Math 213A F23 Homework 3 Solutions

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

We use the following convention: for any $w \in \mathbb{C}$ and $r > 0$, we use the notation

$$\begin{aligned} \Delta(w,r) &= \{z \in \mathbb{C} : |z-w| < r\}, \\ \overline{\Delta(w,r)} &= \{z \in \mathbb{C} : |z-w| \le r\}, \text{ and} \\ S^1(w,r) = \partial\Delta(w,r) &= \{z \in \mathbb{C} : |z-w| = r\}, \end{aligned}$$

and orient the last one counterclockwise when necessary.

Q1. Let $f_n \to f$ uniformly on compact subsets of an open connected set $\Omega \subset \mathbb{C}$, where f_n is analytic, and f is not identically equal to zero. Show if $f(w) = 0$ then we can write $w = \lim z_n$, where $f_n(z_n) = 0$ for all sufficiently large n.

Proof. Since the zeroes of f are isolated, there is an $r > 0$ such that $\overline{\Delta(w,r)} \subset \Omega$ and such that w is the only zero of f in $\overline{\Delta(w,r)}$, so that, in particular, by compactness, we have $\inf_{S^1(w,r)} |f| > 0$. By uniform convergence on compact sets, there is an $N \gg 1$ such that for all $n \ge N$ we have

$$\sup_{S^1(w,r)} |f_n - f| < \inf_{S^1(w,r)} |f|$$

It follows from Rouche's Theorem (Corollary 1.34 in the notes dated 09/28/23) that f and $f_n = f + f_n - f$ have the same number of zeroes in $\Delta(w,r)$, counted with multiplicity, namely $m := \operatorname{ord}_w f(z)$. In particular, by hypothesis we have $m \ge 1$, so that for each $n \ge N$, the function f_n has at least one zero in $\Delta(w,r)$; pick any one and call it z_n. It remains to show $\lim z_n = w$, for which we have to show that if $U \ni w$ is any open subset, then for all but finitely many n, the $z_n \in U$; but this is clear by the same reasoning as above: pick r', N' such that $0 < r' \le r$ and $N' \ge N$ and such that the above statements hold verbatim if we replace Ω, r, N by U, r', N' respectively, so that for $n \ge N'$, all m roots of f_n in the disc $\Delta(w,r)$ actually lie in $\Delta(w,r')$, telling us in particular that $z_n \in \Delta(w,r') \subset U$ for all $n \ge N'$. ∎

Remark 1. In fact, we have shown:

Theorem 0.0.1 (Hurwitz)**.** Let $f_n \to f$ uniformly on compact subsets of an open connected set $\Omega \subset \mathbb{C}$, where f_n is analytic, and suppose f has a zero of order exactly m at w, where $0 \le m < \infty$. Then there exists an $r > 0$ such that for sufficiently large $n \gg 1$, each function f_n has exactly m zeroes in $\Delta(w,r)$, counted with multiplicity, and these zeroes converge to w as $n \to \infty$.

This theorem amounts to a version of the "continuous dependence of the roots of a holomorphic function (e.g. polynomial) on its Taylor coefficients". Here's a fun thing to think about: using the above ideas, try to solve Chapter 1, Exercise 50 in the version of the notes dated 09/28/23; the result of this exercise is also usually attributed to Hurwitz.

Remark 2. Some solutions accidentally "proved" the existence of a sequence $z_k \to w$ such that $f(z_k) = 0$, or such that for a fixed k, we have $f_n(z_k) = 0$ for all sufficiently large n. Neither of these approaches is going to work–quantifiers are important, and care is warranted! Some other solutions "proved" that for all sufficiently large n, the function $f_n(z)$ has a unique zero in some ball around w, which is also false in general, since this fails to account for multiplicities.

Q2. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^6}{(1+x^4)^2} \mathrm{d}x.$$

Solution. Let

$$f(z) := \frac{z^6}{(1+z^4)^2},$$

and note that this expression defines a meromorphic function on the whole plane, which has double poles at the primitive eighth roots of unity (i.e. at $\zeta_8, \zeta_8^3, \zeta_8^5$ and ζ_8^7 where $\zeta_8 := \mathrm{e}^{\pi \mathrm{i}/4}$) and is holomorphic elsewhere.

For any $R > 1$, consider the contour C_R which is the boundary of the rectangle $[-R, R] \times [0, R]\mathrm{i}$ oriented counterclockwise, and note that this contour encloses exactly two poles of $f(z)$, namely those at ζ_8 and ζ_8^3. By the Residue Theorem, we have that

$$\oint_{C_R} f(z)\, \mathrm{d}z = 2\pi \mathrm{i} \left[\operatorname*{Res}_{z=\zeta_8} f(z) + \operatorname*{Res}_{z=\zeta_8^3} f(z) \right].$$

On the other hand, the integral on the left can be written as

$$\left[\int_{-R}^{R} + \int_{C_R'} \right] f(z) \mathrm{d}z,$$

where C_R' is the part of the path not on the real axis. Note that for any path γ lying in the region $|z| \geq R > 1$, we have

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| \leq \ell(\gamma) \sup_{z\in\gamma} \frac{|z|^6}{(|z|^4 - 1)^2} = \ell(\gamma) \frac{d^6}{(d^4-1)^2},$$

where $\ell(\gamma) = \int_\gamma |\mathrm{d}z|$ is the length of the path and $d := \inf_{z\in\gamma} |z|$. It follows that

$$\left| \int_{C_R'} f(z)\, \mathrm{d}z \right| \leq 4R \cdot \frac{R^6}{(R^4-1)^2},$$

which goes to zero as $R \to \infty$. Therefore,

$$\int_{-\infty}^{\infty} \frac{x^6}{(1+x^4)^2} \mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(z)\, \mathrm{d}z = \lim_{R\to\infty} \oint_{C_R} f(z)\, \mathrm{d}z = 2\pi \mathrm{i} \left[\operatorname*{Res}_{z=\zeta_8} f(z) + \operatorname*{Res}_{z=\zeta_8^3} f(z) \right].$$

It remains to compute these residues. For $\zeta \in \{\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7\}$, let $g_\zeta(w) := f(w + \zeta)$. Note that

$$g_\zeta(w) = \frac{(w+\zeta)^6}{\left(1 + (w+\zeta)^4\right)^2} = \frac{1}{16w^2} \cdot \left[\frac{(1+\zeta^7 w)^6}{\left(1 + \frac{3}{2}\zeta^7 w + \zeta^6 w^2 + \frac{1}{4}\zeta^5 w^3\right)^2} \right] = \frac{1}{16w^2} \left(1 + \left(6 - 2 \cdot \frac{3}{2} \right) \zeta^7 w + [w^2] \right),$$

so that the residue, namely the coefficient of w^{-1}, can be read off as

$$\operatorname*{Res}_{z=\zeta} f(z) = \operatorname*{Res}_{w=0} g_\zeta(w) = \frac{3}{16}\zeta^7.$$

Since

$$\zeta_8^7 + (\zeta_8^3)^7 = \zeta_8^7 + \zeta_8^5 = 2\mathrm{i} \sin(7\pi/4) = -\mathrm{i}\sqrt{2},$$

the value of the integral is

$$2\pi \mathrm{i} \cdot \frac{3}{16} \cdot (-\mathrm{i}\sqrt{2}) = \frac{3\pi}{4\sqrt{2}}.$$

∎

Remark 3. Another approach involves first integrating by parts as follows:

$$\int_{-\infty}^{\infty} \frac{x^6}{(1+x^4)^2} \mathrm{d}x = \left[\frac{-x^3}{4(1+x^4)} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\frac{-3x^2}{4(1+x^4)} \right) \mathrm{d}x = \frac{3}{4} \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \mathrm{d}x,$$

and then evaluating the last integral using residues (say). This is easier to do because the latter integrand has only simple poles.

Q3. Compute the Laurent series centered at $z = 0$ such that

$$\sum_{-\infty}^{\infty} a_n z^n = \frac{1}{z(z-1)(z-2)}$$

in the region $1 < |z| < 2$.

Solution. Note first that

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}.^{1}$$

For $1 < |z|$, we may expand

$$\frac{1}{z-1} = \frac{z^{-1}}{1-z^{-1}} = \sum_{n\geq 1} z^{-n}.$$

On the other hand, for $|z| > 2$, we may expand

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-(z/2)} = -\frac{1}{2} \sum_{n\geq 0} \frac{z^n}{2^n}.$$

Putting these together, we get that for $1 < |z| < 2$ we have

$$\frac{1}{z(z-1)(z-2)} = \cdots - z^{-3} - z^{-2} - \frac{1}{2}z^{-1} - \frac{1}{4} - \frac{1}{8}z - \frac{1}{16}z^2 + \cdots ,$$

and this is the required Laurent series. In other words, we have

$$a_n = \begin{cases} -1 & \text{if } n \leq -2, \text{ and} \\ -2^{-n-2} & \text{if } n \geq -1. \end{cases}$$

■

Remark 4. Some solutions expanded $(z-1)^{-1}$ in nonnegative powers of z around the origin as well. Such an expansion would yield the Laurent series of this function at the origin, valid in $|z| < 1$, but such an expansion would not be valid in $|z| > 1$.

[1] In general, we have for any polynomial $p(z)$ of degree ≥ 1 with only simple roots (i.e. for which $p'(\alpha) \neq 0$ for all $\alpha \in p^{-1}(0)$) that

$$\frac{1}{p(z)} = \sum_{\alpha \in p^{-1}(0)} \frac{1}{p'(\alpha)} \cdot \frac{1}{z-\alpha}$$

identically as meromorphic functions on $\mathbb{C}$. This can be proved in many different ways; here's one way using complex analysis: note that the difference of the two functions is a bounded entire function and goes to zero as $z \to \infty$.

Q4. Let $U \subset \mathbb{C}$ be a region with $0 \in U$, and let

$$\mathcal{O}_M(U) := \{f \in \mathcal{O}(U) : \|f\| = \sup_U |f(z)| \leq M\}.$$

(i) Prove that for any sequence $f_n \in \mathcal{O}_M(U)$, the conditions
  (a) $f_n \to 0$ uniformly on compact subsets of U, and
  (b) for each $k \geq 0$, we have $f_n^{(k)}(0) \to 0$,
are equivalent.

(ii) Construct a sequence $f_n \in \mathcal{O}(\Delta)$ such that for each $k \geq 0$, we have $f_n^{(k)}(0) \to 0$, but $|f_n(z)| \to \infty$ for all $z \neq 0$.

Solution.

(i) The implication (a) $\Rightarrow$ (b) doesn't need the f_n to be uniformly bounded; indeed if $f_n \in \mathcal{O}(U)$ is any sequence such that $f_n \to 0$ uniformly on compact subsets of U, then we have for each $k \geq 0$ that $f_n^{(k)}(0) \to 0$. This follows from the Cauchy integral formula: pick any $r > 0$ such that $\overline{\Delta(0,r)} \subset U$, and then use that for each $k \geq 0$ we have

$$\lim_{n\to\infty} f_n^{(k)}(0) = \lim_{n\to\infty} \frac{k!}{2\pi \mathrm{i}} \oint_{S^1(0,r)} \frac{f_n(z)}{z^{k+1}} \mathrm{d}z = \frac{k!}{2\pi \mathrm{i}} \oint_{S^1(0,r)} \left(\lim_{n\to\infty} \frac{f_n(z)}{z^{k+1}} \right) \mathrm{d}z = 0,$$

where the swapping of the limit and the integral is justified by the uniform convergence of f_n to 0 on $S^1(0,r)$. For the implication (b) $\Rightarrow$ (a), suppose that for each $k \geq 0$, we have $f_n^{(k)}(0) \to 0$, but that there is some compact $K \subset U$ such that f_n does not uniformly converge to 0 on K. This means that there is some $\varepsilon > 0$ such that for infinitely many n we have $\sup_K |f_n(z)| \geq \varepsilon$. By replacing f_n by a subsequence, we can assume that in fact $\sup_K |f_n(z)| \geq \varepsilon$ for all n. By Theorem 1.24 in the notes, $\mathcal{O}_M(U)$ is sequentially compact[2], so that there is some further subsequence of f_n which converges uniformly on compact sets to a function f (which is then necessarily holomorphic); again replace f_n by this subsequence to assume that f_n itself converges uniformly on compact subsets to f. Since $f_n \to f$ uniformly on the compact set K, and $\sup_K |f_n(z)| \geq \varepsilon$ for each n, it follows that $\sup_K |f(z)| \geq \varepsilon$ as well[3], so that the function $f(z)$ is not identically zero. However, we know from Corollary 1.22 that for each $k \geq 0$, we have $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets as well; in particular, we have

$$f^{(k)}(0) = \lim_{n\to\infty} f_n^{(k)}(0) = 0.$$

Consequently, the Taylor expansion of f around the origin is zero. Since f agrees with its Taylor expansion in some open disc of positive radius around the origin (in fact in the largest open disc around the origin in U) by Theorem 1.6, this tells us that f is identically zero on some open disc around the origin. Since U is connected, this implies by the identity principle (or equivalently isolation of zeroes, Theorem 1.11) that f is everywhere identically zero, which is a contradiction.

(ii) Any sequence of the form $f_n(z) = p(n)z^n$ works, where $p(n)$ is some superexponential function of n, e.g. any one of $n!, n^n$ or e^{n^2} works.

[2]Technically, the theorem in the notes says that this space $\mathcal{O}_M(U)$, in the topology induced from $\mathcal{O}(U)$ (i.e. topology of uniform convergence on compact sets) is compact. In general for topological spaces, compactness neither implies nor is implied by sequential compactness. However, in this case, $\mathcal{O}_M(U)$ is metrizable in the subspace topology: taking an exhaustion of U by compact subsets $(K_j)_{j\geq 0}$, we may define a metric on $\mathcal{O}_M(U)$ by

$$d(f,g) = \sum_{j\geq 0} 2^{-j} \sup_{K_j} |f - g|.$$

Then the resulting metric topology on $\mathcal{O}_M(U)$ is the same as the subspace topology it inherits from $\mathcal{O}(U)$. Therefore, in this case, compactness and sequential compactness are equivalent. Alternatively, going back and analyzing the proof of this theorem (in particular, the use of the Arzela-Ascoli Theorem) carefully, one can prove directly that $\mathcal{O}_M(U)$ is sequentially compact. This result is usually called Montel's Theorem, in which $\mathcal{O}_M(U)$ is concluded to be a normal family of holomorphic functions on U. For a clear exposition of this result, see for instance [1, Chapter 8, §3.2].

[3]Indeed, given any ρ such that $0 < \rho < \varepsilon$, by uniform convergence we may find an $n \gg 1$ such that $\sup_K |f(z) - f_n(z)| < \rho/2$, and by $\sup_K |f_n(z)| \geq \varepsilon$, we can find a $z \in K$ such that $|f_n(z)| > \varepsilon - (\rho/2)$. It follows then that

$$|f(z)| \geq ||f_n(z)| - |f(z) - f_n(z)|| > \varepsilon - (\rho/2) - (\rho/2) = \varepsilon - \rho.$$

It follows that $\sup_K |f(z)| \geq \varepsilon - \rho$, and since this is true for all $0 < \rho < \varepsilon$, it follows that $\sup_K |f(z)| \geq \varepsilon$ as needed. In fact, the existence of such a $z \in K$ with $f(z) \neq 0$ is all we need, as the rest of the proof shows.

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Remark 5. Here's another way to prove the direction (b) $\Rightarrow$ (a): first show the result for $U = \Delta(0, r)$ for some $r > 0$ by using that the each function f_n agrees with its Taylor expansion around the origin in such a ball, and using that any such K is contained in $\Delta(0, \rho)$ for some $0 < \rho < r$. In general, use connectedness to "propagate": find an integer $m \geq 1$ and a sequence of points z_j and radii $0 < \rho_j < r_j$ for $j = 0, \ldots, m$ such that $z_0 = 0$, we have $\Delta(z_j, r_j) \subset U$ for $j = 0, \ldots, m$ and $z_{j+1} \in \Delta(z_j, \rho_j)$ for $j = 0, \ldots, m-1$, and such that $K \subset \bigcup_{j=0}^{m} \Delta(z_j, \rho_j)$. Then show by induction on $k = 0, \ldots, m$ that $f_n \to 0$ uniformly on $\bigcup_{j=0}^{k} \Delta(z_j, \rho_j)$, where in each inductive step you're using that we know the result for a disc, and that there's nothing special about the choice of $0 \in U$ in the statement in the already shown implication (a) $\Rightarrow$ (b).

Remark 6. It is true that the map $(f, g) \mapsto \|f - g\| = \sup_U |f(z) - g(z)|$ turns $\mathcal{O}_M(U)$ into a metric space, but this metric topology is not the same as (and is, in fact, strictly coarser than) the topology $\mathcal{O}_M(U)$ inherits as a subset of $\mathcal{O}(U)$, i.e. the topology of uniform convergence on compact sets. In particular, this reasoning does not suffice to show that $\mathcal{O}_M(U)$ is metrizable in the subspace topology. In the same vein, it is important to avoid accidentally proving that (b) implies that $f_n \to 0$ uniformly on U, since this is not true, as the example of $f_n(z) = z^n$ on $U = \Delta$ with $M = 1$ shows.

Remark 7. Note that the approach in the second subpart will not work if $p(n)$ is not superexponential, e.g. $p(n) = \mathrm{e}^n$ doesn't work (why?).

Q5. For each $n > 0$, give an example of an analytic function $f : \Delta^* \to \mathbb{C}$ such that

(i) f is nowhere zero,
(ii) f has an essential singularity at $z = 0$, and
(iii) $\mathrm{Res}(f'/f, 0) = n$.

Solution. The function $f(z) := z^n e^{1/z}$ suffices. ■

Remark 8. The idea behind constructing this function is to note that:

- If $g(z), h(z)$ are non-identically-zero holomorphic functions in the punctured disc, and if $g(z)$ does not have an essential singularity at $z = 0$, while $h(z)$ has an essential singularity at $z = 0$, then the product function $f(z) = g(z)h(z)$ still has an essentially singularity at $z = 0$.
- The function $g(z) = z^n$ has a zero order exactly n at $z = 0$, so that $\mathrm{Res}(g'/g, 0) = n$. The function $h(z) = e^{1/z}$ has an essential singularity at $z = 0$ and satisfies $\mathrm{Res}(h'/h, 0) = 0$.
- Finally, for $f(z) = g(z)h(z)$ as above, we have $\mathrm{Res}(f'/f, 0) = \mathrm{Res}(g'/g, 0) + \mathrm{Res}(h'/h, 0)$.

Remark 9. Some solutions suggested functions of the form $f(z) = e^{\pm 1/z^n}$, which don't work–they satisfy $\mathrm{Res}(f'/f, 0) = 0$ always.

Remark 10. Implicit in the above problem statement and solution is $n \in \mathbb{Z}$. What happens for non-integer values of n?

Q6. What is the average value of $1/(1+\cos^2\theta)$ over the interval $[0, 2\pi]$?

Solution. The average value is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1+\cos^2\theta}\mathrm{d}\theta = \frac{1}{2\pi}\oint_{S^1}\frac{1}{1+((z+z^{-1})/2)^2}\frac{\mathrm{d}z}{\mathrm{i}z} = \frac{1}{2\pi\mathrm{i}}\oint_{S^1}\frac{4z}{z^4+6z^2+1}\mathrm{d}z.$$

Now the integrand

$$f(z) := \frac{4z}{z^4+6z^2+1}$$

has 4 simple poles at $\pm\mathrm{i}(\sqrt{2}\pm 1)$ since

$$z^4+6z^2+1 = (z^2+(\sqrt{2}+1)^2)(z^2+(\sqrt{2}-1)^2).$$

Of these, the only two that are in the unit disc Δ are $\pm i(\sqrt{2}-1)$; call these α and β. By the Residue Theorem, it follows that the average value is exactly

$$\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z).$$

To compute these residues, note the following simple but useful fact: if a $f(z)$ is a function meromorphic in a neighborhood of a point $\alpha \in \mathbb{C}$, with a simple pole there, and we write $f(z) = g(z)/h(z)$ for some functions $g(z)$ and $h(z)$ defined and holomorphic in a neighborhood of α such that α is a simple root of $h(z)$ (so that $h'(\alpha) \neq 0$), then we can compute

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}\frac{g(z)}{h(z)}(z-\alpha) = \lim_{z\to\alpha} g(z)\frac{z-\alpha}{h(z)-h(\alpha)} = \frac{g(\alpha)}{h'(\alpha)}.$$

In our case, taking $g(z) = 4z$ and $h(z) = z^4+6z^2+1$, we conclude that

$$\operatorname*{Res}_{z=\alpha} f(z) = \frac{4\alpha}{4\alpha^3+12\alpha} = \frac{1}{\alpha^2+3} = \frac{1}{2\sqrt{2}},$$

where in the last step we've used that $\alpha^2 = -(\sqrt{2}-1)^2 = 2\sqrt{2}-3$. It follows then by symmetry that the residues at α and β are the same and we conclude that

$$\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z) = 2\cdot\frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}},$$

which is therefore the required average value of $1/(1+\cos^2\theta)$ on the interval $[0, 2\pi]$. ■

References

[1] E. M. Stein and R. Shakarchi, *Complex Analysis.* No. 2 in Princeton Lectures in Analysis, Princeton University Press, 2007.