Math 213A F23 Homework 2 Solutions

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

I want to collect here a couple of remarks about the problem sets that I make below:

(i) In general, you are allowed to quote standard theorems whose proofs are unrelated to the problem at hand (e.g. the Stone-Weierstrass Theorem), but I will ask you to precisely state the theorems you quote. For instance, the statement "the other direction follows from Stone-Weierstrass" by itself will not get you full credit on a problem. The only exception to this general rule is when quoting the theorem trivializes the problem entirely (e.g. quoting Abel's Theorem on the fifth problem below)–then I will ask you to at least sketch how a proof of this theorem goes. I recognize that this is a little subjective–if you are unsure, please come and ask me whether you can quote a particular theorem on the solutions.

(ii) Even if you are unsure about a particular problem (even after coming to sections/OH/problem sessions), I would recommend against leaving your answer blank entirely. Usually, I will give partial credit to partial progress or an explanation of how you tried to approach the problem. I cannot give any points to nonexistent answers.

Q1. Let $v(z)$ be a smooth vector field in $\mathbb{C}$. Express $\dfrac{\mathrm{d}v}{\mathrm{d}z}$ in terms of $\operatorname{div}(v)$ and $\operatorname{curl}(v)$.

Solution. Writing $v(z) = a(z) + \mathrm{i}\, b(z)$ for smooth real-valued functions a and b, we have

$$\frac{\partial v}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\frac{\partial}{\partial y}\right)(a + \mathrm{i}\, b) = \frac{1}{2}\left\{\left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right) + \mathrm{i}\left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)\right\} = \frac{1}{2}\left(\operatorname{div}(v) + \mathrm{i}\operatorname{curl}(v)\right).$$

∎

Q2. Let $f(z)$ be one-to-one and analytic on a neighborhood of the unit circle $S^1 \subset \mathbb{C}$. Relate

$$\oint_{S^1} \overline{f(z)} f'(z)\, \mathrm{d}z$$

to the area of the region enclosed by $f(S^1)$.

Solution. The restriction of f to S^1 is a smooth[1] proper[2] injective[3] immersion[4] and hence a smooth embedding. By the Jordan Curve Theorem, this separates the plane $\mathbb{C}$ into two connected components, of which exactly one is bounded–call it Ω. Then $\Omega \subset \mathbb{C}$ is a bounded connected open subset with smooth boundary $\partial\Omega = f(S^1)$; what is meant by the area of the region enclosed by $f(S^1)$ is exactly the quantity

$$\operatorname{area}(\Omega) = \int_\Omega \mathrm{d}x \wedge \mathrm{d}y.$$

Now we use the magic of differential forms: we know that

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{\mathrm{i}}{2} \mathrm{d}z \wedge \mathrm{d}\overline{z} = \mathrm{d}\left(-\frac{\mathrm{i}}{2}\overline{z}\,\mathrm{d}z\right),$$

so by Stokes' Theorem we have

$$\operatorname{area}(\Omega) = \int_\Omega \mathrm{d}x \wedge \mathrm{d}y = \int_\Omega \mathrm{d}\left(-\frac{\mathrm{i}}{2}\overline{z}\,\mathrm{d}z\right) = \oint_{\partial\Omega}\left(-\frac{\mathrm{i}}{2}\overline{z}\,\mathrm{d}z\right) = -\frac{\mathrm{i}}{2}\oint_{f(S^1)} \overline{z}\,\mathrm{d}z.$$

Finally, we use the diffeomorphism $f : S^1 \to f(S^1)$ to write

$$\oint_{f(S^1)} \overline{z}\,\mathrm{d}z = \pm \oint_{S^1} f^*(\overline{z}\,\mathrm{d}z) = \pm \oint_{S^1} \overline{f(z)}\,\mathrm{d}(f(z)) = \pm \oint_{S^1} \overline{f(z)} f'(z)\,\mathrm{d}z,$$

where the sign $\pm$ is determined by whether or not $f : S^1 \to f(S^1)$ preserves orientation, where the latter is given the boundary orientation as the boundary of Ω (which we need to do to apply Stokes' theorem as written). It follows that

$$\oint_{S^1} \overline{f(z)} f'(z) \mathrm{d}z = \pm 2\mathrm{i} \cdot \operatorname{area}(\Omega),$$

with again the sign $\pm$ determined by whether or not $f : S^1 \to f(S^1)$ preserves orientation.

Remark 1. None (!) of the submitted solutions included the sign in their final answer. Remember that only oriented diffeomorphisms can be used to compute integrals via pulling forms back. Note that the maps $f(z) = z$ and $f(z) = 1/z$ give examples to show that both signs are possible.

■

[1] Note that if $U \subset \mathbb{C}$ is an open subset containing S^1 such that f extends to an injective holomorphic function on f, then $f : U \to \mathbb{C}$ is holomorphic and hence smooth, and the inclusion $S^1 \hookrightarrow U$ is smooth.

[2] Any continuous map $g : S^1 \to \mathbb{C}$ is proper since S^1 is compact and $\mathbb{C}$ is Hausdorff.

[3] By hypothesis, f is injective on U and so certainly on S^1.

[4] If a holomorphic function $f : U \to \mathbb{C}$ is injective, then f is an immersion, i.e. $f'(z) \neq 0$ on U. This follows, for instance, from the local normal form of holomorphic maps (see Theorem 1.43 in the notes), since if $f'(z)$ has a zero of order $n \geq 0$ at a point, then up to holomorphic changes of coordinates f looks locally like $z \mapsto z^{n+1}$ around that point, so that if $n \geq 1$, the function f cannot be injective in a neighborhood of this point.

Q3. Suppose $f(z)$ is analytic. Compute

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}|f(z)|^2.$$

Solution. We have for a holomorphic function $f(z)$ that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}|f(z)|^2 = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}(f(z)\overline{f(z)}) = \frac{\partial}{\partial z}\left(f(z)\frac{\partial}{\partial \overline{z}}\overline{f(z)}\right),$$

where we have used the product rule and that

$$\frac{\partial}{\partial \overline{z}}f(z) = 0.$$

Next note that

$$\frac{\partial}{\partial \overline{z}}\overline{f(z)} = \overline{\frac{\partial f}{\partial z}(z)} = \overline{f'(z)}.$$

Therefore, the required quantity equals

$$\frac{\partial}{\partial z}(f(z)\overline{f'(z)}) = \overline{f'(z)}\frac{\partial}{\partial z}f(z) = \overline{f'(z)}f'(z) = |f'(z)|^2,$$

where again in the first step we have used the product rule and the fact that

$$\frac{\partial}{\partial z}\overline{f'(z)} = 0.$$

Therefore, we have shown that for holomorphic f, we have

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}|f(z)|^2 = |f'(z)|^2.$$

■

Remark 2. One could also do this calculation by resorting to x-y coordinates, but you will eventually want to learn how to do such calculations without doing so. This is because the approach using z and $\overline{z}$ generalizes most easily and cleanly to similar calculations in the theory of several complex variables.

Q4. Let $f_1, \dots, f_n$ be analytic functions on Δ. Suppose $\sum_{i=1}^n |f_i(z)|^2 = 1$ for all $z \in \Delta$.

(i) Show all f_i are constant functions, by taking the Laplacian of both sides of the equation.

(ii) Show all f_i are constant functions, by applying the maximum principle (or open mapping theorem) to suitable linear combinations of these functions.

(iii) Let $H : \mathbb{C}^n \to \mathbb{R}$ be a strictly convex smooth function. Using method (i) or (ii), prove that if $H(f_1(z), \dots, f_n(z))$ is constant on Δ, then each function $f_i(z)$ is constant.

Proof.

(i) Recall that the Laplacian is the operator defined by

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

The previous problem shows that if $f : U \to \mathbb{C}$ is holomorphic, where $U \subset \mathbb{C}$ is open, then we have that

$$\Delta |f(z)|^2 = 4|f'(z)|^2$$

on U. On the other hand, the Laplacian of any constant function is clearly zero. Therefore, it follows from taking $U = \Delta$[5] and applying the Laplacian operator to both sides of

$$\sum_{i=1}^n |f_i(z)|^2 = 1$$

that

$$4\sum_{i=1}^n |f_i'(z)|^2 = 0.$$

Since the $|f_i(z)|^2$ are nonnegative real numbers, this forces that $f_i'(z) = 0$ for each $i = 1, \dots, n$ and $z \in \Delta$, forcing each f_i to be locally constant and hence constant as needed.

(ii) Let $f := (f_1, \dots, f_n) : \Delta \to \mathbb{C}^n$. Recall that the usual Hermitian inner product on $\mathbb{C}^n$ is defined as

$$\langle z, w \rangle := \sum_{i=1}^n z_i \overline{w}_i$$

for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. For any point $w \in \mathbb{C}^n$, define the map $g_w : \mathbb{C}^n \to \mathbb{C}$ by

$$g_w(z) = \langle z, w \rangle.$$

Now fix any $t \in \Delta$, and consider the function $h_t : \Delta \to \mathbb{C}$ given by $h_t(z) := g_{f(t)}(f(z))$; this is a holomorphic function. It follows that the function $\operatorname{Re} h_t : \Delta \to \mathbb{R}$ is a harmonic function which satisfies

$$\operatorname{Re} h_t(z) \leq |h_t(z)| = |\langle f(z), f(t) \rangle| \leq |f(z)| \cdot |f(t)| = 1,$$

where in the third step we have used the Cauchy-Schwarz inequality. Since equality holds when $z = t$, it follows from the Maximum Principle that equality holds everywhere in the above chain of inequalities for every $z \in \Delta$. Equality at the first step tells us that

$$\langle f(z), f(t) \rangle \in \mathbb{R}_{\geq 0}. \tag{1}$$

Equality at the second step tells us that there is an $\alpha \in \mathbb{C}$ (a priori possibly dependent on z and t) such that

$$f(z) = \alpha f(t). \tag{2}$$

By putting equations (1) and (2) together and using $|f(t)|^2 = 1$, we get

$$\mathbb{R}_{\geq 0} \ni \langle \alpha f(t), f(t) \rangle = \alpha |f(t)|^2 = \alpha,$$

[5] Here Δ is the unit disc–apologies for this overloaded notation! Hopefully, this will not cause any confusion.

which combined with

$$1 = |f(z)|^2 = |\alpha|^2 |f(t)|^2 = |\alpha|^2$$

tells us that $\alpha = 1$, so that in fact $f(z) = f(t)$. Since this holds for all $z \in \Delta$, it follows that $f(z)$ is the constant function $f(t)$.[6]

Remark 3. Here's what's happening geometrically: for any $w \in \mathbb{C}^n$ of length $r := |w| > 0$, the map $\operatorname{Re} g_w : \mathbb{C}^n \to \mathbb{R}$ is a real linear functional whose level sets are parallel to the geometric tangent plane to the sphere $S^{2n-1}(r)$ of radius r in $\mathbb{C}^n$ at the point w, which is defined by the equation $\operatorname{Re} g_w = r$. Therefore, fixing a $t \in \Delta$ and $w = f(t)$, then the map $z \mapsto \operatorname{Re} h_t(z)$ measures the value of this functional precomposed with f. Since the image of z under this harmonic map lies in one half-space determined by the geometric tangent plane $\operatorname{Re} g_{f(t)} = 1$ and actually meets it at the unique point $z = t$, this contradicts that images of nonconstant holomorphic maps look like "saddles", unless f happens to be constant. Note that $S^{2n-1}(r)$ is the level set of the strictly convex function $H : z \mapsto |z|$ passing through w, and that the geometric tangent plane $\operatorname{Re} g_w = r$ is the unique supporting hyperplane of the convex set $\Omega_r := \{z : |z| \leq r\}$ at the point w. This observation allows us to generalize (ii) to (iii) below.

(iii) We show how to generalize (ii).[7] Suppose that H and f are as given, with the function $H \circ f \equiv r$ for some constant $r \in \mathbb{R}$. Then the set $\Omega_r := H^{-1}(-\infty, r]$ is a convex subset of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with boundary $\partial\Omega_r = H^{-1}(r)$, which is either a point or a smoothly embedded real submanifold of dimension $2n-1$ in $\mathbb{C}^n$. Note that f takes Δ to $\partial\Omega_r$, so if it is a point, we are done; hence assume that we are in the second case. Fix any $t \in \Delta$, and pick any supporting hyperplane Λ to Ω_r at $f(t)$, and a real linear map $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $\rho(f(z)) \leq \rho(f(t))$ for all $z \in \Delta$ and $\Lambda := \{w \in \mathbb{C}^n : \rho(w) = \rho(f(t))\}$ (such a ρ is given by taking the dot product with the outward normal vector to $\partial\Omega_r$, or equivalently its tangent hyperplane, at $f(t)$). Then the composite function $\rho \circ f : \Delta \to \mathbb{R}$ is a harmonic function that achieves a maximum value at $z = t$, and is hence constant, so that f takes Δ to Λ and hence to $\Lambda \cap \partial\Omega_r$. But now, since the boundary $\partial\Omega_r$ is smooth and H is strictly convex, we must have $\Lambda \cap \partial\Omega_r = \{f(t)\}$, and this tells us that f is constant.

■

Remark 4. It is not true that if $H : \mathbb{C}^{2n} \to \mathbb{R}$ is a strictly convex smooth function, then the real Hessian $[\partial^2 H/\partial q^i \partial q^j]$ is necessarily positive definite at all points of $\mathbb{C}^n = \mathbb{R}^{2n}_{q^1,\ldots,q^{2n}}$; some submitted solutions made this error. A simple counterexample is given by taking $n = 1$ and $H(z) := x^4 + y^4 = \frac{1}{8}\left(z^4 + \overline{z}^4 + 6z^2\overline{z}^2\right)$. Therefore, the naive approach to generalizing (i), which computes

$$\Delta(H \circ f)(z) = \langle f'(z), (\nabla^2 H)(f(z)) \cdot f'(z) \rangle,$$

where $\nabla^2 H$ is the complex Hessian (also called Levi matrix)

$$\nabla^2 H := \left[\frac{\partial^2 H}{\partial z_j \partial \overline{z}_k}\right]_{j,k}$$

and then says that $\nabla^2 H$ is everywhere positive definite on the image of f to conclude that $f'(z) \equiv 0$ will not quite work the same way.[8] However, if it impose the slightly stronger hypothesis that $\nabla^2 H$ is everywhere nondegenerate, then this generalization works. In this case, the subset $\operatorname{Int} \Omega_r = H^{-1}(-\infty, r)$ is called strongly (Levi) pseudoconvex. The image of $f : \Delta \to \mathbb{C}^n$ for a holomorphic f is usually called an analytic disc, and this result is often stated as: the boundary of a strongly pseudoconvex domain cannot contain a nonconstant analytic disc (see [1, Ex 1, §3.2, p. 137]). If you're really interested in this, see [1, Chapter 3] for much more on pseudoconvexity and (pluri)subharmonicity.

Remark 5. Some people turned in absolutely nothing for part (iii). I understand that this is a challenging problem, but, in the future, please try to turn in partial progress to get partial credit for problems you can't solve completely.

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[6] Alternatively, in this case, one could conclude directly from $|h_t(z)| \leq 1$ on $z \in \Delta$ with $h_t(t) = 1$ using the Maximum Principle for holomorphic functions that $h_t \equiv 1$ identically on $z \in \Delta$. Then equation (1) follows immediately, and equation (2) follows again from the equality condition in the Cauchy-Schwarz inequality, and the rest of the proof proceeds as before.

[7] For the generalization of (i), see Remark 4.

[8] One could try to make the argument that the set of points where the Hessian is not positive definite is "small" in some sense, but that seems hard to quantify at this level of generality.

Q5.

(i) Show that $\log(2) = 1 - 1/2 + 1/3 - 1/4 + \cdots$.

(ii) Show that $\log(2) = \sum_1^\infty 2^{-k}/k$. (Hint: this is obtained from (i) by Euler's method, but a much simpler proof can be given.)

(iii) Sum the first 5 terms of each series, and compare the results to the actual value of $\log(2)$.

Solution.

(i) We use Abel's Theorem.

Theorem 0.0.1 (Abel)**.** Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a power series with radius of convergence $R = 1$. Suppose that the series converges at $z = 1$, so that $f(1) := \sum_{n\geq 0} a_n$ is well-defined. Then we have that $f(z)$ converges to $f(1)$ as z approaches 1 in a Stolz angle, i.e. such that $|1-z|/(1-|z|)$ remains bounded. In particular, this is the case when z approaches 1 from the left on the real axis, denoted $z \to 1^-$.

Proof. See [2, Thm. 3, §2.2.5]. ■

Now simply use the above for the power series

$$\log(1+z) = \sum_{n\geq 1} (-1)^{n-1} \frac{z^n}{n}$$

which has radius of convergence $R = 1$. Note that this series converges at $z = 1$ by the alternating series test. It follows therefore that

$$\log(2) = \lim_{z\to 1^-} \log(1+z) = \lim_{z\to 1^-} \sum_{n\geq 1} (-1)^{n-1} \frac{z^n}{n} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n},$$

where the first step uses that $\log(1+z)$ can be extended to be defined and continuous (and in fact holomorphic) in a neighborhood of $z = 1$ (and the value of this extension at $z = 1$ is what we mean by $\log(2)$), and the last step uses Abel's Theorem. Another way to go about doing this would be to use Theorems 1.9-1.10 from the notes, along with the identity principle.

(ii) The proof using Euler's method (Theorem 1.9 in the notes) uses that (in the same notation), we have

$$F(y) = \log\left(1 + \frac{y}{1-y}\right) = -\log(1-y) = \sum_{k\geq 1} \frac{y^k}{k}$$

so that $b_0 = 0$ and $b_k = 1/k$ for $k \geq 1$. Then Theorem 1.10 tells us that

$$e_n = \sum_{k=1}^{n} \frac{b_k}{2^k} = \sum_{k=1}^{n} \frac{1}{k2^k} \xrightarrow{n\to\infty} \sum_{n\geq 1} (-1)^{n-1} \frac{1}{n},$$

so that given the result in (i) we conclude that $e_n \xrightarrow{n\to\infty} \log(2)$ as needed. The "much simpler proof" is given by observing that

$$\log(2) = -\log\left(1 - \frac{1}{2}\right) = \sum_{k\geq 1} \frac{1}{k2^k},$$

where in the last step we use that $-\log(1-y)$ has radius of convergence $R = 1 > |1/2|$.

(iii) The first five terms of the first series sum to $47/60 = 0.78\overline{3}$, and the first five terms of the second series sum to $661/960 = 0.688541\overline{6}$. The actual value of $\log(2) \approx 0.69314...$, and so the first estimate has a relative error of about 13.01%, whereas the second estimate has a much lower relative error of -0.66%. The second estimate is much better because the geometric convergence of the latter series.

■

Remark 6. Some solutions to (i) simply plugged in $z = 1$ into the power series for $\log(1+z)$ around the origin to deduce the result. This proof is incorrect (or at least incomplete), since the power series has radius of convergence 1 and you can a priori only evaluate the value of a function at a point by plugging it into its power series around another point when the former lies strictly within the relevant disc of convergence (i.e. not on the boundary). Here's another way to state the problem with this attempted solution. Let $f : \mathbb{C} \smallsetminus (-\infty, -1] \to \mathbb{C}$ be the unique holomorphic function satisfying $e^{f(z)} = 1 + z$ and $f(0) = 0$; this is what we mean by $f(z) = \log(1+z)$.[9] We want to evaluate $f(1)$. Next, let $g : \Delta \to \mathbb{C}$ be the function defined by the power series

$$g(z) = \sum_{n\geq 0} (-1)^n \frac{z^n}{n+1}.$$

Then we have $f(z) = g(z)$ on Δ, but this does not mean that $f(1) = g(1)$, since a priori $g(1)$ is not even defined. Even if we define $g(1) := \sum_{n\geq 0}(-1)^n/(n+1)$, then it is not clear that this defines a continuous extension of g to $\Delta \cup \{1\}$, and so we can't conclude that $f(1) = g(1)$ without some additional argument.

Remark 7. Some solutions claimed that the fact that $\log(1+z)$ can be defined to be holomorphic in a neighborhood of $z = 1$ implies that this power series expansion is valid (and in particular converges conditionally) in some neighborhood of $z = 1$. This is false: even though $\log(1+z)$ can be defined to be holomorphic in a neighborhood of $z = 1$, there is no neighborhood of $z = 1$ on which this power series expansion for $\log(1+z)$ is valid (why?). If we consider the example $f(z) = 1/(1+z)$, we see that this function can be extended to be holomorphic in a neighborhood of $z = 1$, but its value at 1, namely $f(1) = 1/2$, is not obtained by plugging it into the power series, which yields the divergent $1 - 1 + 1 - 1 + \cdots$. However, Abel's Theorem says that

- if a power series converges conditionally at a boundary point p,
- and the corresponding function can be extended to be continuous in a neighborhood of p,

then the value of the extension at p can be obtained from the conditionally convergent series. Both hypotheses are necessary.

Remark 8. Note that, as discussed in section/OH, you could not simply quote Abel's theorem without at least indicating a proof sketch because the argument that goes into the proof is an essential step in proving this result (and many more). Some solutions "proved" a version of Abel's Theorem that does not include the hypothesis that z approaches 1 in a Stolz angle, and this version of the theorem is false, e.g. consider the example on Wikipedia where

$$f(z) = \sum_{n\geq 1} \frac{z^{3^n} - z^{2\cdot 3^n}}{n}.$$

Then this power series converges at $z = 1$ to 0, but the function $f(z)$ is unbounded on the sequence of points $z_n := e^{3^{-n}\pi i}$. Note that this sequence does not approach 1 in a Stolz angle.

[9] Here we could replace $\mathbb{C} \smallsetminus (-\infty, -1]$ by any simply connected open subset containing $\Delta \cup \{1\}$.

Q6. Let $C(\overline{\Delta})$ denote the Banach space of continuous complex-valued functions on the [closed] unit disk, with $\|f\| = \sup|f(z)|$. Let $f \in C(\overline{\Delta})$ be analytic on Δ.

(i) Prove that the $\mathbb{C}$-algebra $\mathbb{C}[f]$ generated by f is not dense in $C(\overline{\Delta})$.
(ii) Prove that $\mathbb{C}[f, \overline{f}]$ is dense in $C(\overline{\Delta}) \Leftrightarrow f$ is injective on $\overline{\Delta}$. (Here $\overline{f}(z) := \overline{f(z)}$.)

Proof. We use the following convention from class here and below: we say that a function $f : \overline{\Delta} \to \mathbb{C}$ is holomorphic and write $f \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ if it is "holomorphic on Δ and continuous up to the boundary", i.e. if it is a continuous map and the restriction $f|_\Delta$ is holomorphic. The set $\mathcal{O}(\Delta) \cap C(\overline{\Delta})$ of such functions is clearly a complex subalgebra of $C(\overline{\Delta})$.

(i) Since $\mathbb{C}[f] \subseteq \mathcal{O}(\Delta) \cap C(\overline{\Delta})$, it follows that

$$\overline{\mathbb{C}[f]} \subseteq \overline{\mathcal{O}(\Delta) \cap C(\overline{\Delta})} = \mathcal{O}(\Delta) \cap C(\overline{\Delta}),$$

where in the last step we have used that $\mathcal{O}(\Delta) \cap C(\overline{\Delta})$ is a closed subspace of $C(\overline{\Delta})$ (this follows from Corollaries 1.22 and 1.23 in the notes). Therefore, it suffices to observe that

$$\mathcal{O}(\Delta) \cap C(\overline{\Delta}) \subset C(\overline{\Delta})$$

is a proper subspace; and indeed, complex conjugation σ defined by $\sigma(z) = \overline{z}$ is a function such that

$$\sigma \in C(\overline{\Delta}) \smallsetminus \big(\mathcal{O}(\Delta) \cap C(\overline{\Delta})\big) .$$

(ii) The function f is not injective on $\overline{\Delta}$ iff there are $z, w \in \overline{\Delta}$ such that $z \neq w$ but $f(z) = f(w)$; suppose this is the case and pick such z, w. In this case, we have that $p(z) = p(w)$ for all $p \in \mathbb{C}[f, \overline{f}]$. Now, if $\mathbb{C}[f, \overline{f}]$ is dense in $C(\overline{\Delta})$, then there is a function $p \in \mathbb{C}[f, \overline{f}]$ that approximates the identity function $\mathrm{id} \in C(\overline{\Delta})$ with an error less than $\varepsilon := |z - w|/2 > 0$ in the supremum norm, i.e. satisfies $\|p - \mathrm{id}\| < \varepsilon$. But then we get that

$$|z - w| = |(p(z) - z) - (p(w) - w)| \leq |p(z) - z| + |p(w) - w| \leq 2\,\|p - \mathrm{id}\| < 2\varepsilon = |z - w|,$$

which is a contradiction, proving one direction. For the other direction, suppose that $f \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ is injective on $\overline{\Delta}$. We use the Stone-Weierstrass Theorem:

Theorem 0.0.2 (Stone-Weierstrass)**.** Let X be a compact topological space, and let $C(X)$ be the algebra of continuous complex-valued functions on X. If $\mathcal{A} \subset C(X)$ is any complex subalgebra

(a) that is invariant under complex conjugation, and

(b) that separates points (i.e. for all distinct $z, w \in X$ there is a $p \in \mathcal{A}$ such that $p(z) \neq p(w)$),

then $\mathcal{A}$ is dense in $C(X)$ in the supremum norm.

Proof. See [3, Theorem 7.33]. ■

Now take $X = \overline{\Delta}$ and $\mathcal{A} = \mathbb{C}[f, \overline{f}]$. Note that (a) is trivially satisfied[10] and (b) is satisfied because $\{f\} \subset \mathcal{A}$ already separates points, since f is injective. It follows then from the Stone-Weierstrass Theorem that $\mathbb{C}[f, \overline{f}]$ is dense in $C(\overline{\Delta})$ as needed.

■

Remark 9. You were allowed to simply quote the Stone-Weierstrass theorem, as long as you stated it precisely (this applies more generally to any theorem you quote in the solutions). For instance, the statement that "the reverse direction follows from Stone-Weierstrass" is not sufficient by itself.

[10]This uses that we are including $\overline{f}$, and is exactly what goes wrong in (i) above.

Q7. Show that for any polyonomial $p(z)$ there is a z with $|z| = 1$ such that $|p(z) - 1/z| \geq 1$.

Solution. Note that the assumption that $p(z)$ is polynomial is not needed: we only need $p(z)$ to be holomorphic on $\overline{\Delta}$.[11] We show:

Theorem 0.0.3. Let $p : \overline{\Delta} \to \mathbb{C}$ be holomorphic. Then there is a $z \in S^1$ such that $|p(z) - 1/z| \geq 1$.

Proof. Consider the holomorphic function $q : \overline{\Delta} \to \mathbb{C}$ defined by $g(z) = zp(z) - 1$. Then we have

$$1 = |g(0)| \leq \sup_{z \in \overline{\Delta}} |g(z)| = \sup_{z \in \partial\Delta} |g(z)| = \max_{z \in S^1} |g(z)|,$$

where we have used in the penultimate step the Maximum Principle and in the last step that $\partial\Delta = S^1$ is compact. The statement

$$1 \leq \max_{z \in S^1} |g(z)|$$

is equivalent to the statement there is a $z \in S^1$ such that $|g(z)| \geq 1$, and then for this z we have

$$\left| p(z) - \frac{1}{z} \right| = |z| \cdot \left| p(z) - \frac{1}{z} \right| = |zp(z) - 1| = |g(z)| \geq 1$$

as needed. ■

Aliter. Suppose to the contrary that $|p(z) - 1/z| < 1$ is for all $z \in S^1$. Apply Rouché's Theorem (Corollary 1.32 in the notes) to $U = \Delta$, $f(z) \equiv 1$, $g(z) = zp(z) - 1$: note that we have for all $z \in S^1$ that

$$|f(z)| = 1 > \left| p(z) - \frac{1}{z} \right| = |zp(z) - 1| = |g(z)|,$$

so Rouché's Theorem applies and tells us that $f(z) \equiv 1$ and $f(z) + g(z) = zp(z)$ have the same number of zeroes on Δ. This is clearly a contradiction since the former has none and the latter has at least one, namely $z = 0$. ■

[11] It is only included so that the statement reads: linear combinations of powers of z cannot approximate $\overline{z}$ on $\overline{\Delta}$ uniformly.

References

[1] S. G. Krantz, *Function Theory of Several Complex Variables.* AMS Chelsea Publishing, second ed., 2000.

[2] L. V. Ahlfors, *Complex Analysis.* International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., third ed., 1979.

[3] W. Rudin, *Principles of Mathematical Analysis.* International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., third ed., 1976.