Math 213A F23 Homework 11 Solutions

Dhruv Goel

November 2023

If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Let $\Lambda \subset \mathbb{C}$ be a lattice, let $X = \mathbb{C}/\Lambda$ and let $\mathrm{End}(\Lambda) = \{\alpha \in \mathbb{C} : \alpha\Lambda \subset \Lambda\}$.

(a) Show that for each $\alpha \in \mathrm{End}(\Lambda)$, the formula $[f(z)] = [\alpha z]$ defines an analytic covering map $f : X \to X$ of degree $|\alpha|^2$.

(b) For what values of $\alpha \in \mathbb{C}$ does there exist a lattice Λ with $\alpha \in \mathrm{End}(\Lambda)$?

(c) Conclude that $\mathrm{End}(\mathbb{Z} \oplus \mathbb{Z}\tau) = \mathbb{Z}$ for almost all values of τ.

Solution.

(a) From the inclusion $\alpha\Lambda \subset \Lambda$ and the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & X \end{array}$$

we get that f is a well-defined map; since X is compact, f is proper, and since π and α are local homeomorphisms, so is f. It follows from a previous homework problem that f is a covering map. Further, the degree of f is

$$\#f^{-1}(0) = \#\alpha^{-1}\Lambda/\Lambda = [\alpha^{-1}\Lambda : \Lambda] = \frac{\mathrm{covol}(\Lambda)}{\mathrm{covol}(\alpha^{-1}\Lambda)} = |\alpha|^2,$$

where we have used the following lemma:

Lemma 0.0.1. Let $\Lambda \subset \mathbb{C}$ be a full rank lattice. Then:

(i) Given any $\alpha \in \mathbb{C}^\times$, we have $\mathrm{covol}(\alpha\Lambda) = |\alpha|^2 \mathrm{covol}(\Lambda)$.

(ii) If $\Lambda' \subset \mathbb{C}$ is any other lattice such that $\Lambda \subset \Lambda'$, then the index of Λ in Λ' as an abelian group satisfies

$$[\Lambda' : \Lambda] = \frac{\mathrm{covol}(\Lambda)}{\mathrm{covol}(\Lambda')}.$$

Proof. Recall that the covolume of a full rank lattice $\Lambda \subset \mathbb{R}^n$ is defined to be the volume of any of its fundamental parallelopipeds, which can be expressed as the absolute value of the determinant of any matrix whose rows form a basis of Λ (and that this is well-defined uses that any two bases are related by a matrix in $\mathrm{SL}_n(\mathbb{Z})$).

(a) This is clear from our definition of the covolume, since multiplication by α takes a fundamental parallelogram of Λ to a fundamental parallelogram of $\alpha\Lambda$ and stretches areas by a factor of $|\alpha|^2$.

(b) By a linear isomorphism, it suffices to consider the case $\Lambda' = \mathbb{Z}^2$; let Λ be generated by $\lambda_1 = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\lambda_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ for $a, b, c, d \in \mathbb{Z}$. Then, by performing elementary row operations on the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which amounts to picking a different basis of Λ, we may assume that Λ has a basis of the form $\lambda_1' = \begin{bmatrix} \delta \\ 0 \end{bmatrix}$ and $\lambda_2' = \begin{bmatrix} b' \\ d' \end{bmatrix}$, where $\delta = \gcd(a, c)$ and $b', d' \in \mathbb{Z}$ with $\delta \cdot d' \neq 0$ (because λ_1 and λ_2 are linearly independent). It is then clear that

$$[\mathbb{Z}^2 : \Lambda] = \delta \cdot d' = \mathrm{covol}(\Lambda),$$

and $\mathrm{covol}(\mathbb{Z}^2) = 1$.

(b) We claim:

Lemma 0.0.2. Given an $\alpha \in \mathbb{C}$, there exists a lattice Λ with $\alpha \in \operatorname{End}(\Lambda)$ iff $\alpha \in \mathbb{Z}$ or α is a nonreal quadratic algebraic integer.

Proof. If $\alpha \in \mathbb{Z}$, then any lattice works; if α is a nonreal quadratic algebraic integer, then we may take $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\alpha$ (why?). Conversely, if such a lattice Λ exists; then with respect to some basis of Λ, the map given by multiplication by α is represented by a 2×2 integer matrix M. Therefore, by the Cayley-Hamilton Theorem, it follows that α satisfies

$$\alpha^2 - \alpha \operatorname{tr} M + \det M = 0,$$

so that α is a(n) (at most) quadratic algebraic integer. It remains to note that if $\alpha \notin \mathbb{Z}$, then $\alpha \notin \mathbb{R}$; indeed, if $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, then taking any vector $\lambda \in \Lambda$ of minimal length in $\Lambda \cap \mathbb{R}\lambda$ and noting that $\alpha\lambda - \lfloor \alpha \rfloor \lambda \in \Lambda \cap \mathbb{R}\lambda$ gives us the required contradiction.

(c) If $\alpha \in \operatorname{End}(\mathbb{Z} \oplus \mathbb{Z}\tau) \setminus \mathbb{Z}$, then α is a nonreal quadratic algebraic integer. Since $\alpha \notin \mathbb{R}$, it follows from $\alpha \in \mathbb{Z} \oplus \mathbb{Z}\tau$ that $\tau \in \mathbb{Q}(\alpha)$. In particular, τ is algebraic, and hence the set of such τ is countable.

Remark 1. Note that every endomorphism $X \to X$ is of this form by covering space theory (why?). The above problem gets you some way along proving the theorem that the endomorphism ring $\operatorname{End}(X)$ of an elliptic curve in characteristic zero is either $\mathbb{Z}$ or an order in a nonreal qudratic number field.[1] This can be used to prove that, over an algebraically closed base field of characteristic 0, the automorphism group of an elliptic curve is either trivial (if $j \notin \{0, 1728\}$), a cyclic group of order 4 (if $j = 1728$) or a cyclic group of order 6 (if $j = 0$). See [1, §III.9-10].

[1] By the way, this is not true in positive characteristic, where these endomorphism rings can be orders in noncommutative quaternionic algebras.

Q2. What are the zeroes of the $\wp$-function for the lattice $\mathbb{Z} \oplus \mathbb{Z}\mathrm{i}$? For the lattice $\mathbb{Z} \oplus \mathbb{Z}\omega$ (here $\omega := \zeta_3 = (-1 + \sqrt{-3})/2$)?[2]

Solution. Let $\wp_\tau(z)$ denote the $\wp$-function associated to the lattice $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$ for $\tau \in \mathbb{H}$. Note that $\wp_\tau$ has a pole of order 2 at $z = 0$ on $X_\tau := \mathbb{C}/\Lambda_\tau$, and hence has either a double zero or two distinct simple zeroes on X_τ, which then correspondingly lift to zeroes on $\mathbb{C}$. Since $\wp_\tau(-z) = \wp_\tau(z)$, these zeroes must be of the form $\pm\alpha_\tau$ on X_τ (this can also be seen from the Argument Principle, or the Abel-Jacobi condition). It therefore suffices to determine the $\pm\alpha_\tau$ on X_τ for $\tau \in \{\mathrm{i}, \omega\}$.

(a) Note that

$$\wp_{\mathrm{i}}(\mathrm{i}z) = -\wp_{\mathrm{i}}(z), \tag{1}$$

which follows from the fact that $\wp_{\mathrm{i}}(\mathrm{i}z)$ is a doubly periodic function with respect to the lattice $\mathbb{Z} \oplus \mathbb{Z}\mathrm{i}$, with poles only at the lattice points, and the polar part of $\wp_{\mathrm{i}}(\mathrm{i}z)$ at $z = 0$ is the negative of that of $\wp_{\mathrm{i}}(z)$, since $(1/\mathrm{i})^2 = -1$. If $z = (1 + \mathrm{i})/2$, then $\mathrm{i}z = z \in X_{\mathrm{i}}$, and hence it follows from the identity (1) that $\wp_{\mathrm{i}}((1+\mathrm{i})/2) = 0$. Since this z is a critical point of $\wp_{\mathrm{i}}$, it follows that $\wp_{\mathrm{i}}$ has a double zero at $\alpha_{\mathrm{i}} := (1 + \mathrm{i})/2$.

(b) Similarly, we have

$$\wp_\omega(z) = \omega\wp_\omega(z). \tag{2}$$

If $z = (\omega + 2)/3$, the $\omega z = z \in X_\omega$, and hence it follows from (2) that $\wp_\omega((\omega + 2)/3) = 0$. Since this root is not a point of order 2, it follows that $\wp_\omega$ has two distinct roots on X_ω given by $\pm\alpha_\omega = \pm(\omega + 2)/3$.

[2] The problem statement specifies $\omega = (1 + \sqrt{-3})/2$, but the letter ω is usually reserved for ζ_3 and not for ζ_6. Of course, they generate the same lattice by virtue of $\zeta_6 = 1 + \zeta_3$, and so you may choose to work with whichever–I slightly prefer working with $\omega = \zeta_3$.

Q3. Prove that there exists a pair of nonconstant meromorphic functions on $\mathbb{C}$ such that $f(z)^3 + g(z)^3 = 1$. (Hint: show the equation $x^3 + y^3 = 1$ defines the same elliptic curve in $\mathbb{P}^2$ as $y^2 = 4x^3 - 1$.)

Solution. That such a parametrization is plausible follows from the fact that $u^3 + v^3 = 1$ defines a smooth plane cubic in $\mathbb{CP}^2$, and hence by our uniformization theorem for elliptic curves can be parametrized by using Weierstrass $\wp$-function for a suitable lattice. To find such a parametrization explicitly, note that the (projective) transformation of coordinates

$$u = \frac{\sqrt{3} + y}{2\sqrt{3}x}, \quad v = \frac{\sqrt{3} - y}{2\sqrt{3}x}$$

takes the curve $y^2 = 4x^3 - 1$ isomorphically to the curve $u^3 + v^3 = 1$; therefore, it suffices to parametrize $y^2 = 4x^3 - 1$ using nonconstant meromorphic functions on $\mathbb{C}$. Now $y^2 = 4x^3 - 1$ is in Weierstraß form.

For a lattice $\Lambda \subset \mathbb{C}$, the Weierstraß $\wp$-function satisfies

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

For $\mathbb{Z} \oplus \mathbb{Z}\omega$ where $\omega = \zeta_3 = e^{2\pi i/3}$, we have $g_2 = 0$ and $g_3 \neq 0$. Take any $t \in \mathbb{C}$ such that $t^6 = g_3(\omega)$, we have $g_3(t\omega) = t^{-6}g_3(\omega) = 1$, whereas $g_2(t\omega) = 0$. Therefore, if $\wp(z)$ denotes the Weierstrass $\wp$-function corresponding to the lattice $\Lambda = \mathbb{Z}t \oplus \mathbb{Z}t\omega$, then $(\wp(z), \wp'(z))$ is a pair of nonconstant meromorphic functions on $\mathbb{C}$ that map to the curve $y^2 = 4x^3 - 1$.

Remark 2. The isomorphism between the curves $x^3 + y^3 = 1$ and $y^2 = 4x^3 - 1$ in the above solution can be obtained by standard methods used in dealing with plane cubics: move a flex and the tangent line to $[0 : 1 : 0]$ and the line at infinity respectively, and then normalize in the finite plane; see [2, Appendix B]. This, by the way, is the only class of solutions to this equation; see [3].

Q4. Given $\tau \in \mathbb{H}$, let $\wp(z)$ be the Weiestraß $\wp$-function for the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, and let

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{\pi^2}{\sin^2\left(\pi(z - n\tau)\right)}.$$

Prove that $f(z) = \wp(z) + C$, and express the constant C in terms of the function $G_1(\tau)$ defined by [the forbidden Eisenstein series].

Proof. The absolute and uniform convergence on compact subsets of the series defining $f(z)$ on $\mathbb{C} \setminus \Lambda$ is clear from the inequality

$$|\sin z| \geq \sinh\left(|\mathrm{Im}\, z|\right)$$

for $z \in \mathbb{C}$ along with $\mathrm{Im}\,\tau > 0$ (details left to the reader). From this, all rearrangement is justified in concluding that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$, so that f is doubly periodic for the lattice Λ. To show the result, we use the identity

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{m \in \mathbb{Z}} \frac{1}{(z-m)^2},$$

where the series converges absolutely and uniformly on from compact subsets of $\mathbb{C} \setminus \mathbb{Z}$. From this, it follows that $f(z)$ has a double pole at $z = 0$ with polar part $1/z^2$, so it follows that $f(z) - \wp(z)$ is a doubly periodic holomorphic function and hence constant, say C. To compute this constant, recall that $\wp(z) = z^{-2} + O(z)$, so that

$$C = \lim_{z \to 0}\left(f(z) - \frac{1}{z^2}\right) = \lim_{z \to 0}\left[\left(\frac{\pi^2}{\sin^2(\pi z)} - \frac{1}{z^2}\right) + \sum_{n \in \mathbb{Z}}{}' \frac{\pi^2}{\sin^2\left(\pi(z - n\tau)\right)}\right] = \frac{\pi^2}{3} + \sum_{n \in \mathbb{Z}}{}' \frac{\pi^2}{\sin^2(n\tau)},$$

where the prime means as usual that we omit zero. On the other hand, we have

$$G_1(\tau) := \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}}{}' \frac{1}{(m+n\tau)^2},$$

where the prime denotes in this case that for $n = 0$ we omit the summand $m = 0$. Separating the term $n = 0$, we get

$$G_1(\tau) = \sum_{m \in \mathbb{Z}}{}' \frac{1}{m^2} + \sum_{n \in \mathbb{Z}}{}' \left(\sum_{m \in \mathbb{Z}} \frac{1}{(m+n\tau)^2}\right) = \frac{\pi^2}{3} + \sum_{n \in \mathbb{Z}}{}' \frac{\pi^2}{\sin^2(n\tau)}$$

as well, so that $C = G_1(\tau)$.

Aliter. We may also directly argue as follows:write

$$f(z) = \sum_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} \frac{1}{(z - m - n\tau)^2}\right) = \frac{1}{z^2} + \sum_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}}{}' \frac{1}{(z - m - n\tau)^2}\right).$$

On the other hand,

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda}{}' \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right) = \frac{1}{z^2} + \sum_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}}{}' \frac{1}{(z - m - n\tau)^2} - \frac{1}{(m+n\tau)^2},\right)$$

where we have used that the series defining $\wp$ converges absolutely and uniformly on compact sets of $\mathbb{C} \setminus \Lambda$. Subtracting, we are left with

$$\begin{aligned}
f(z) - \wp(z) &= \sum_{n\in\mathbb{Z}} \left({\sum_{m\in\mathbb{Z}}}' \frac{1}{(z-m-n\tau)^2} \right) - \sum_{n\in\mathbb{Z}} \left({\sum_{m\in\mathbb{Z}}}' \frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2} \right) \\
&= \sum_{n\in\mathbb{Z}} \left[\left({\sum_{m\in\mathbb{Z}}}' \frac{1}{(z-m-n\tau)^2} \right) - \left({\sum_{m\in\mathbb{Z}}}' \frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2} \right) \right] \\
&= \sum_{n\in\mathbb{Z}} \left[{\sum_{m\in\mathbb{Z}}}' \left(\frac{1}{(z-m-n\tau)^2} - \frac{1}{(z-m-n\tau)^2} + \frac{1}{(m+n\tau)^2} \right) \right] \\
&= \sum_{n\in\mathbb{Z}} \left({\sum_{m\in\mathbb{Z}}}' \frac{1}{(m+n\tau)^2} \right) =: G_1(\tau),
\end{aligned}$$

where the second equality is justified because of the absolute and locally uniform convergence of the outer sum (over n), whereas the third equality is justified because of the absolute and locally uniform convergence of the inner sum (over m, for a fixed n). This gives us $C = G_1(\tau)$.

Remark 3. As above, you have to be very careful in justifying rearrangements in the order of summation; a lot of solutions were sloppy and used the expression ${\sum}'_{\lambda\in\Lambda} \lambda^{-2}$, which is both meaningless, since the sum is not absolute convergent, and incorrect as an expression for $G_1(\tau)$. This is similar to saying, for instance, that

$$ {\sum_{n\in\mathbb{Z}}}' \frac{1}{n} $$

is meaningless, whereas

$$ \lim_{N\to\infty} {\sum_{n=-N}^{N}}' \frac{1}{n} = 0. $$

Q5.

(a) What is the fundamental group of the real projective plane $\mathbb{RP}^2$?
(b) Let $V \subset \mathbb{RP}^2$ be the closed cubic curve defined by $y^2 = x^3 - x$ (including its unique point at infinity). Show that there is no topological disk $\mathbb{D}^2$ such that $V \subset \mathbb{D}^2 \subset \mathbb{RP}^2$. Conclude that there is no automorphism g of $\mathbb{RP}^2$ such that $g(V) \subset \mathbb{R}^2 \subset \mathbb{RP}^2$.

Solution.

(a) The fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2$. One way to show this is to use that for any covering space $\Gamma \to E \xrightarrow{\pi} X$ with path connected E and basepoints $e \in E$ and $x \in X$ with $x = \pi(e)$, we have the short exact sequence of pointed sets

$$0 \to \pi_1(E, e) \xrightarrow{\pi} \pi_1(X, x) \xrightarrow{\delta} \pi_0(\Gamma, e) = \Gamma \to 0,$$

where the map δ is given by path lifting.[3] In particular, if E is simply connected, then $\delta : \pi_1(X, x) \to \Gamma$ is a bijection. Now apply this to the $2:1$ covering map $S^2 \to \mathbb{RP}^2$, and use that S^2 is simply connected, and that there is a unique group with two elements up to isomorphism.[4] An explicit generator of $\pi_1(\mathbb{RP}^2, x)$ is given by any loop based at $x \in \mathbb{RP}^2$ that lifts to a path joining antipodal points $\pm\tilde{x}$ on S^2.

(b) Note that V consists of two loops, one contained in the finite plane $\mathbb{R}^2 \subset \mathbb{RP}^2$ and another containing the point at infinity $\mathsf{O} := [0 : 1 : 0]$. It suffices to show that the latter is not contractible in $\mathbb{RP}^2$ (why?). By part (a), it suffices to show that it lifts to a path joining $\pm\tilde{\mathsf{O}} = (0, \pm 1, 0)$ on S^2, for which we write down an explicit lift. To do this, note that there is a unique continuous function $f : \mathbb{R} \to [1, \infty)$ such that $f(t)^2 = t^3 - t$; this f satisfies $f(t) \to \pm\infty$ as $t \to \pm\infty$.[5] Let $t : (0, 1) \to \mathbb{R}$ be the homeomorphism defined by $t(s) = \tan(\pi(s - 1/2))$. Then the function $\tilde{\gamma} : [0, 1] \to S^2$ defined by

$$\tilde{\gamma}(s) := \begin{cases} (0, -1, 0), & s = 0, \\ (0, 1, 0), & s = 1, \\ \dfrac{(f(t(s)), t(s), 1)}{\sqrt{f(t(s))^2 + t(s)^2 + 1}}, & s \in (0, 1) \end{cases}$$

gives a lift of the loop of V through O, and this clearly connects the antipodal points $\pm\tilde{\mathsf{O}} = (0, \pm 1, 0)$ on S^2.[6]

Remark 4. A few solutions claimed that to show that this loop is not contractible, it suffices to argue that it intersects the line at infinity exactly once. This is incorrect, as the example of the

[3] Explicitly, given a $[\gamma] \in \pi_1(X, x)$, pick a representative $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x$, lift to a map $\tilde{\gamma} : [0, 1] \to E$ with $\tilde{\gamma}(0) = e$, and define $\delta[\gamma] = [\tilde{\gamma}(1)]$. This is well-defined by the homotopy lifting property.

[4] One could also use the Siefert-van Kampen theorem to deduce this result.

[5] The exact formula for f does not matter at all, but is not too hard to write down; namely

$$f(t) = \frac{1}{\sqrt{3}} \left(\lambda_t + \frac{1}{\lambda_t} \right),$$

where

$$\lambda_t := \left(\frac{\sqrt{81t^4 - 12} + 9t^2}{2\sqrt{3}} \right)^{1/3}.$$

This formula holds on the nose (i.e. using the real roots) for $|t| \geq (4/27)^{1/4}$ and for a suitable choice of roots otherwise.

[6] The noncontractibility of this loop can also be shown by observing that projecting to the y-axis gives a homotopy between this loop and the (extended) y-axis, and this clearly defines a noncontractible loop in $\mathbb{RP}^2$ (because it is clear what its lift to S^2 is, say).

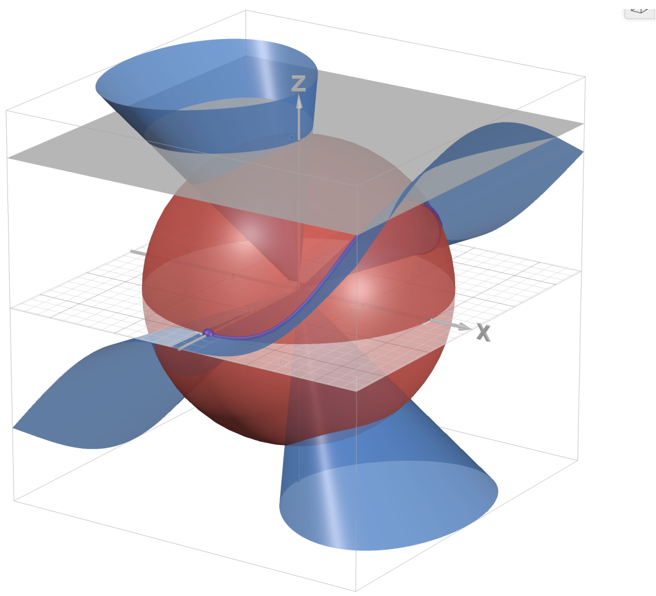

Figure 1: A Desmos diagram representing the curve V. The surface in red is the sphere $x^2 + y^2 + z^2 = 1$, the surface in blue is $y^2 z = x^3 \quad xz^2$, and the surface in gray is the plane $z = 1$ (so that the intersection of the blue and gray surfaces is the 2D plot of the curve $y^2 = x^3 \quad x$).

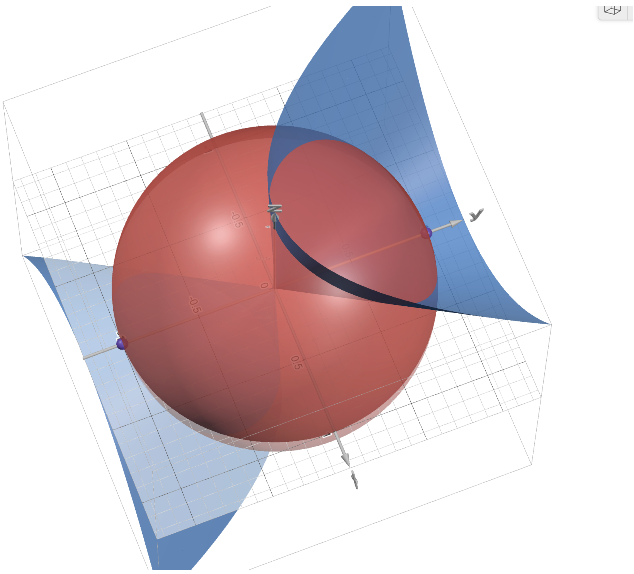

Figure 2: A Desmos diagram representing the parabola $y = x^2$ in RP^2. Here, the surface in red is the sphere $x^2 + y^2 + z^2 = 1$, and the surface in blue is the cone $yz = x^2$.

loop defined by $y = x^2$ shows: this equation defines a parabola in $\mathsf{R}^2 \quad \mathsf{RP}^2$, and appending the point at infinity O yields a contractible closed loop on RP^2 which intersects the line at infinity in the unique point O. See Figures 1 and 2. The key difference in our case is that the equation $y^2 = x^3 \quad x$ defines a curve that intersects the line at infinity with algebraic multiplicity 3 at O, and hence "crosses over", whereas the parabola $y = x^2$ is simply tangent to the line at infinity, i.e. has algebraic multiplicity 2 and hence "stays on the same side". This mod 2 parity of the intersection multiplicity with the line at infinity is another way of interpreting the result $\pi_1(\mathsf{RP}^2) = \mathsf{Z}/2$.

Q6. Let $\wp(z)$ be the Weierstraß $\wp$-function for $E = \mathbb{C}/\Lambda$, and consider the meromorphic 1-form

$$\omega = \frac{\wp(z)\wp''(z)}{\wp'(z)}\mathrm{d}z$$

on E.

(a) Prove that $\mathrm{Res}_0(\omega) = 0$.
(b) Using the residue theorem, that $e_1 + e_2 + e_3 = 0$. Here $e_i = \wp(c_i)$ are the images of the nonzero points of order 2 in the group E.

Solution.

(a) The coefficient function $\wp(z)\wp''(z)/\wp'(z)$ is odd and has a pole of order 3 at the origin, so its polar part around zero looks like

$$\frac{a_{-3}}{z^3} + \frac{a_{-1}}{z} + O(1)$$

for some $a_{-3}, a_{-1} \in \mathbb{C}$.[7] From the expression

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

it follows that the polar parts of $\wp(z)$ and its first few derivatives around $z = 0$ are given by

$$\wp(z) = z^{-2} + O(z), \quad \wp'(z) = -2z^{-3} + O(1), \quad \wp''(z) = 6z^{-4} + O(1).$$

It follows that

$$\frac{\wp(z)\wp''(z)}{\wp'(z)} = \frac{(z^{-2} + O(z))(6z^{-4} + O(1))}{-2z^{-3} + O(1)} = -3z^{-3}\frac{(1 + O(z^3))(1 + O(z^4))}{1 + O(z^3)},$$

so that $a_{-3} = -3$ and $\mathrm{Res}_0(\omega) = a_{-1} = 0$.

(b) The other poles of ω on E are located at the zeroes of $\wp'(z)$, namely at the c_i. Since each zero c_i is simple, it follows from the Argument Principle that

$$\mathrm{Res}_{c_i}\,\omega = \mathrm{Res}_{c_i}\,\wp(z)\left(\frac{\wp''(z)}{\wp'(z)}\,\mathrm{d}z\right) = \wp(c_i) = e_i.$$

Therefore, the Residue Theorem gives us that

$$0 = \sum_{p \in E} \mathrm{Res}_p\,\omega = \mathrm{Res}_0\,\omega + \sum_i \mathrm{Res}_{c_i}\,\omega = 0 + \sum_i e_i$$

as needed.

[7] Here, as always, $O(z^k)$ for $k \in \mathbb{Z}$ represents some function $f(z)$ defined in a punctured neighborhood of zero such that $z^{-k}f(z)$ extends to a holomorphic function at zero.

References

[1] J. H. Silverman, *The Arithmetic of Elliptic Curves*. 106, Graduate Texts in Mathematics, Springer, second ed., 2008.

[2] J. H. Silverman and J. H. Tate, *Rational Points on Elliptic Curves*. Undergraduate Texts in Mathematics, Springer, second ed., 2015.

[3] I. N. Baker, "On a Class of Meromorphic Functions.," *Proceedings of the American Mathematical Society*, vol. 17, no. 4, pp. 819–822, 1966.