Math 213A F23 Homework 1 Solutions

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Let $T \subset \mathbb{R}^3$ be the spherical triangle defined by $x^2 + y^2 + z^2 = 1$ and $x, y, z \geq 0$. Let $\alpha = z\,\mathrm{d}x\,\mathrm{d}z$.

(a) Find a smooth 1-form β on $\mathbb{R}^3$ such that $\alpha = \mathrm{d}\beta$.
(b) Define consistent orientations for T and ∂T.
(c) Using your choices in [(b)], compute $\int_T \alpha$ and $\int_{\partial T} \beta$ directly, and check that they agree. (Why should they agree?)

Solution. Many solutions are possible.

(a) One solution is $\beta = -\frac{1}{2} z^2\,\mathrm{d}x$. All possible solutions are obtained by setting $\beta = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$, and solving the triple of PDE's in the smooth functions f, g, h obtained by requiring $\mathrm{d}\beta = \alpha$.

(b) One approach: $S^2 \subset \mathbb{R}^3$ obtains an orientation as the boundary of the closed disc

$$\mathbb{D}^3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$$

(which has an orientation as a codimension-0 submanifold-with-boundary of the oriented $\mathbb{R}^3$); namely, if u is any outward normal vector at a point $p \in S^2$, then a basis given by $v, w \in \mathrm{T}_p S^2$ is an oriented basis iff u, v, w is an oriented basis of $\mathrm{T}_p \mathbb{R}^3$ in that order. The spherical triangle T obtains an orientation as a codimension-0 submanifold-with-corners of S^2. Given this, to give ∂T the boundary orientation (away from the corners), we do the same as before: if v is any outward normal vector at a boundary point p (so for instance on the arc $T \cap \{z = 0\}$ by the vector $(0, 0, -1)^t$), then $w \in \mathrm{T}_p(\partial T)$ is an oriented basis iff $v, w \in \mathrm{T}_p(T)$ give an oriented basis in that order. For the solution, it suffices to provide a picture that looks like Figure 1.

(c) Let $D \subset \mathbb{R}^2$ be the region given by

$$D = \{(x, y) : 0 \leq x, y \leq 1 \text{ and } x^2 + y^2 \leq 1\}$$

with the orientation on D induced by the orientation on $\mathbb{R}^2$, and the boundary ∂D given the boundary orientation. Then the map $\varphi : D \to T$ given by $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$ is an oriented parametrization of D (and its restriction to ∂D an oriented parametrization of ∂T), as one can check explicitly. Therefore, this can be used to compute the integrals above. Namely, we have

$$\varphi^*(\mathrm{d}z) = -\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{1 - x^2 - y^2}}$$

and so

$$\int_T \alpha = \int_D \varphi^*(\alpha) = \int_D \sqrt{1 - x^2 - y^2}\,\mathrm{d}x \wedge \left(-\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{1 - x^2 - y^2}}\right) = -\int_D y\,\mathrm{d}x \wedge \mathrm{d}y = -\int_D y\,\mathrm{d}x\,\mathrm{d}y.$$

This can be computed by various ways (e.g. a parametrization by polar coordinates); we just use Fubini's Theorem to write this as

$$-\int_0^1 \left(\int_0^{\sqrt{1-x^2}} y\,\mathrm{d}y\right) \mathrm{d}x = -\int_0^1 \frac{1 - x^2}{2} \mathrm{d}x = -\frac{1}{3}.$$

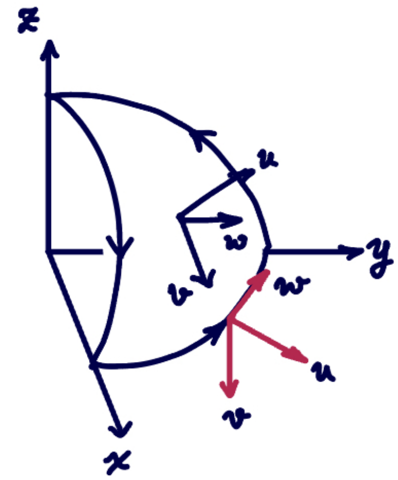

Figure 1: Consistent Orientations on T and ∂T.

On the other hand, pulling β back to ∂D via $\varphi|_{\partial D}$, note that the only term of the three that survives is the integral along the interval $[0, 1]$ with its usual orientation (why?), and so

$$\int_{\partial T} \beta = \int_{\partial D} \varphi|_{\partial D}^*(\beta) = \int_0^1 -\left(\frac{1-x^2}{2}\right) \mathrm{d}x = -\frac{1}{3}.$$

These are the same, as they should be, because of Stokes' Theorem. ■

Remark 1. Note that you are only allowed to compute integrals of forms via oriented parameterizations. A common oversight in the solutions was the failing to check or justify that your parametrization was orientation-preserving.

Q2. Let $f(z) = (az+b)/(cz+d)$ be a Möbius transformation. Show [that] the number of rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$g(g(g(g(g(z))))) = f(z)$$

is $1, 5$ or ∞. Explain how to determine which alternative holds for a given f.

Solution. Many solutions are possible. Clearly there is nothing special about 5; we work with a general $n \geq 1$, which is to say, given a Möbius transformation $f(z)$, we find the number of rational maps $g(z)$ such that $g^{\circ n}(z) = f(z)$, where $g^{\circ n}(z)$ means the n-fold iteration of g.

- Any non-identity Möbius transformation has either exactly 1 or 2 fixed points. Indeed, write $f(z) = (az+b)/(cz+d)$ with $ad - bc \neq 0$ (this is part of the definition of a Möbius transformation). If $c \neq 0$, then solving $f(z) = z$ amounts to solving a quadratic equation (which can have exactly one or two distinct roots), and ∞ is not a fixed point because $f(\infty) = ac^{-1}$. If $c = 0$, then $ad \neq 0$, and so replacing a, b by ad^{-1}, bd^{-1} respectively, we can write $f(z) = az + b$ with $a \neq 0$; in this case, $f(\infty) = \infty$, so ∞ is at least one fixed point. Then $f(z)$ has no more fixed points if $a = 1$ and $b \neq 0$, exactly one more fixed point if $a \neq 1$, and in the remaining case $a = 1, b = 0$, it is in fact the identity map. The above reasoning shows that, in fact, Möbius transformation f, we have that f fixes infinity (i.e. $f(\infty) = \infty$) iff $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$. In this case, ∞ is the only fixed point iff $a = 1$ and $b \neq 0$, and ∞ and 0 are the only fixed points iff $a \neq 1$ and $b = 0$.[1]
- Now, let's move on to the problem. First note that any such g must be a Möbius transformation itself, by the fact that

$$1 = \deg f = \deg g^{\circ n} = (\deg g)^n.$$

  This degree can be the topological degree or the degree as a rational function–we showed in class that these are equivalent. Note also that any fixed point of g is a fixed point of f.
- Suppose $f(z) = z$ is first the identity. Then for any $c \in \mathbb{C}$, the Möbius transformation $g(z) = e^{2\pi i/n} z + c$ is an n^{th} functional root of f. In particular, there are infinitely many such g.
- Next, suppose f has exactly one fixed point. Then g cannot have 2 or more fixed points, and hence g has exactly one fixed point, namely that of f. By a change of coordinates (i.e. conjugating by a Möbius transformation that sends this fixed point to ∞), we can assume that f and g fix only ∞. By our observation above, this implies that $f(z) = z + \lambda$ and $g(z) = z + \mu$ for some $0 \neq \lambda, \mu \in \mathbb{C}$. Then $g^{\circ n}(z) = z + n\mu$, and so we have $g^{\circ n}(z) = f(z)$ iff $n\mu = \lambda$, and this tells us that there is exactly one such g.
- Finally, suppose that f has exactly two fixed points. Then g must have exactly two fixed points as well; indeed, we showed above that if g has only one fixed point, then so does every iterate $g^{\circ n}$ for $n \geq 1$. Again, by a change of coordinates, we can assume that both f and g fix only ∞ and 0, and then $f(z) = \lambda z$, $g(z) = \mu z$ with $\lambda, \mu \in \mathbb{C} \smallsetminus \{0, 1\}$. Then $g^{\circ n}(z) = \mu^n z$, and so we have $g^{\circ n}(z) = f(z)$ iff $\mu^n = \lambda$, and this tells us there are exactly n such g.

This proves:

Theorem 0.0.1. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Möbius transformation. Given any integer $n \geq 1$, the number of rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $g^{\circ n}(z) = f(z)$ is exactly $1, n$, or ∞. More precisely,

(a) it is 1 iff f is in some coordinate system a nonzero translation $z \mapsto z + \lambda$ with $\lambda \in \mathbb{C}^*$ (or equivalently iff any matrix representing f is non-diagonalizable, or equivalently iff f has only one fixed point),
(b) it is n iff f is in some coordinate system a non-identity similarity $z \mapsto \lambda z$ with $\lambda \in \mathbb{C} \smallsetminus \{0, 1\}$ (or equivalently iff f is not the identity but any matrix representing f is diagonalizable, or equivalently iff f has two distinct fixed points), and
(c) it is ∞ iff $f = \mathrm{id}$.

■

[1] The same result can also be obtained by looking at eigenvalues of any matrix representing f or considering its Jordan canonical form.

Q3. Let $\sum_n a_n z^n$ be the Taylor series for $\tanh(z)$ at $z = 0$.

(a) What is the radius of convergence of this power series?
(b) Show that $a_5 = 2/15$.
(c) Give an explicit value of N such that $\tanh(1)$ and $\sum_0^N a_n$ agree to 1000 decimal places. Justify your answer. [Correction: we only ask for you to given an N such that

$$\left|\tanh 1 - \sum_0^N a_n\right| < 10^{-1000}.$$

In fact, this suffices, because $\tanh 1$ doesn't have tons of consecutive 9's or 0's at the end of its first 1000 decimal digits, but you don't need to prove that.]

Solution.

(a) The meromorphic function

$$\tanh(z) = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{\mathrm{e}^z + \mathrm{e}^{-z}}$$

on $\mathbb{C}$ has poles exactly at $\{(2n+1)\pi \mathrm{i}/2\}_{n\in\mathbb{Z}}$. In particular, the poles closest to zero are $\pm\pi\mathrm{i}/2$, so the radius of convergence of this series must be $\pi/2$, since the power series of a holomorphic function around a point in a region in the complex plane is valid (i.e. converges absolutely and locally uniformly) in the largest open disc contained in that region (this follows from the Cauchy integral formula; see [1, Thm. 1.1.2.3]).

(b) One possible solution: differentiate a few times and plug in 0 (and divide by 5!). Alternatively, compute a suitable integral; namely

$$a_5 = \operatorname*{Res}_{z=0} z^{-6}\tanh(z) = \frac{1}{2\pi\mathrm{i}}\oint_{S^1} z^{-6}\tanh(z)\mathrm{d}z.$$

Alternatively, you could expand

$$\begin{aligned}
\tanh(z) &= \frac{\sinh(z)}{\cosh(z)} \\
&= \frac{\left(z + \frac{z^3}{6} + \frac{z^5}{120} + [z^6]\right)}{\left(1 + \frac{z^2}{2} + \frac{z^4}{24} + [z^6]\right)} \\
&= \left(z + \frac{z^3}{6} + \frac{z^5}{120}\right)\left(1 - \left(\frac{z^2}{2} + \frac{z^4}{24}\right) + \left(\frac{z^2}{2} + \frac{z^4}{24}\right)^2\right) + [z^6] \\
&= \left(z + \frac{z^3}{6} + \frac{z^5}{120}\right)\left(1 - \frac{z^2}{2} + \frac{5z^4}{24}\right) + [z^6],
\end{aligned}$$

where we use the notation of [1, §5.1.2]. At this point, collecting coefficients of z^5 yields

$$a_5 = \frac{5}{24} - \frac{1}{12} + \frac{1}{120} = \frac{2}{15}.$$

(c) In general, we use the following version of Taylor's Theorem with remainder:

Theorem 0.0.2. Let $\Omega \subset \mathbb{C}$ be an open subset and let $f : \Omega \to \mathbb{C}$ be holomorphic. If $\sum_n a_n(z-a)^n$ is the Taylor series for $f(z)$ at $z = a$ for some $a \in \Omega$, i.e. if $a_n = \dfrac{f^{(n)}(a)}{n!}$ for $n \geq 0$, then for any integer $N \geq 0$, we can write

$$f(z) = \sum_{n=0}^N a_n(z-a)^n + f_{N+1}(z)(z-a)^N$$

for some holomorphic function $f_{N+1} : \Omega \to \mathbb{C}$. If $r > 0$ is chosen such that the open disk $\Delta(a;r) = \{z \in \mathbb{C} : |z-a| < r\}$ is contained in Ω and is $S^1(a;r) = \partial\Delta(a;r)$ is the circle of radius r at a oriented counterclockwise, then for any $z \in \Delta(a;r)$, we have the representation

$$f_{N+1}(z) = \frac{1}{2\pi \mathrm{i}} \oint_{S^1(a;r)} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta-a)^{N+1}(\zeta-z)}.$$

In particular, if $M_r := \sup_{\zeta\in S^1(a;r)} |f(\zeta)|$, then for all $z \in \Delta(a;r)$ we have

$$|f_{N+1}(z)| \le \frac{M_r}{r^N(r-|z-a|)}.$$

Proof. See [1, §4.3.1]. ■

Let's apply this to our case, with $\Omega = \Delta(\pi/2)$ and $f(z) = \tanh(z)$ around $a = 0$. Then for any r with $1 < r < \pi/2$, we have for all $z \in \Delta(r) = \Delta(0;r)$ that

$$|\sinh(z)| \le \frac{\mathrm{e}^{\operatorname{Re} z} + \mathrm{e}^{-\operatorname{Re} z}}{2} \le \mathrm{e}^{\pi/2}$$

and

$$|\cosh(z)| = |\cosh x \cos y + \mathrm{i} \sinh x \sin y| \ge |\cosh x \cos y| \ge |\cos y|$$

if $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ because $\cosh x \ge 1$ for all $x \in \mathbb{R}$. It follows that for $1 < r < \pi/2$ we have

$$M_r \le \mathrm{e}^{\pi/2} \sec r.$$

Applying the above bound for $z = 1$ gives us that for any $N \ge 0$, we have

$$\left|\tanh(1) - \sum_{n=0}^{N} a_n\right| = |f_{N+1}(1)| \le \frac{\mathrm{e}^{\pi/2}\sec r}{r^N(r-1)}.$$

Therefore, for any r with $1 < r < \pi/2$, given any $\varepsilon > 0$, we can take

$$N > \frac{\pi/2 + \ln\sec r - \ln(r-1) - \ln\varepsilon}{\ln r}$$

to give us an error

$$\left|\tanh(1) - \sum_{n=0}^{N} a_n\right| < \varepsilon.$$

Now take any $r \in (1, \pi/2)$ and $\varepsilon = 10^{-1000}$ to get your N, e.g. taking $r = 3/2$ tells us that $N = 5691$ suffices.

■

Remark 2. For $\varepsilon = 10^{-1000}$, the expression on the right as a function of r achieves a minimum at around $r \approx 1.5705$ of around 5123.773, so $N = 5124$ certainly suffices and is the best bound we get from this method. A few lines of Sage code tell us that the least N for which the problem statement is true is $N = 5099$, so our estimate using complex analysis is not bad at all!

Remark 3. A lot of solutions either did not bound $\tanh(z)$ at all, or bounded it incorrectly, claiming for instance that $|\tanh(z)| \le 1$ for all $|z| < \pi/2$. This statement is not true; indeed, $|\tanh x| \le 1$ for all $x \in \mathbb{R}$, but $\tanh(z)$ has a pole, and is in particular unbounded in absolute value, as $z \to \pm\mathrm{i}\pi/2$.

Q4. Let $f : U \to V$ be a proper local homeomorphism between a pair of open sets $U, V \subset \mathbb{C}$. Prove that f is a covering map.

Solution. Here's the definition of covering space that I use:

Definition 0.0.3. A map $p : X \to Y$ of topological spaces is a covering space if for each $y \in Y$ there is an open neighborhood W of y in Y, a discrete space Γ, and a homoemorphism $\Phi : p^{-1}(W) \to W \times \Gamma$ that commutes with projection to W, i.e. satisfies $p(x) = \mathrm{pr}_W(\Phi(x))$ for all $x \in p^{-1}(W)$, where $\mathrm{pr}_W : W \times \Gamma \to W$ is the first projection.

Remark 4. Note that $\Gamma = \emptyset$ is allowed. We call any such W an evenly covered neighborhood of y. When our spaces locally path-connected Hausdorff spaces, we can assume by shrinking W that W is connected, and then the second condition above is equivalent to saying that p maps each connected component of $p^{-1}(W)$ homeomorphically onto W. The cardinality of any such Γ is called the local degree of this covering map at y; this is a locally constant function of $y \in Y$. Usually, we require that the local covering degree be positive or at most countable (the former is equivalent to surjectivity, and the latter is automatic if X and Y are manifolds by the second countability/paracompactness hypothesis), but neither is strictly logically necessary.

The only property of open subsets of $\mathbb{C}$ we'll use is that they are locally compact Hausdorff. We show:

Theorem 0.0.4. Let $p : X \to Y$ be a proper local homeomorphism between locally compact Hausdorff spaces. Then p is a covering map.

Proof. For any $y \in Y$, the fiber $p^{-1}(y)$ is discrete (because p is a local homeomorphism) and compact (because p is proper), and hence a finite set. Fix a $y \in Y$, and let $p^{-1}(y) = \{x_1, \ldots, x_n\}$ for some $n \geq 0$ and x_i distinct. For each $i = 1, \ldots, n$, there is an open neighborhood $U_i \subset X$ such that $p(U_i) \subset Y$ is open and $p|_{U_i} : U_i \to p(U_i)$ is a homeomorphism. Since X is Hausdorff and n is finite, we may shrink the U_i to be pairwise disjoint. It is not true that any open neighborhood W of y contained in $O := \bigcap_{i=1}^n p(U_i)$ is evenly covered, because there is no guarantee that $p^{-1}(W) \subset \bigcup_{i=1}^n U_i$. However, when p is proper, there is some neighborhood W of y contained in O which satisfies $p^{-1}(W) \subset \bigcup_{i=1}^n U_i$, and then this neighborhood is evenly covered, finishing the proof.

There are various ways to produce such a W. We mention three:

- Take any precompact open neighborhood $V \subset Y$ of y (i.e. with $\overline{V}$ compact) satisfying $\overline{V} \subset O$, and define
$$W := V \smallsetminus p\left(p^{-1}(\overline{V}) \smallsetminus \bigcup_{i=1}^n U_i\right).$$
Note that $p^{-1}(\overline{V})$ is compact because p is proper, and hence so is $p^{-1}(\overline{V}) \smallsetminus \bigcup_{i=1}^n U_i$, and hence so is its image in p. It follows that this image is closed (since Y is Hausdorff), and hence W is an open neighborhood of y in Y with the required property.
- Another, similar, way to proceed is to use the following helpful lemma:

  Lemma 0.0.5. Let $p : X \to Y$ be a proper map between topological spaces with Y locally compact and Hausdorff. Then p is a closed map.

  Proof. Let $C \subset X$ be closed, and let $y \in Y \smallsetminus p(C)$. Pick a precompact neighborhood U of y (i.e. an open neighborhood with compact closure $\overline{U}$). Since $p^{-1}(\overline{U})$ is compact, so is its closed subset $K := C \cap p^{-1}(\overline{U})$, and therefore $p(K)$. Since Y is Hausdorff, $p(K)$ is closed in Y, and then $y \in U \smallsetminus p(K) \subset Y \smallsetminus p(C)$, proving $Y \smallsetminus p(C)$ is open. ■

  Having done this, look at $p\left(X \smallsetminus \bigcup_{i=1}^n U_i\right)$, which is a closed subset of Y because p is closed, and note that taking
$$W := O \smallsetminus p\left(X \smallsetminus \bigcup_{i=1}^n U_i\right)$$
  works.

- This third method works only in our setting of open subsets of the plane[2] Note that for any $y' \in O$, the fiber $p^{-1}(y')$ has cardinality at least n (why?). It suffices to show that there is some neighborhood W of y in Y contained in O such that for all $y' \in W$, the fiber $|p^{-1}(y')|$ has cardinality exactly n, because this would prove that $p^{-1}(W) \subset \bigcup_{i=1}^{n} U_i$ (why?). Suppose this is not the case, so there is a sequence of points $\{y_k\}_{k\geq 1}$ in O converging to y, where each y_k has at least $n+1$ preimages. Let $K := \{y_k\}_{k\geq 1} \cup \{y\}$, and note that this is a compact subset of Y. We claim that the preimage $p^{-1}(K) \subset X$ is not compact, and this is the required contradiction. Indeed, for each y_k, pick a preimage x_k that is not contained in $\bigcup_{i=1}^{n} U_i$; this exists by hypothesis (think about this). Then $\{x_k\}_{k\geq 1} \subset f^{-1}(K)$ is a sequence with no convergent subsequence: if a subsequence $\{x_{k_j}\}$ converged to say $x' \in X$, then by continuity we would have $p(x') = y$ and so $x' = x_i$ for some i, which tells us that all but finitely many (in particular at least one) $\{x_{k_j}\}$ lies in U_i, which is a contradiction.[3]

$\blacksquare$

Remark 5. Note that if we assume that U is nonempty and V is connected, then the map $f : U \to V$ is necessarily surjective (and so $n \geq 1$ in the above). This follows because then the image $f(U)$ is open (it's the image of a local homeomorphism), closed (by Lemma 0.0.5), and nonempty (since U is nonempty); therefore, by connectedness, we would conclude that $f(U) = V$. In general, surjectivity of f is neither a necessary nor sufficient condition for it to be a covering map.

Remark 6. A lot of the submissions failed to do the final step, and ended up showing the false statement that a local homeomorphism with finite fibers (even between open subsets of the complex plane) is a covering map. A counterexample to this statement would be the inclusion $\mathbb{C}^\times \hookrightarrow \mathbb{C}$. It is instructive to see what goes wrong in the above argument for this example. This is false also if we add the word "surjective"; here's a surjective continuous map between open subsets of the plane with finite fibers that is not a covering map: let $\Delta(a;r) = \{z \in \mathbb{C} : |z-a| < r\}$ for $a \in \mathbb{C}, r \in \mathbb{R}_{>0}$, and take $U = \Delta(0;1) \smallsetminus \{0\} \cup \Delta(3;1)$ and $V := \Delta(0;1)$, with the map $f : U \to V$ given by the identity on the first (punctured) disc and $z \mapsto z-3$ on the second disc.

[2] Or at least needs additional hypotheses like first countability and the equivalence of compactness and sequential compactness, etc....

[3] This reasoning shows more generally that a local homeomorphism between Hausdorff spaces with fibers of locally constant finite cardinality is a covering map.

Q5. Let $f : \mathbb{C} \to \mathbb{C}$ be given by a polynomial of degree 2 or more. Let

$$V_1 = \{f(z) : f'(z) = 0\} \subset \mathbb{C}$$

be the set of critical values of f, let $V_0 = f^{-1}(V_1)$ and let $U_i = \mathbb{C} \smallsetminus V_i$ for $i = 0, 1$. Prove that $f : U_0 \to U_1$ is a covering map.

Proof. We show that f is a proper local homeomorphism and use the previous result.

First note that if $f : \mathbb{C} \to \mathbb{C}$ is a polynomial map, then it is proper because if $\{z_n\}$ is any sequence escaping to infinity, then so is $(f(z_n))$, because f is eventually dominated by its leading term, for which this statement is clearly true (this needs to be more precise in your solution). Alternatively, we may use the Heine-Borel theorem to establish this fact. Next, note that if $f : X \to Y$ is a proper map between topological spaces and $V \subset Y$ any subspace, then

$$f|_{X \smallsetminus f^{-1}(V)} : X \smallsetminus f^{-1}(V) \to Y \smallsetminus V$$

is also proper (why?). In our situation, we can take $X = Y = \mathbb{C}$, $V = V_1$, and f the given polynomial map to conclude that $f : U_0 \to U_1$ is proper.

Under the identification of $\mathbb{C} \cong \mathbb{R}^2$, the smooth map f at the point z has Jacobian matrix

$$J_z f = \begin{bmatrix} \partial_x y & \partial_x v \\ \partial_y y & \partial_y v \end{bmatrix} = \begin{bmatrix} \partial_x u & \partial_x v \\ -\partial_x v & \partial_x v \end{bmatrix},$$

where in the second equality we have used the Cauchy-Reimann equations. It follows that the Jacobian determinant

$$\det J_z f = (\partial_x u)^2 + (\partial_x v)^2 = |f'(z)|^2.$$

By definition, we have for each $z \in U_0$ that $f'(z) \neq 0$, which tells us that the Jacobian matrix $J_z f$ of f at z is invertible. It follows from the Inverse Function Theorem that f is a C^∞-diffeomorphism in a neighborhood of z. This shows that $f : U_0 \to U_1$ is a local homeomorphism.

Another way of showing that f is a local homeomorphism whenever $f'(z) \neq 0$ (without resorting to x, y coordinates) is to note that for any smooth map g the total differential $\mathrm{d}g$ evaluated at a tangent vector $v \in T_z U_0$ is given by

$$\mathrm{d}g(v) = \frac{\partial g}{\partial z} v + \frac{\partial g}{\partial \overline{z}} \overline{v}.$$

In particular, if f is holomorphic and so $\partial f / \partial \overline{z} = 0$, this then reduces to

$$\mathrm{d}f(v) = f'(z) v,$$

so that if $f'(z) \neq 0$, then this is an isomorphism, and we may apply the Inverse Function Theorem as before. ■

Remark 7. In fact, $f|_{U_0}$ is a local biholomorphism. This follows from the fact that the inverse of an invertible holomorphic function still satisfies the Cauchy-Riemann equations, and is hence holomorphic. This can be proved in one dimension by the Cauchy integral formula (how?).

Remark 8. A lot of students applied the Heine-Borel Theorem incorrectly. A metric space X is said to be a Heine-Borel space or have the Heine-Borel property if a subset $K \subset X$ is compact iff it is closed and bounded. For any metric space, one implication is always true (which one? why?); it's the other implication that defines a Heine-Borel space. The Heine-Borel Theorem states that $\mathbb{R}^n$ for any $n \geq 0$ is a Heine-Borel space. It is, however, not true above (for any f) that U_1 is a Heine-Borel space (why?). Therefore, you cannot simply quote the Heine-Borel theorem and that the preimage of bounded subsets is bounded under f to conclude that $f : U_0 \to U_1$ is proper; this needs, the additional argument of the sort made above, i.e. that properness descends when restricting the codomain.

Q6. Given an example where U_0/U_1 is a normal (or Galois) covering, i.e. where $f_*(\pi_1(U_0))$ is a normal subgroup of $\pi_1(U_1)$.

Solution. Many solutions are possible. One possible line of approach: this is automatically the case if $\pi_1(U_1)$ is abelian, which happens iff V_1 is a single point (proof: exercise). Therefore, it suffices to take $f(z) = z^n$ for any $n \geq 2$.

Remark 9. For an example where the covering $f : U_0 \to U_1$ is not a Galois cover, consider $f(z) = z^3 - 3z$. In this case, $\pi_1(U_1)$ is the free group on two generators, say α and β, and (by taking 0 as the basepoint for both domain and codomain), the subgroup $f_*\pi_1(U_0)$ is $\langle \alpha^2, \beta^2, \alpha\beta\alpha, \beta\alpha\beta \rangle$, which is not a normal subgroup. (Exercise: justify all of these assertions. What cover of the doubly-punctured plane, or equivalently the figure-8, does this correspond to? We'll talk more about this cover in class.)

Remark 10. Can you classify all covers of the multiply-punctured complex plane (or equivalently Riemann sphere) that arise in this way?

References

[1] L. V. Ahlfors, *Complex Analysis*. International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., third ed., 1979.