graduate Texts in Mathematics, Springer, 2020.

[5] S. L. Segal, *Nine Introductions in Complex Analysis.* No. 208 in North-Holland Mathematics Studies, Eselvier, revised ed., 2008.

[6] R. Miranda, *Algebraic Curves and Riemann Surfaces.* Graduate Studies in Mathematics, Vol. 5, American Mathematical Society, 1995.

[7] R. Vakil, "The Rising Sea: Foundations of Algebraic Geometry." `https://math.stanford.edu/~vakil/216blog/`.