Complex Analysis Notes

Math 213a — Harvard University
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1 Basic complex analysis; the simply-connected Riemann surfaces

1. Background in real analysis and basic differential topology (such as covering spaces and differential forms) is a prerequisite.

2. Relations of complex analysis to other fields include: algebraic geometry, complex manifolds, several complex variables, Lie groups and homogeneous spaces ($\mathbb{C}, \mathbb{H}, \widehat{\mathbb{C}}$), geometry (Platonic solids; hyperbolic geometry in dimensions two and three), Teichmüller theory, elliptic curves and algebraic number theory, $\zeta(s)$ and prime numbers, dynamics (iterated rational maps).

3. Algebraic origins of complex analysis; solving cubic equations $x^3+ax+b=0$ by Tchirnhaus transformation to make $a=0$. This is done by introducing a new variable $y=cx^2+d$ such that $\sum y_i=\sum y_i^2=0$; even when a and b are real, it may be necessary to choose c complex (the discriminant of the equation for c is $27b^2+4a^3$.) It is negative when the cubic has only one real root; this can be checked by looking at the product of the values of the cubic at its max and min.

4. Elements of complex analysis: $\mathbb{C}=\mathbb{R}[i]$, $\overline{z}$ (and related examples for $\mathbb{Q}[\sqrt{2}]$). Geometry of multiplication: why is it conformal? (Because $|ab|=|a||b|$, so triangles are mapped to similar triangles!) Visualizing $e^z=\lim(1+z/n)^n$. A pie slice centered at $-n$ and with angle π/n is mapped to the upper semi-circle; in the limit we find $\exp(\pi i)=-1$.

5. Definition: $f(z)$ is analytic if $f'(z)$ exists. Note: we do not require continuity of f'! Cauchy's theorem: $\int_\gamma f(z)dz=0$. Plausibility: $\int_\gamma z^n dz=\int_a^b \gamma(t)^n\gamma'(t)dt=\int_a^b (\gamma(t)^{n+1})/(n+1)'dt=0$.

   Proof 1: $f(z)dz$ is a closed form, when f is holomorphic, assuming $f(z)$ is smooth. Discussion: f is holomorphic iff $idf/dx=df/dy$; from this $d(fdz) = 0$. Moreover $df/dx = df/dz$, where $d/dz = 1/2(d/dx + (1/i)d/dy)$. We have f analytic if and only if $df/d\overline{z} = 0$. Then $df = (df/dz)dz + (df/d\overline{z})d\overline{z}$ and we see $d(fdz) = 0$ iff $df/d\overline{z} = 0$ iff f is holomorphic.

Proof 2: (Goursat), assuming only complex differentiability.

6. Analyticity and power series. The fundamental integral $\int_\gamma dz/z$. The fundamental power series $1/(1-z) = \sum z^n$. Put these together with Cauchy's theorem,
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z},$$
to get a power series.

   Theorem: $f(z) = \sum a_n z^n$ has a singularity (where it cannot be analytically continued) on its circle of convergence $|z| = R = 1/\limsup |a_n|^{1/n}$.

7. Infinite products: $\prod(1 + a_n)$ converges if $\sum |a_n| < \infty$. The proof is in two steps: first, show that when $\prod(1+|a_n|)$ converges, the differences in successive partial products bound the differences for $\prod(1+a_n)$. Then, show $\sum |a_n| \le \prod(1 + |a_n|) \le \exp(\sum |a_n|)$.

   Example: evaluation of $\zeta(2) = \prod(1/(1 - 1/p^2))$, where p ranges over the primes, converges (to $\pi^2/6$).

8. Cauchy's bound $|f^{(n)}(0)| \le n!M(R)/R^n$. Liouville's theorem; algebraic completeness of $\mathbb{C}$. Compactness of bounded functions in the uniform topology. Parseval's inequality (which implies Cauchy's:)
$$\sum |a_n|^2 R^2 = \frac{1}{2\pi}\int_{|Z|=R} |f(z)|^2 d\theta \le M(R)^2.$$

9. Morera's theorem (converse to Cauchy's theorem). Definition of $\log(z) = \int_1^z d\zeta/\zeta$. Analytic continuation, natural boundaries, $\sum a_n z^{n!}$. Laurent series: $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ where $a_n = (1/2\pi i)\int_C f(z)/z^{n+1}dz$. Classification of isolated singularities; removability of singularities of bounded functions. Behavior near an essential singularity (Weierstrass-Casorati): $\overline{f(U)} = \mathbb{C}$.

10. Generating functions and $\sum F_n z^n$, F_n the nth Fibonacci number. A power series represents a rational function iff its coefficients satisfiy a

recurrence relation. Pisot numbers, the golden ratio, and why are 10:09 and 8:18 such pleasant times.

11. Kronecker's theorem: one need only check that the determinants of the matrices $a_{i,i+j}$, $0 \le i, j \le n$ are zero for all n sufficiently large.

12. Residue theorem and evaluation of definite integrals. Three types: $\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta$, $\int_{-\infty}^{\infty} R(x) dx$, and $\int_0^{\infty} x^a R(x) dx$, $0 < a < 1$, R a rational function.

13. Hardy's paper on $\int \sin(x)/x \, dx$.

14. The argument principle: number of zeros - number of poles is equal to

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

Similary, the weighted sum of $g(z)$ over the zeros and poles is given by multiplying the integrand by $g(\zeta)$.

Winding numbers of the topological nature of the argument principle: if a continuous $f : \overline{\Delta} \to \mathbb{C}$ has nonzero winding number on the circle, then f has a zero in the disk.

15. Rouché's theorem: if $|g| < |f|$ on $\partial\Omega$, then $f + g$ and g have the same number of zeros-poles in Ω. Example: $z^2 + 15z + 1$ has all zeros of modulus less than 2, but only one of modulus less than $3/2$.

16. Open mapping theorem: if f is nonconstant, then it sends open sets to open sets. Cor: the maximum principle ($|f|$ achieves its maximum on the boundary).

17. Invertibility. (a) If $f : U \to V$ is injective and analytic, then f^{-1} is analytic. (b) If $f'(z) \neq 0$ then f is locally injective at z. Formal inversion of power series.

18. Phragmen -Lindelöf type results: if f is bounded in a strip $\{a < \mathrm{Im}(z) < b\}$, and f is continuous on the boundary, then the sup on the boundary is the sup on the interior.

19. Hadamard's 3-circles theorem: if f is analytic in an annulus, then $\log M(r)$ is a convex function of $\log r$, where $M(r)$ is the sup of $|f|$ over $|z| = r$. Proof: a function $\phi(s)$ of one real variable is convex if and only if $\phi(s) + ar$ satisfies the maximum principle for any constant a. This holds for $\log M(\exp(s))$ by considering $f(z)z^a$ locally.

20. The concept of a Riemann surface; the notion of isomorphism; the three simply-connected Riemann surface $\mathbb{C}$, $\widehat{\mathbb{C}}$ and $\mathbb{H}$.

21. A nonconstant map between compact surfaces is surjective, by the open mapping theorem.

22. Theorem: $\mathrm{Aut}(\mathbb{C}) = \{az + b\}$.

23. The complex plane $\mathbb{C}$. The notion of metric $\rho(z)|dz|$. The automorphism group is solvable. Inducing a metric $|dz|/|z|$ on $\mathbb{C}^*$. The cone metric $|dz|/|\sqrt{z}|$ giving the quotient by z^2.

24. The Riemann sphere $\widehat{\mathbb{C}}$ and its automorphisms. Theorem: $\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C})$. A particularly nice realization of this action is as the projectivization of the linear action on $\mathbb{C}^2$.

25. Möbius transformations: invertible, form a group, act by automorphisms of $\widehat{\mathbb{C}}$, triply-transitive, sends circles to circles. Proof of last: a circle $x^2 + y^2 + Ax + By + C = 0$ is also given by $r^2 + r(A\cos\theta + B\sin\theta) + C = 0$, and it is easy to transform the latter under $z \mapsto 1/z$, which replaces r by $1/\rho$ and θ by $-\alpha$.

26. Classification of Möbius transformations and their trace squared: (a) identity, 4; (b) parabolic (a single fixed point) 4; (c) elliptic (two fixed points, derivative of modulus one) $[0, 4)$; (d) hyperbolic (two fixed points, one attracting and one repelling) $\mathbb{C} - [0, 4]$.

27. Stereographic projection preserves circles and angles. Proof for angles: given an angle on the sphere, construct a pair of circles through the north pole meeting at that angle. These circles meet in the same angle at the pole; on the other hand, each circle is the intersection of the sphere with a plane. These planes meet $\mathbb{C}$ in the same angle they meet a plane tangent to the sphere at the north pole, QED.

28. Four views of $\widehat{\mathbb{C}}$: the extended complex plane; the Riemann sphere; the Riemann surface obtained by gluing together two disks with $z \mapsto 1/z$; the projective plane for $\mathbb{C}^2$.

29. The spherical metric $2|dz|/(1+|z|^2)$. Derive from the fact "Riemann circle" and the map $x = \tan(\theta/2)$, and conformality of stereographic projection.

30. Some topology of projective spaces: $\mathbb{RP}^2$ is the union of a disk and a Möbius band; the Hopf map $S^3 \to S^2$.

31. Gauss-Bonnet for spherical triangles: area equals angle defect. Prove by looking at the three lunes (of area 4θ) for the three angles of a triangle. General form: $2\pi\chi(X) = \int_X K + \int_{\partial X} k$.

32. Dynamical application of Schwarz Lemma: let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. Then the immediate basin of any attracting cycle contains a critical point.

    Cor. The map f has at most $2d-2$ attracting cycles.

33. Theorem: $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$. Schwarz Lemma and automorphisms of the disk. The hyperbolic metric $|dz|/\operatorname{Im}(z)$ on $\mathbb{H}$, and its equivalence to $2|dz|/(1-|z|^2)$ on Δ.

34. Classification of automorphisms of $\mathbb{H}^2$, according to translation distance.

35. Hyperbolic geometry: geodesics are circles perpendicular to the circle at infinity. Euclid's fifth postulate (given a line and a point not on the line, there is a unique parallel through the point. Here two lines are parallel if they are disjoint.)

36. Gauss-Bonnet in hyperbolic geometry. (a) Area of an ideal triangle is $\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}(1/y^2)dydx = \pi$. (b) Area $A(\theta)$ of a triangle with two ideal vertices and one external angle θ is additive ($A(\alpha+\beta) = A(\alpha)+A(\beta)$) as a diagram shows. Thus $A(\alpha) = \alpha$. (c) Finally one can extend the edges of a general triangle T in a spiral fashion to obtain an ideal triangle containing T and 3 other triangles, each with 2 ideal vertices.

37. Harmonic functions. A real-value function $u(z)$ is harmonic iff u is locally the real part of an analytic function; indeed, harmonic means $d(*du) = 0$, and $v = \int *du$. Harmonic functions are preserved under analytic mappings. Examples: electric potential; fluid flow around a cylinder.

    The mean-value principle ($u(0) =$ average over the circle) follows from Cauchy's formula, as does the Poisson integral formula ($u(p) =$ visual average of u).

38. The Schwarz reflection principle: if $U = U^*$, and f is analytic on $U \cap \overline{\mathbb{H}}$, continuous and real on the boundary, then $\overline{f(\overline{z})}$ extends f to all of U. This is easy from Morera's theorem. A better version only requires that $\mathrm{Im}(f) \to 0$ at the real axis, and can be formulated in terms of harmonic functions (cf. Ahlfors):

    If v is harmonic on $U \cap \overline{\mathbb{H}}$ and vanishes on the real axis, then $v(\overline{z}) = -v(z)$ extends v to a harmonic function on U. For the proof, use the Poisson integral to replace v with a harmonic function on any disk centered on the real axis; the result coincides with v on the boundary of the disk and on the diameter (where it vanishes by symmetry), so by the maximum principle it is v.

39. Reflection gives another proof that all automorphisms of the disk extend to the sphere.

40. Quotient of the cylinder: $s(z) = z + 1/z$ gives the quotient isomorphism, $\mathbb{C}^*/(\mathbb{Z}/2) \cong \mathbb{C}$. It has critical points at ± 1 and critical values at ± 2.

    Since $z \mapsto z^n$ commutes with $z \mapsto 1/z$, there are unique polynomial $p_n(z)$ such that $s(z^n) = p_n(s(z))$. Writing $z = e^{i\theta}$, these are essentially the Chebyshev polynomials; they satisfy $2\cos(n\theta) = p_n(2\cos\theta)$.

41. Classification of polynomials. Let us say $p(z)$ is equivalent to $q(z)$ is there are $A, B \in \mathrm{Aut}(\mathbb{C})$ such that $Bp(Az) = q(z)$. Then every polynomial is equivalent to one which is monic and centered (the sum of its roots is zero). Every quadratic polynomial is equivalent to $p(z) = z^2$.

42. Solving the cubic. Every cubic polynomial is equivalent to $p_3(z) = z^3 - 3z$. But this polynomial arises as a quotient of $z \mapsto z^3$; that is, it satisfies $s(z^3) = p_3(s(z))$. Thus we can solve $p_3(z) = w$ easily: the solution is $s(u)$, where $s(u^3) = w$.

The mapping $p_3(z)$ (and indeed all the maps p_n) preserve the ellipses and hyperbolas with foci ± 2, since $z \mapsto z^n$ preserves circles centered at $z = 0$ and lines through the origin. This facilitates visualizing these polynomials as mappings of the plane to itself.

2 Entire and meromorphic functions

1. Weierstrass/Hadamard factorization theory. (A good reference for this material is Titchmarsh, *Theory of Functions*). We will examine the extent to which an entire function is determined by its zeros. We beginning by showing that any discrete set in $\mathbb{C}$ arises as the zeros of an entire function.

2. Weierstrass factor. Inspired by the fact that $(1-z)\exp\log 1/(1-z) = 1$ and that $\log 1/(1-z) = z + z^2/2 + z^3/3 + \ldots$, we set

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

   By convention $E_0(z) = (1-z)$.

   Theorem: For $|z| < 1$, we have $|E_p(z) - 1| \le |z|^{p+1}$.

   Proof: Writing $E_p(z) = 1 + \sum a_k z^k$, one may check (by computing $E_p'(z)$) that all $a_k < 0$, $\sum |a_k| = 1$, and $a_1 = a_2 = \ldots = a_p = 0$. Then $|E_p(z) - 1| = |\sum a_k z^k| \le |z|^{p+1} \sum |a_k| \le |z|^{p+1}$. QED

3. Theorem: If $\sum (r/|a_n|)^{p_n+1} < \infty$ for all $r > 0$, then $f(z) = \prod E_{p_n}(z/a_n)$ converges to an entire function with zeros exactly at the a_n.

   Cor: Since $p_n = n$ works for any $a_n \to \infty$, we have shown any discrete set arises as the zeros of an entire function.

4. Blaschke products. Let $f : \Delta \to \Delta$ be a proper map of degree d. Then

$$f(z) = e^{i\theta} \prod_1^d \left(\frac{z - a_i}{1 - \overline{a_i} z}\right)$$

   where the a_i enumerate the zeros of f.

5. Jensen's formula. Let $f(z)$ be holomorphic on the disk of radius R about the origin. Then the average of $\log|f(z)|$ over the circle of radius R is given by:
$$\log|f(0)| + \sum_{f(z)=0;\ |z|<R} \log\frac{R}{|z|}.$$
Proof: Suffices to assume $R = 1$. Clear if f has no zeros, because $\log|f(z)|$ is harmonic. Clear for a Blaschke factor $(z-a)/(1-\overline{a}z)$. But the formula is true for fg if it is true for f and g, so we are done.

Cor: Let $n(r)$ be the number of zeros of f inside the circle of radius r. Then
$$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|.$$
Remark: We have used the mean value property of harmonic functions. This holds for any harmonic function u on the disk by writing $u = \mathrm{Re}(f)$, f holomorphic, and then applying Cauchy's integral formula for $f(0)$.

The physical idea of Jensen's formula is that $\log|f|$ is the potential for a set of unit point charges at the zeros of f.

6. Entire functions of finite order. An entire function $f : \mathbb{C} \to \mathbb{C}$ is of finite order if there is an $A > 0$ such that $|f(z)| = O(\exp|z|^A)$. The least such A is the *order* ρ of f.

Examples: Polynomials have order 0; $\sin(z), \cos(z), \exp(z)$ have order 1; $\cos(\sqrt{z})$ has order 1/2; $\exp(\exp(z))$ has infinite order.

7. Theorem. Let $f(z)$ be an entire function of finite order with no zeros. Then $f(z) = e^{Q(z)}$ where $Q(z)$ is a polynomial.

Proof. Since f has no zeros, $f(z) = e^{Q(z)}$ for some entire function $Q(z)$. Since f has finite order, $|f(z)| = O(e^{|z|^d})$ for some d, and thus $\mathrm{Re}\, Q(z) \le |z|^d + O(1)$. Thus there is a constant $A > 0$ such that Q maps $\Delta(2R)$ into the half-plane $U(R) = \{z : \mathrm{Re}\, z < AR^d\}$, for $R > 1$.

By the Schwarz Lemma, Q is distance-decreasing from the hyperbolic metric on $\Delta(2R)$ to the hyperbolic metric on $U(R)$. Since $\Delta(R) \subset \Delta(2R)$ has bounded hyperbolic diameter, the same is true for $Q(\Delta(R)) \subset U(R)$. Therefore in the Euclidean metric,
$$\operatorname{diam} Q(\Delta(R)) = O(d(Q(0), \partial U(R))) = O(R^d).$$

Therefore $|Q(z)| = O(|z|^d)$ for $|z| > 1$. By Cauchy's integral formula, we find $Q^{d+1}(z) = 0$ and thus Q is a polynomial (of degree at most d).

8. Number of zeros. By Jensen's formula, if f has order ρ, then $n(r) = O(r^{\rho+\epsilon})$, where $n(r)$ is the zero counting function for f. Corollary: $\sum 1/|a_i|^{\rho+\epsilon} < \infty$, where a_i enumerates the zeros of f (other than zero itself).

   In other words, $\rho(a_i) \le \rho(f)$, where $\rho(a_i)$ is the exponent of convergence of the zeros of f, i.e. the least ρ such that $\sum 1/|a_i|^{\rho+\epsilon} < \infty$.

9. Definition: a *canonical product* is an entire function of the form

$$f(z) = z^m \prod E_p(z/a_i)$$

   where p is the least integer such that $\sum |z/a_i|^{p+1} < \infty$ for all z.

10. Hadamard's Factorization Theorem. Let f be an entire function of order ρ. Then $f(z) = P(z) \exp Q(z)$, where P is a canonical product with the same zeros as f and Q is a polynomial of degree less than or equal to ρ.

11. To prove Hadamard's theorem we develop two estimates. First, we show a canonical product $P(z)$ is an entire function of order $\rho = \rho(a_i)$. This is the least order possible for the given zeros, by Jensen's theorem. Second, we show a canonical product has $m(r) \ge \exp(-|z|^{\rho+\epsilon})$ for most radii r, where $m(r)$ is the minimum of $|f(z)|$ on the circle of radius r. More precisely, for any $\epsilon > 0$, this lower bound holds for all r outside a set of finite total length. In particular it holds for r arbitrarily large.

    Given these facts, we observe that $f(z)/P(z)$ is an entire function of order ρ with *no zeros*. Thus $Q(z) = \log f(z)$ is an entire function satisfying $\operatorname{Re} Q(z) = O(1 + |z|^{\rho+\epsilon})$. This implies Q is a polynomial. Indeed, for any entire function $f(z) = \sum a_n z^n$, we have $|a_n| r^n \le \max(4A(r), 0) - 2\operatorname{Re}(f(0))$, where $A(r) = \max \operatorname{Re} f(z)$ over the circle of radius r.

12. Theorem. A canonical product has order $\rho = \rho(a_i)$ and also satisfies $m(r) \ge \exp(-r^{\rho+\epsilon})$ for infinitely many r.

    Proof. Let $r_n = |a_n|$ and $r = |z|$. Recall that p is the least integer such that $\sum (1/r_n)^{p+1} < \infty$. Thus $p \le \rho \le p+1$. For convenience we will assume $\sum(1/r_n)^\rho < \infty$. (For the general case, just replace ρ with $\rho + \epsilon$.)

By construction, the Weierstrasss factor

$$E_p(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right)$$

satisfies

$$\log E_p(u) = \frac{u^{p+1}}{p+1} + \frac{u^{p+2}}{p+2} + \cdots$$

for $|u| < 1$. Thus for 'u small', meaning $|u| < 1/2$, we have

$$|\log E_p(u)| = O(|u|^{p+1}). \tag{2.1}$$

On the other hand, for 'u large', meaning $1/2 \le |u|$, we have

$$|\log |E_p(u)|/|1-u|| = O(|u|^p). \tag{2.2}$$

The $|1-u|$ can be ignored unless u is very close to 1, making the log very negative.

Combining these estimates for $f(z) = \prod E_p(z/a_n)$, we get the upper bound:

$$\log |f(z)| = O\left(r^{p+1} \sum_{r/r_n < 1/2} \frac{1}{r_n^{p+1}} + r^p \sum_{r/r_n \ge 1/2} \frac{1}{r_n^p}\right).$$

Now in the second sum, since $p \le \rho$, we have

$$\sum_{r/r_n \ge 1/2} \frac{1}{r_n^p} = \sum_{r_n \le 2r} \frac{r_n^{\rho-p}}{r_n^\rho} \le (2r)^{\rho-p} \sum \frac{1}{r_n^\rho} = O(r^{\rho-p}).$$

Similarly in the first sum, since $\rho \le p+1$, we have

$$\sum_{r/r_n < 1/2} \frac{1}{r_n^{p+1}} = \sum_{r_n > 2r} \frac{1}{r_n^{p+1-\rho}} \frac{1}{r_n^\rho} \le (2r)^{\rho-p-1} \sum \frac{1}{r_n^\rho} = O(r^{\rho-p-1}).$$

Altogether this gives

$$\log|f(z)| = O(r^\rho),$$

and thus f has order ρ.

The lower bound works the same way, since equations (2.1) and (2.2) give bounds for $|logE_p(u)|$. All that remains is to prove that

$$\sum_{r/r_n>1/2} \log|1-z/a_n| = O(r^{\rho+\epsilon})$$

for infinitely many values of r.

Note that the number of terms in the sum above is $N(2r)$, the number of n such that $r_n < 2r$. We have

$$N(2r)(2r)^{-\rho} \leq \sum r_n^{-\rho} < \infty,$$

and thus $N(2r) = O(r^\rho)$. Also when $\log|1-z/a_n|$ is positive it is no bigger than $O(\log r)$ (the constant depending on the smallest a_n), and so the positive terms contribute at most $N(r)\log r$ to the sum, which is $O(r^{\rho+\epsilon})$.

We now have to worry about the terms where z/a_n is close to one. To this end, we require that $|r-r_n| > 1/r_n^\rho$ for all n. Since $\sum 1/r_n^\rho < \infty$, a set of r of infinite measure remains. Then for any such r, if z/a_n is close to one, we still have

$$|1-z/a_n| = |a_n - z|/|a_n| \geq r_n^{-1-\rho},$$

and thus $\log|1-a/z_n| = O(\log r_n) = O(\log r)$. Thus the cases where z/a_n is close to one also contribute $O(N(r)\log r) = O(r^{\rho+\epsilon})$. ∎

13. Example:

$$\sin(\pi z) = \pi z \prod_{n\neq 0}\left(1-\frac{z^2}{n^2}\right).$$

Indeed, the right hand side is a canonical product, and $\sin(\pi z)$ has order one, so the formula is correct up to a factor $\exp Q(z)$ where $Q(z)$ has degree one. But since $\sin(\pi z)$ is odd, we conclude Q has degree zero, and by checking the derivative at $z=0$ of both sides we get $Q=0$.

14. Alternative proof. We have

$$\frac{\pi^2}{\sin^2(\pi z)} = \frac{1}{z^2} + \frac{\pi^2}{3} + \frac{\pi^4}{15}z^2 + \cdots = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}$$

because both sides are periodic and tend to zero as $|\operatorname{Im} z| -> \infty$. Integrating, we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{1}^{\infty} \frac{1}{z-n} + \frac{1}{z+n},$$

using the fact that both sides are odd to fix the constant of integration.

On the other hand, the logarithmic derivative of $f(z) = \sin(\pi z)$ is $\pi \cot(\pi z)$. Using the fact the logarithmic derivative of $1 - z^2/n^2$ is the same as that of $(z-n)(z+n)$, namely $(z-n)^{-1} + (z+n)^{-1}$ we deduce that $g'/g = f'/f$ where

$$g(z) = \pi z \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Since $\sin(\pi z)/(\pi z) = g(z)/(\pi z) = 1$ at $z = 0$, we conclude that $f = g$.

15. The Gamma Function:

$$\Gamma(z) = \frac{\exp(-\gamma z)}{z} \prod_{1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} \exp(z/n).$$

Note that this expression contains the reciprocal of the canonical product associated to the non-positive integers. The constant γ is chosen so that $\Gamma(1) = 1$; it can be given by:

$$\gamma = \lim (\sum_{1}^{n} 1/k) - \log(n+1).$$

This expression is the error in an approximation to $\int_1^{n+1} dx/x$ by the area of n rectangles of base one lying over the graph.

16. Let $G(z) = \prod_1^\infty (1 + z/n)e^{-z/n}$ be the canonical product for the set of negative integers. Evidentally

$$G(z)G(-z) = \frac{\sin(\pi z)}{\pi z}. \tag{2.3}$$

Moreover, we find $G(z-1) = z\exp(az+b)G(z)$ since both sides have the same zeros and are of order one. Taking the logarithmic derivative shows $a = 0$ and $b = \gamma$; that is,

$$G(z-1) = ze^{\gamma}G(z).$$

Defining

$$\Gamma(z) = \frac{1}{ze^{\gamma z}G(z)},$$

we obtain $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(0) = \Gamma(1) = 1$, $\Gamma(z)$ has no zeros, $\Gamma(z)$ has simple poles at $-1, -2, -3, \ldots$; and $\Gamma(n+1) = n!$ for $n \geq 0$. Using the fact that

$$\Gamma(z-n) = \frac{\Gamma(z)}{(z-n)\cdots(z-1)},$$

we find

$$\mathrm{Res}_{-n}(\Gamma(z)) = \frac{(-1)^n}{n!}.$$

17. Sine/Gamma formula: from 2.3 we obtain:

$$\frac{\pi}{z\sin(\pi z)} = \Gamma(z)\Gamma(-z).$$

18. Gauss's formula:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

This is obtained by examining the product formula directly for $ze^{\gamma z}G(z) = 1/\Gamma(z)$.

Corollary: $|\Gamma(z)| \leq \mathrm{Re}\, z$ when $\mathrm{Re}\, z > 0$.

19. Uniqueness Theorem (Wielandt, 1939): If $F(z+1) = zF(z)$ for $\mathrm{Re}\, z > 0$, $F(1) = 1$ and $F(z)$ is bounded on the strip $\{\mathrm{Re}\, z \in [1, 2]\}$, then $F(z) = \Gamma(z)$.

    Proof. The functional equation allows one to extend $F(z)$ to a meromorphic function on the whole plane, whose poles and their residues agree with those of $\Gamma(z)$. Thus $G(z) = F(z) - \Gamma(z)$ is entire, $G(0) = G(1) = 0$ and $G(z+1) = zG(z)$. Our boundedness assumptions now imply that $G(z)$ is bounded in the strip $S = \{\mathrm{Re}\, z \in [0, 1]\}$. Thus $H(z) = G(z)G(1-z)$ is also bounded in S. The functional equation for G implies $H(z+1) = H(z)$ (as in the sine formula), and thus H is a constant, which must be zero. ■

20. The integral representation: for $\mathrm{Re}(z) > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}.$$

    In other words, $\Gamma(z)$ is the *Mellin transform* of the function e^{-t} on $\mathbb{R}^*$. The Mellin transform is an integral against characters $\chi : \mathbb{R}^* \to \mathbb{C}^*$ (given by $\chi(t) = t^z$), and as such it can be compared to the Fourier transform (for the group $\mathbb{R}$ under addition) and to Gauss sums. Indeed the Gauss sum

$$\sigma(\chi) = \sum_{(n,p)=1} \chi(n) e^{2\pi i n/p}$$

    is the analogue of the Gamma function for the group $(\mathbb{Z}/p)^*$.

    Proof. Apply the uniqueness theorem above.

21. Duplication formula: $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$. Note that both sides have poles at $-n/2$, $n > 0$.

22. Periodic functions: If $f : \mathbb{C} \to X$ has period λ, then $f(z) = F(\exp(2\pi i z/\lambda)$, where $F : \mathbb{C}^* \to X$. Example:

$$\sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)}.$$

    From this we get $\sum_1^\infty 1/n^2 = \pi^2/6$.

23. The Mittag-Leffler theorem. Let $a_n \to \infty$ in $\mathbb{C}$ be a sequence of *distinct* points, and let $p_n(z)$ be principal parts centered at a_n. Then there exists a memorphic function $f(z)$ with poles exactly at a_n, and with the prescribed principal parts.

    Proof. Let $q_n(z)$ be the polynomial given by the power series for $p_n(z)$ about $z = 0$, truncated so $|p_n(z) - q_n(z)| < 2^{-n}$ when $|z| < |a_n|/2$. Then $f(z) = \sum(p_n(z) - q_n(z))$ is the desired function. ■

    The Weierstrass Theorem — that there exists a function $g(z)$ with prescribed zeros — is a corollary, by taking $p_n(z) = m_n/(z - a_n)$ and $g(z) = \exp(\int f(z)\,dz)$.

3 Conformal mapping

1. Normal families: any bounded family of analytic functions is normal, by Arzela-Ascoli.

2. Riemann mapping theorem: given a simply-connected region $U \subset \mathbb{C}$, $U \neq \mathbb{C}$, and a basepoint $u \in U$, there is a unique conformal homeomorphism $f : (U, u) \to (\Delta, 0)$ such that $f'(u) > 0$. Proof: let $\mathcal{F}$ be the family of univalent maps $(U, u) \to (\Delta, 0)$. Using a square-root and an inversion, show $\mathcal{F}$ is nonempty. Also $\mathcal{F}$ is closed under limits. By the Schwarz Lemma, $|f'(u)|$ has a finite maximum over all $f \in \mathcal{F}$. Let f be a maximizing function. If f is not surjective to the disk, then we can apply a suitable composition of a square-root and two automorphisms of the disk to get a $g \in \mathcal{F}$ with $|g'(u)| > |f'(u)|$, again using the Schwarz Lemma. QED.

3. The length-area method. Let $f : R(a, b) \to Q$ be a conformal map of a rectangle to a Jordan region $Q \subset \mathbb{C}$, where $R(a, b) = [0, a] \times [0, b] \subset \mathbb{C}$, Then there is a horizontal line $(0, a) \times \{y\}$ whose image has length $L^2 \le (a/b)\,\text{area}(Q)$. Similarly for vertical lines.

    Proof: apply the Cauchy-Schwarz inequality: there exists a horizontal length whose image has length at most L where

$$L^2 = \left(b^{-1} \int |f'|\right)^2 \le \frac{1}{b^2} \left(\int 1^2\right) \left(\int |f'|^2\right) = \frac{a}{b}\,\text{area}(Q).$$

■

Corollary: given any quadrilateral Q, the product of the minimum distances between opposite sides is a lower bound for area(Q).

4. Theorem: The Riemann map to a Jordan domain extends to a homeomorphism on the closed disk.

   Proof: given a point $z \in \partial\Delta$, map Δ to an infinite strip, sending z to one end. Then there is a sequence of disjoint squares in the strip tending towards that end. The images of these squares have areas tending to zero, so there are cross-cuts whose lengths tend to zero as well, by the length-area inequality. This gives continuity at z. Injectivity is by contradiction: if the map is not injective, then it is constant on some interval along the boundary of the disk. ■

5. Schwarz reflection: a Riemann mapping $f : \Delta \to U$ can be analytically continued past $p \in S^1$ whenever ∂U is a real-analytic arc near p. When the arc is a straight line, the continuation is easily given by reflection.

6. Uniformization of annuli: any doubly-connected region in the sphere is conformal isomorphic to $\mathbb{C}^*$, Δ^* or $A(R) = \{z : 1 < |z| < R\}$. The map from $\mathbb{H}$ to $A(R)$ is $z \mapsto z^\alpha$, where $\alpha = \log(R)/(\pi i)$. The deck transformation is given by $z \mapsto \lambda z$, where $\lambda = \exp(2\pi^2/\log(R))$.

7. Uniformization of quadrilaterals: any quadrilateral is equivalent to a unique rectangle. Sketch of the proof, by continuity.

8. PDE Application: harmonic functions. The function $\log((1+z)/(1-z))$ on the disk, and the harmonic function =0 on top hemisphere, =1 on the bottom. fluid flow around a cylinder, and the inverse of $z + 1/z$.

9. The class S of univalent maps $f : \Delta \to \mathbb{C}$ such that $f(0) = 0$ and $f'(0) = 1$. Compactness of S. The Bieberbach Conjecture/de Brange Theorem: $f(z) = \sum a_n z^n$ with $|a_n| \leq n$.

10. The area theorem: if $f(z) = z + \sum b_n/z_n$ is univalent on $\{z : |z| > 1\}$, then $\sum n|b_n|^2 < 1$. The proof is by integrating $\overline{f}df$ over the unit

circle and observing that, since $|dz|^2 = (-i/2)d\overline{z}\,dz$, the area A of complement of the image of f is given by:

$$A = -\frac{i}{2}\int_{S^1} \overline{f}df = -\frac{i}{2}(1 - \sum n|b_n|^2)\int_{S^1}\frac{dz}{z} = \pi\left(1 - \sum n|b_n|^2\right).$$

Remark: little is known about optimal bounds for $|b_n|$ over Σ. The area theorem gives $|b_n| = O(1/n1/2)$ and it is conjectured that $|b_n| = O(1/n^{3/4})$; see [CJ].

11. Proof that $|a_2| \leq 2$: first, apply the area theorem to conclude $|a_2^2 - a_3| \leq 1$. Then consider $g(z) = \sqrt{f(z^2)} = z + (a_2/2)z^3 + \cdot$ for $f(z) = z + a_2 z^2 + \cdots \in S$

12. The Koebe 1/4 Theorem: if $f \in S$ then $f(\Delta) \supset \Delta(1/4)$. Proof: if w is omitted from the image, then $f(z)/(1 - f(z)/w) \in S$; now apply $|a_2| \leq 2$.

13. Corollary. The hyperbolic metric on a simply-connected region $U \subset \mathbb{C}$ satisfies $\rho(z) \in [1/2, 2] \cdot d(z, \partial U)$.

14. The distortion lemmas. Given $f : \Delta \to \mathbb{C}$, univalent, investigate the invariant quantity $\delta(z) = |f'(z)|/\rho(z)$. This measures the derivative from the hyperbolic metric to the Euclidean metric.

    We claim along any hyperbolic path, we have $d(\log\delta)/ds \leq 2$. It suffices to treat the case $z = 0$. Then $\rho(z) = 2/(1 - |z|^2)$ is stationary, while $|f''(z)/f'(z)| \leq |2a_2|/|a_1| \leq 4$. But $|f''/f'|$ measures $d/|dz|$ which is twice d/ds, since $\rho(0) = 2$, so we get the desired bound.

    It follows that $\delta(z)/\delta(0) \leq \exp(2d(0, r))$, $r = |z|$. But $\exp(d(0, r)) = (1 + r)/(1 - r)$, as can be seen by using $i(1 - z)/(1 + z)$ to map to $\mathbb{H}$. Thus $\delta(z)/\delta(0) \leq (1 + r)^2/(1 - r)^2$. It follows that for $f \in S$, since $f'(0) = 1$, we have:

$$|f'(z)| = \frac{|f'(z)|}{|f'(0)|} \leq \frac{(1+r)^2}{(1-r)^2}\cdot\frac{\rho(r)}{\rho(0)} = \frac{(1+r)}{(1-r)^3}.$$

    Similarly we have the reciprocal lower bound

$$\frac{(1-r)}{(1+r)^3} \leq |f'(z)|.$$

We may finally conclude that S is compact, and indeed we can obtain by integration the upper and lower bounds:

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}.$$

15. Riemann surfaces and holomorphic 1-forms. The naturality of the residue, and of df.

16. The Residue Theorem: the sum of the residues of a meromorphic 1-form on a compact Riemann surface is zero. Application to df/f, and thereby to the degree of a meromorphic function.

17. Remarks on 1-forms: a holomorphic 1-form on the sphere is zero, because it integrates to a global analytic function. Moreover a meromorphic 1-form on the sphere always has 2 more poles than zeros.

18. The Schwarz-Christoffel formula. Let $f : \mathbb{H} \to U$ be the Riemann mapping to a polygon with vertices p_i, $i = 1, \ldots, n$ and exterior angles $\pi\mu_i$. Then

$$f(z) = \alpha \int \frac{d\zeta}{\prod_1^n (\zeta - q_i)^{\mu_i}} d\zeta \ + \beta,$$

where $f(q_i) = p_i$.

Proof: Notice that $g(z) = (f(z) - p_i)^{\alpha_i}$ extends by Schwarz reflection across q_i, where $\alpha_i = 1/(1-\mu_i)$. This implies

$$f(z) = p_i + (z - q_i)^{1-\mu_i} h_i(z),$$

where $h_i(z)$ is holomorphic near q_i and $h_i(q_i) \neq 0$, and thus $f'(z)$ behaves like $(z - q_i)^{-\mu_i}$.

Next we compute the nonlinearity $N(f) = f''(z)/f'(z)dz = d \log f'(z)$. By Schwarz reflection, it extends to a meromorphic 1-form on the sphere with simple poles at $z = q_i$ and at $z = \infty$. (Near infinity, $f(z)$ behaves like $1/z$ which has nonlinearity $-2dz/z$ and hence residue 2 at infinity.)

Using the local expression above, one finds that $N(f)$ has residue $-\mu_i$ at q_i. Since a 1-form is determined by its singularities, we have

$$N(f) = \sum \frac{-\mu_i dz}{(z - q_i)},$$

and the formula results by integration. Note that the residue theorem gives $\sum \pi\mu_i = 2\pi$ which is also consistent geometrically (the sum of the exterior angles in π.)

19. Examples of Schwarz-Christoffel: $\log(z) = \int d\zeta/\zeta$ (maps to a bigon with external angles of π); $\sin^{-1}(z) = \int d\zeta/\sqrt{1-\zeta^2}$ (maps to a triangle with external angles $\pi/2$, $\pi/2$ and π.)

20. Regular polygons. The same reasoning applies to Riemann maps on the disk and shows $f(z) = \int dz/(z^n - 1)^{1/n}$ maps the unit disk to a regular n-gon.

21. Bloch's Theorem: there exists a universal $R > 0$ such that for any $f : \Delta \to \mathbb{C}$ with $|f'(0)| = 1$, *not* necessarily univalent, there is an open set $U \subset \Delta$ (perhaps a tiny set near S^1) such that f maps U univalently to a ball of radius r.

    It is known that the best value of R, *Bloch's constant*, satisfies $0.433 < \sqrt{3}/4 \le R < 0.473$. The best-known upper bound comes from the Riemann surface branched with order 2 over the vertices of the hexagonal lattice.

22. Little Picard Theorem. A nonconstant entire function omits at most one value in the complex plane. (This is sharp as shown by the example of $\exp(z)$.)

23. These apparently unrelated theorems can both be proved using the same idea. (Cf. [BD] and references therein.)

    We first consider Bloch's theorem. Given $f : \Delta \to \mathbb{C}$, let

    $$\|f'(z)\| = \|f'(z)\|_{\Delta,\mathbb{C}} = (1/2)|f'(z)|(1 - |z|^2)$$

    denote the norm of the derivative from the hyperbolic metric to the Euclidean metric. By assumption, $\|f'(0)\| = 1/2$. We can assume (using $f(rz)$) that f is smooth on S^1; then $\|f'(z)\| \to 0$ as $|z| \to 1$, and thus $\sup |\|f'(z)\|$ is achieved at some $p \in \Delta$.

    Now replace f with $f \circ r$ where $r \in \mathrm{Aut}(\Delta)$ moves p to zero. We can also shrink and translate f so $f(0) = 0$ and $\|f'(0)\| = 1$; this will only decrease the size of its unramified disk. Then $\|f'(z)\| \le \|f'(0)\| = 1$, and thus $f|\Delta(1/2)$ ranges in a compact family of nonconstant anayltic functions. Thus the new f has an unramified disk of definite radius; but then the old f does as well. ∎

24. Given $g : \mathbb{C} \to \widehat{\mathbb{C}}$, we now let

$$\|g'(z)\|_\infty = \frac{|g'(z)|}{(1 + |g(z)|^2)}.$$

This is the norm of the derivative from the Euclidean metric $|dz|$ to the spherical metric of curvature 4. Note that $g(z) = \exp(z)$ has $\|g'\|_\infty = 1/2$; a function with bounded derivative can be rather wild.

Theorem. Let $f_n : \Delta \to \widehat{\mathbb{C}}$ be a sequence of analytic functions such that $\|f_n'(0)\|_\infty \to \infty$. There after passing to a subsequence, there is a nonconstant entire function $g : \mathbb{C} \to \mathbb{C}$ such that $g = \lim f_n \circ r_n$. Moreover, $\sup \|g'(z)\|_\infty < \infty$.

Here r_n are Möbius transformations such that any compact set $K \subset \mathbb{C}$ satisfies $r_n(K) \subset \Delta$ for all $n \gg 0$.

Proof. The main idea is to consider, for $g : \Delta \to \widehat{\mathbb{C}}$, the norm

$$\|g'(z)\|_1 = \frac{|g'(z)|(1 - |z|^2)}{(1 + |g(z)|^2)}$$

from the hyperbolic metric on Δ to the spherical metric on $\widehat{\mathbb{C}}$, scaled for convenience to get rid of factors of 2; and more generally, for $g : \Delta(R) \to \widehat{\mathbb{C}}$, the norm

$$\|g'(z)\|_R = \frac{|g'(z)|(1 - |z/R|^2)}{(1 + |g(z)|^2)}$$

using the hyperbolic metric $\rho_R = |dz|/(1 - |z/R|^2)$ of curvature -4.

We can assume f_n is analytic on $\overline{\Delta}$; then $\|f_n'\|_1 \to 0$ near S^1 so the maximum is achieved somewhere, and greater than $\|f_n'(0)\|_1$. Using Möbius transformations $r_n : \Delta \to \Delta$, we can normalize so the maximum is achieved at $z = 0$. Then we still have $R_n = \|f_n'(0)\|_1 \to \infty$, but now $\|f_n(z)\|_1 \le \|f_n'(0)\|_1$.

Now let $g_n(z) = f_n(z/R_n) : \Delta(R_n) \to \widehat{\mathbb{C}}$. Then $\|g_n'(z)\|_R \le 1$ and the value 1 is again achieved at $z = 0$.

Now for any compact set K, we have $\rho_{R_n} \to |dz|$ uniformly on K. Thus $\|g_n'(z)\|_\infty$ is uniformly bounded on K. Thus by Arzela-Ascolia, we can extract a uniformly convergent subsequence. This gives the desired limit $g(z)$. ■

25. Proof of the Little Picard Theorem. Suppose if $f : \mathbb{C} \to \mathbb{C}$ is nonconstant and omits 0 and 1. Then $f_n(z) = f_n^{1/n}(z)$ omits more and more points on the unit circle. We can of rescale in the domain so the spherical derivative satisfies $\|f_n'(0)\|_\infty \to \infty$. Passing to a subsequence and reparameterizing, we obtain in the limit a nonconstant entire function that omits the unit circle. This contradicts Liouville's theorem.

26. Classical Proof of Little Picard: The key fact is that the universal cover of $\mathbb{C} - \{0, 1\}$ can be identified with the upper halfplane.

    To see this, it is useful to start by considering the subgroup $\Gamma_0 \subset \mathrm{Isom}(\Delta)$ generated by reflections in the sides of the ideal triangle T with vertices $\{1, i, -1\}$. For example, $z \mapsto \overline{z}$ is one such reflection, sending T to $-T$. By considering billiards in T, one can see that its translates tile the disk and thus T is a fundamental domain for Γ_0. Thus the quadratilateral $F = T \cup (-T)$ is a fundamental domain for the orientation-preserving subgroup $\Gamma \subset \Gamma_0$, and the edges of $-T$ are glued to the edges of T to give a topological triply-punctured sphere as quotient.

    Now let $\pi : T \to \mathbb{H}$ be the Riemann mapping sending T to $\mathbb{H}$ and its vertices to $\{0, 1, \infty\}$. Developing in both the domain and range by Schwarz reflection, we obtain a covering map $\pi : \Delta \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

    Given this fact, we lift an entire function $f : \mathbb{C} \to \mathbb{C} - \{0, 1\}$ to a map $\widetilde{f} : \mathbb{C} \to \mathbb{H}$, which is constant by Liouville's theorem.

27. Uniformization of planar regions. Once we know that $\widehat{\mathbb{C}} - \{0, 1, \infty\}$ is uniformized by the disk, it is straightforward to prove that any planar region U omitting two points is covered by the disk. To do this, we consider a basepoint p in the abstract universal cover $\pi : \widetilde{U} \to U$, and let $\mathcal{F}$ be the family of all holomorphic maps

$$f : (\widetilde{U}, p) \to (\Delta, 0)$$

that are covering maps to their image. Using the uniformization of the triply-punctured shere, we have that $\mathcal{F}$ is nonempty. It is also a closed, normal family of functions in $\mathcal{O}(\widetilde{U})$; and by the classical square-root trick, it contains a surjective function (which maximizes $|f'(p)|$). By the theory of covering spaces, this extremal map must be bijective.

28. Great Picard Theorem. Near an essential singularity, a meromorphic function assumes all values on $\widehat{\mathbb{C}}$ with at most two exceptions.

29. Proof of Great Picard. Consider a loop γ around the puncture of the disk. If f sends γ to a contractible loop on the triply-punctured sphere, then f lifts to a map into the universal cover $\mathbb{H}$, which implies by Riemann's removability theorem that f extends holomorphically over the origin.

    Otherwise, by the Schwarz lemma, $f(\gamma)$ is a homotopy class that can be represented by an arbitrarily short loop. Thus it corresponds to a puncture, which we can normalize to be $z = 0$ (rather than 1 or ∞). It follows that f is bounded near $z = 0$ so again the singularity is not essential.

4 Elliptic curves

1. Theorem: Any discrete subgroup of $(\mathbb{C}, +)$ is isomorphic to $\mathbb{Z}^n$ with $n = 0, 1$ or 2. In the last case, the generators are linearly independent over $\mathbb{R}$.

2. Entry 1: Let $\Lambda \subset \mathbb{C}$ be a lattice. How can we describe the Riemann surface $X = \mathbb{C}/\Lambda$? Answer will be: $K(X) \cong \mathbb{C}(x, y)$ where $y^2 = 4x^3 + ax + b$. This will involve the construction of functions $x(t), y(t)$ on $\mathbb{C}$ that sweep out or *uniformize* the an elliptic curve, and are periodic under Λ.

    Compare the curves $x^2 + y^2 = 1$, $x^2 - y^2 = 1$ and $xy = 1$, which are isomorphic to $\mathbb{C}/2\pi\mathbb{Z}$ or $\mathbb{C}/2\pi i\mathbb{Z}$, and are uniformized by $(\cos(t), \sin(t))$, $(\cosh(t), \sinh(t))$ and (e^t, e^{-t}) respectively. Note that all 3 curves have, over $\mathbb{C}$, two asymptotes, corresponding to the ends of $\mathbb{C}^*$.

3. Entry 1': let f be a nonconstant meromorphic function on $\mathbb{C}$, and let $\Lambda \subset \mathbb{C}$ be the group of translations that leave f invariant. If Λ is

nontrivial then either $\Lambda \cong \mathbb{Z}$ or Λ is a lattice, and in both cases we can ask to construct all functions with the given periods.

For $\Lambda = 2\pi i\mathbb{Z}$ the answer is: $f(t) = g(\exp(t))$, where g is meromorphic on $\mathbb{C}^*$. For Λ a lattice we will find $f(t) = g(\wp(t), \wp'(t))$, where g is a rational function of two variables.

4. Properties of elliptic (doubly-periodic) functions. Such functions can be considered as holomorphic maps $f : X \to \widehat{\mathbb{C}}$, where $X = \mathbb{C}/\Lambda$. By general easy results on compact Riemann surfaces: An entire elliptic function is constant. The sum of the residues of f over X is zero.

   More interesting is the fact that, if f is nonconstant with zeros a_i and poles p_i, then $\sum a_i = \sum p_i$ in the group law on X.

   Proof: working in a fundamental parallelogram $F = [0, \lambda_1] \times [0, \lambda_2]$ in $\mathbb{C}$, we have
$$\sum a_i - \sum p_i = (2\pi i)^{-1} \int_{\partial F} \frac{z f'(z)\, dz}{f(z)}.$$
   The integrals over opposite edges cancel, up to a terms of the form $\lambda_i (2\pi i)^{-1} \int_e f'(z)/f(z)\, dz$. Since the f is periodic, it has an integral winding number $N(e)$ on each edge, and these terms have the form $N(e)\lambda_i \in \Lambda$.

   We will see that we may *construct* an elliptic function with given zeros and poles subject only to this constraint.

5. Construction of elliptic functions. For $n \geq 3$,
$$\zeta_n(z) = \sum_{\Lambda} \frac{1}{(z-\lambda)^n}$$
   defines an elliptic function of degree n. For degree two, we adjust the factors in the sum so they vanish at the origin, and obtain the definition of the Weierstrass function:
$$\wp(z) = \frac{1}{z^2} + \sum_{\Lambda}{}' \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$
   To see $\wp(z)$ is elliptic, use the fact that $\wp(z) = \wp(-z)$ and $\wp'(z) = -2\zeta_3(z)$ (which implies $\wp(z+\lambda) = \wp(z) + A_\lambda$).

Other ways to construct elliptic functions of degree two: if Λ is generated by (λ_1, λ_2), first sum over one period to get:

$$f_1(z) = \sum_{-\infty}^{\infty} \frac{1}{(z - n\lambda_1)^2} = \frac{\pi^2}{\lambda_1^2 \sin(\pi z/\lambda_1)^2};$$

then

$$f(z) = \sum_{-\infty}^{\infty} f_1(z - n\lambda_2)$$

converges rapidly, and defines an elliptic function of degree two. Similarly, if we write $X = \mathbb{C}^*/ < z \mapsto \alpha z >$, then

$$f(z) = \sum \frac{\alpha^n z}{(\alpha^n z - 1)^2}$$

converges to an elliptic function of degree two.

These functions are not quite canonnical; there is a choice $\mathbb{Z}\lambda_1$ of which direction in the lattice to sum over first. As a consequence they agree with $\wp(z)$ only up to a constant. This constant is a multiple of the important *quasimodular form* G_1 we will discuss below.

6. Laurent expansion and differential equation. Expanding $1/(z-\lambda)^2$ in a Laurent series about $z = 0$ and summing, we obtain

$$\wp(z) = \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots = \frac{1}{z^2} + \sum_1^{\infty} (2n+1) z^{2n} G_{n+1}$$

where

$$G_n = G_n(\Lambda) = \sum_{\Lambda}{}' \frac{1}{\lambda^{2n}}.$$

By a straightforward calculation (using the fact that an elliptic function with no pole is constant) we have:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3,$$

where $g_2 = 60G_2$ and $g_3 = 140G_3$.

7. Geometry of the Weierstrass map $\wp : X \to \widehat{\mathbb{C}}$. We have $\wp(x) = \wp(y)$ iff $x = \pm y$. Thus the critical points of $\wp$, i.e. the zeros of $\wp'$, are the points of order two: $\{0, c_1, c_2, c_3\}$.

   Since $\wp$ has degree two, all its critical points are simple and its critical values, ∞ and $\wp(c_i) = e_i$ are distinct. Here r_i are the roots of the cubic equation $p(x) = 4x^3 - g_2 x - g_3$.

   The map $z \mapsto (\wp(z), \wp'(z))$ sends $X - \{0\}$ bijectively to the affine cubic curve $y^2 = p(x)$. Thus $\wp$ "uniformizes" this plane curve of genus 1.

8. Real case. Now suppose $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ with $\lambda_1 > 0$ and $\lambda_2 \in i\mathbb{R}_+$. Then Λ is invariant under both negation and complex conjugation. Thus we have
$$\wp(\overline{z}) = \wp(-\overline{z}) = \overline{\wp(z)}.$$

   It follows that $\wp$ must be *real* on the locus $R \subset X$ which is fixed under $z \mapsto \overline{z}$ or $z \mapsto -\overline{z}$. This locus is covered in $\mathbb{C}$ by the vertical and horizontal lines through the points of $(1/2)\Lambda$. In particular it the rectangle S with vertices the points of order two, $\{0, c_1, c_2, c_3\}$.

   It is now easy to check that $R = \wp^{-1}(\mathbb{R} \cup \{\infty\})$, and (using the fact that $\wp(z) = 1/z^2 + \cdots$) that $\wp$ is decreasing along $[0, \lambda_1/2]$ and hence maps S homeomorphically to $-\mathbb{H}$.

   The inverse map on the upper halfplane, $f : \mathbb{H} \to e_1 + S$, satisfies $f(\wp(z)) = z$ and thus $f'(\wp(z))\wp'(z) = 1$. Consequently:
$$f'(\wp) = (\wp')^{-1/2} = \frac{1}{\sqrt{4\wp^3 - g_2\wp - g_3}},$$

   which we recognize as the Schwarz-Christoffel formula for the Riemann map to a rectangle.

9. Function fields. Given any Riemann surface X, the meromorphic functions on X form a field $K(X)$. Then K is a contravariant functor from category of Riemann surfaces with non-constant maps to the category of fields with extensions. Example: $K(\widehat{\mathbb{C}}) = \mathbb{C}(z)$.

   Theorem: For $X = \mathbb{C}/\Lambda$, $K(X) = \mathbb{C}(x, y)/(y^2 - 4x^3 + g_2 x + g_3)$.

   To see that $\wp$ and $\wp'$ generate $K(X)$ is easy. Any even function $f : X \to \widehat{\mathbb{C}}$ factors through $\wp$: $f(z) = F(\wp(z))$, and so lies in $\mathbb{C}(\wp)$. Any odd function becomes even when multiplied by $\wp'$; and any function is a sum of one even and one odd.

To see that the field is exactly that given is also easy. It amounts to showing that $K(X)$ is of degree exactly two over $\mathbb{C}(\wp)$, and $\wp$ is transcendental over $\mathbb{C}$. The first assertion is obvious, and if the second fails we would have $K(X) = \mathbb{C}(\wp)$, which is impossible because $\wp$ is even and $\wp'$ is odd.

10. Addition law on an elliptic curve.

    Consider the curve $E \subset \mathbb{P}^2$ defined by $y^2 = 4x^3 + g_2 x + g_3$ and parameterized by the Weierstrass $\wp$-function via $(x, y) = (\wp(z), \wp'(z))$.

    Theorem. For any line L, the intersection $L \cap E = \{a, b, c\}$ where $a + b + c = 0$ on $E = \mathbb{C}/\Lambda$.

    Proof. The line L is simply the zero set of $A\wp' + B\wp + C$ for some (A, B, C). This function has all its poles at $z = 0$. Since the sum of the zeros and poles is zero, its zeros (a, b, c) also sum to zero.

    Cor. The map $p \mapsto -p$ on E is given by $(x, y) \mapsto (x, -y)$.

    Proof. Then the line passes through ∞ which is the origin of E, consistent with the equation $p + (-p) + 0 = 0$.

    Cor. The point $c = a + b$ is constructed geometrically by drawing the line L through (a, b), finding its third point of intersection $(-c) = (x, -y)$ on E and then negating to get $c = (x, y)$.

    Cor. To construct the point $2a$, take the line tangent to E at a, find its other point of intersection and then negate.

11. An elliptic function with given poles and zeros. It is natural to try to construct an elliptic function by forming the Weierstrass product for the lattice Λ:

$$\sigma(z) = z \prod_{\Lambda}{}' \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}\right).$$

    Then $(\log \sigma)'' = -\wp(z)$, from which it follows that $\sigma(z+\lambda) = \sigma(z) \exp(a_\lambda + b_\lambda z)$. From this it is easy to see that

$$\frac{\sigma(z - a_1) \ldots \sigma(z - a_n)}{\sigma(z - p_1) \ldots \sigma(z - p_n)}$$

defines an elliptic function whenever $\sum a_i = \sum p_i$, and therefore this is the only condition imposed on the zeros and poles of an elliptic function.

12. The *moduli space of complex tori*, $\mathcal{M}_1$, classifies Riemann surfaces of genus 1 up to isomorphism.

    Since any elliptic curve has the form $X = \mathbb{C}/\Lambda$, we see that $\mathcal{M}_1$ is the same as the lattice in $\mathbb{C}$ modulo the action of $\mathbb{C}^*$. If a lattice $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ is given an oriented basis, then it is similar to $\mathbb{Z} \oplus \mathbb{Z}\tau$ where $\tau \in \mathbb{H}$. Thus $\mathbb{H}$ is the space of *similarity classes of framed lattices.*

    Any two oriented bases of $\Lambda \cong \mathbb{Z}^2$ are related by an element of $\mathrm{SL}_2(\mathbb{Z})$. That is, if
    $$\mathbb{Z} \oplus \mathbb{Z}\tau_1 = \alpha(\mathbb{Z} \oplus \mathbb{Z}\tau_2)$$
    with $\alpha \in \mathbb{C}^*$, then we can write
    $$\begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} a\tau_2 + b \\ c\tau_2 + d \end{pmatrix}$$
    for some $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$, and thus
    $$\tau_1 = (a\tau + 2 + b)/(c\tau_2 + d).$$
    (The fact that $ad - bc = 1$ instead of -1 insures that $\operatorname{Im}\tau_1 > 0$ if $\operatorname{Im}\tau_2 > 0$.) Conversely, any two points of $\mathbb{H}$ in the same orbit of $\mathrm{SL}_2(\mathbb{Z})$ give similar lattices and hence isomorphic complex tori. This shows:

    Theorem 4.1 *The moduli space $\mathcal{M}_1$ is naturally isomorphic to the complex orbifold $\mathbb{H}/\,\mathrm{SL}_2(\mathbb{Z})$.*

    Now we can always normalize a lattice Λ by $\mathbb{C}^*$ so that its shortest nonzero vector is $z = 1$ and its next shortest vector is $\tau \in \mathbb{H}$. Then $|\tau| \geq 1$, and $\mathbb{Z} + \tau \subset \Lambda$ so $|\operatorname{Re}\tau| \leq 1/2$. Moreover $\mathbb{Z} \oplus \mathbb{Z}\tau = \Lambda$; otherwise there would be a lattice point $v = a + b\tau$ with $a, b \in [0, 1/2]$, but then
    $$|v|^2 \leq 1/4 + |\tau|^2/4 \leq 1/2$$
    contrary to our assumption that the shortest vector in Λ^* has length 1. This shows:

Theorem 4.2 *The region* $|\operatorname{Re}\tau| \le 1/2$, $|\tau| > 1$ *in* $\mathbb{H}$ *is a fundamental domain for* $\mathrm{SL}_2(\mathbb{Z})$.

13. To study $\mathcal{M}_1$ further, we now introduce the space of cross-ratios

$$\mathcal{M}_{0,4}^{\mathrm{ord}} = \widehat{\mathbb{C}} - \{0, 1, \infty\}$$

and its quotient orbifold

$$\mathcal{M}_{0,4} = \mathcal{M}_{0,4}^{\mathrm{ord}} / S_3.$$

Here $\mathcal{M}_{0,4}^{\mathrm{ord}}$ is the moduli space of ordered quadruples of distinct points on $\widehat{\mathbb{C}}$, up to the action of $\mathrm{Aut}(\widehat{\mathbb{C}})$. Any such quadruple has a unique representative of the form $(\infty, 0, 1, \lambda)$, giving a natural coordinate for this moduli space.

If we reorder the quadruple, the cross-ratio changes, ranging among the six values

$$\{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\}.$$

(There is a natural action of S_4, but the Klein 4-group $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts trivially.) There are 5 points fixed under the action of S_3: the points -1, $1/2$ and 2 each have stabilizer $\mathbb{Z}/2$, and correspond to the vertices of a square; while the points

$$\pm\omega = 1/2 \pm \sqrt{-3}/2$$

have stabilizer $\mathbb{Z}/3$ and correspond to the vertices of a tetrahedron.

The degree 6 rational map

$$F(\lambda) = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (1-\lambda)^2}$$

is invariant under S_3, and gives a natural bijection

$$F : (\widehat{\mathbb{C}} - \{0, 1, \infty\})/S_3 \cong \mathbb{C}$$

satisfying $F(0) = F(1) = F(\infty) = \infty$, $F(\omega) = F(\overline{\omega}) = 0$, and $F(-1) = F(1/2) = F(2) = 1$ (hence the factor for $4/27$). We should really think of the image as $\mathcal{M}_{0,4}$ and in particular remember the orbifold structure: $\mathbb{Z}/2$ at $F = 1$ and $\mathbb{Z}/3$ at $F = 0$.

14. Orbifolds. Here are some other examples of quotients of $\widehat{\mathbb{C}}$. Let $\mathbb{Z}/n$ act on $\widehat{\mathbb{C}}$ by $z \mapsto \exp(2\pi i k/n)z$. Extend this to an action of the dihedral group D_n by adding in $z \mapsto 1/z$. Then the quotient in both cases is $\widehat{\mathbb{C}}$ again: the (n,n)-orbifold and the $(2,2,n)$-orbifold respectively. The quotient map can be given by $F(z) = z^n$ in the first case and by $f(z) = z^n + z^{-n}$ in the second.

15. The modular function. We now define a map $J : \mathcal{M}_1 \to \mathcal{M}_{0,4}$ by associating to any complex torus, the four critical values of the Weierstrass $\wp$-function. (Note: any degree two map $f : X = \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ is equivalent, up to automorphisms of domain and range, to the Weierstrass $\wp$ function, so the associated point in $\mathcal{M}_{0,4}$ is canonically determined by X.)

    More concretely, given $\tau \in \mathbb{H}$ we define the half-integral points $(c_1, c_2, c_3) = (1/2, \tau/2, (1+\tau)/2)$ and the corresponding critical values by $e_i = \wp(p_i)$. Then their cross-ratio (together with the critical value at infinity) is given by:

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}.$$

    This depends on an ordering of the points of order two. Now $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Aut}(\Lambda)$ acts on the three points of order two through the natural quotient

$$0 \to \Gamma(2) \to \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2) \to 0.$$

    In particular, λ is invariant under the subgroup $\Gamma(2)$ of matrices equivalent to the identity modulo two.

    Now any elliptic element in $\mathrm{SL}_2(\mathbb{Z})$ has trace $-1, 0$ or 1, while the trace of any element in $\Gamma(2)$ must be even. Moreover, trace zero cannot arise: if $g = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \Gamma(2)$ then $-a^2 - bc = 1$ implies $-a^2 = 1 \bmod 4$ which is impossible.

    Summing up, we have:

    Theorem 4.3 *The group $\Gamma(2)$ is torsion-free, with fundamental domain the ideal quadratilateral with vertices $\{\infty, -1, 0, 1\}$.*

    To remove the ambiguity of ordering, we now define

$$J(\tau) = F(\lambda(\tau)) = \frac{4}{27}\frac{\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2}.$$

Theorem 4.4 *The map*

$$J : \mathcal{M}_1 = \mathbb{H}/\operatorname{SL}_2(\mathbb{Z}) \to \mathcal{M}_{0,4} = (\widehat{\mathbb{C}} - \{0, 1, \infty\})/S_3$$

is a bijection, and a isomorphism of orbifolds.

Proof. The value of $J(\tau)$ determines a quadruple $B \subset \widehat{\mathbb{C}}$ which in turn determines a unique Riemann surface $X \to \widehat{\mathbb{C}}$ of degree two, branched over B, with $X \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. Thus $J(\tau_1) = J(\tau_2)$ iff the corresponding complex tori are isomorphic iff $\tau_1 = g(\tau_2)$ for some $g \in \operatorname{SL}_2(\mathbb{Z})$. Thus J is injective.

To see it is surjective, we first observe that $J(\tau + 1) = J(\tau)$. Now if $\operatorname{Im} \tau = y \to \infty$, then on the region $|\operatorname{Im} z| < y/2$ we have

$$\wp(z) = \frac{1}{z^2} + \sum{}' \frac{1}{(z-n)^2} - \frac{1}{n^2} + \epsilon(z) = \frac{\pi^2}{\sin^2(\pi z)} + C + \epsilon(z),$$

where $\epsilon(z) \to 0$ as $y \to \infty$. (In fact we have

$$\epsilon(z) = \sum_n{}' \frac{\pi^2}{\sin^2(\pi(z + n\tau))} = O(e^{-\pi y}).$$

We really only need that it tends to zero; and we will not need the exact value of the constant $C = -\pi^2/3$).

Since $\sin(z)$ grows rapidly as $|\operatorname{Im} z|$ grows, we have

$$(e_1, e_2, e_3) = (\pi^2 + C, C, C) + O(\epsilon)$$

and thus $\lambda(\tau) = (e_3 - e_2)/(e_1 - e_2) \to 0$ and hence $J(\tau) \to \infty$.

Thus J is a *proper* open map, which implies it is surjective (its image is open and closed).

For the orbifold assertion, just note that the square torus has $J(\tau) = 1$ and the hexagonal torus has $J(\tau) = 0$; symmetries of X are reflected in symmetries of the 4-tuple (e_1, e_2, e_3, ∞). ∎

16. Picard's theorem revisited. It is now straightforward, by the theory of covering spaces, to show:

Theorem 4.5 *The map*

$$\lambda : \mathbb{H}/\Gamma(2) \to \mathcal{M}_2^{\mathrm{ord}} = \widehat{\mathbb{C}} - \{0, 1, \infty\}$$

is a bijection, presenting $\mathbb{H}$ *as the universal cover of the triply-punctured sphere.*

This gives the one line proof of the Little Picard Theorem: 'consider $\lambda^{-1} \circ f : \mathbb{C} \to \mathbb{H}$.' Cf. [Bol, p.39–40]:

> The question recently arose in conversation whether a dissertation of 2 lines could deserve and get a Fellowship... in mathematics the answer is yes....
>
> (Theorem.) An integral function never 0 or 1 is a constant. (Proof.) $\exp\{i\Omega(f(z))\}$ is a bounded integral function.
>
> ...But I can imagine a referee's report: 'Exceedingly striking and a most original idea. But, brilliant as it undoubtedly is, it seems more odd than important; an isolated result, unrelated to anything else, and not likely to lead anywhere.'

Here Ω is our λ^{-1}.

17. Periods. The lattice Λ associated to 4 points can also be covered explicitly (up to similarity) as follows.

Let $f : X = \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ be any degree two map, with branch locus $B \subset \widehat{\mathbb{C}}$, such that $f(z) = f(-z)$. Note that the quadratic differential dz^2 on $X = \mathbb{C}/\Lambda$ is invariant under $z \mapsto -z$. Thus it descends to a well-defined differential $q(w)\, dw^2$ on $\widehat{\mathbb{C}}$. It has poles at the branch points B, and these four points uniquely determine q up to scale.

In the case $w = \wp(z)$, we have $dw^2 = \wp'(z)^2\, dz^2$ and hence

$$q(w)\, dw^2 = \frac{dw^2}{\wp'(z)^2} = \frac{dw^2}{4w^3 - g_2 w - g_3}.$$

Now we have a map $\pi_1(\widehat{\mathbb{C}} - B) \to \mathbb{Z}/2$ defining the covering X. Let K be the kernel of this map. Then $\sqrt{q}$ has a single-valued branch along any $[\gamma] \in K$, and $[\gamma]$ lifts to a loop $\delta \in \pi_1(X)$. Consequently:

$$\int_\gamma \sqrt{q} = \pm \int_\delta dz \in \Lambda.$$

All elements of Λ arise in this way, and thus Λ can be recovered as the periods of $\sqrt{q}$.

18. Modular forms: algebraic perspective. We now wish to classify holomorphic k-forms $\omega = \omega(z)\, dz^k$ on the *orbifold*

$$\mathcal{M}_1 \cong \mathcal{M}_{0,4} \overset{J}{\cong} \mathbb{C},$$

which are 'holomorphic' at the orbifold and cusp points. The set of all such ω forms the finite-dimensional vector space M_k, called the space of *modular forms* of weight $2k$.

By removable singularities, ω can be considered an ordinary rational k-form on $\widehat{\mathbb{C}}$. The condition that it is holomorphic at an orbifold point then becomes the following.

Theorem 4.6 *At an orbifold point of order n, a holomorphic k-form has a pole, in classical coordinates, of order at most $k(1 - 1/n)$.*

Proof. If the orbifold point is $z = 0$, we want the form $z^i\, dz^k$ to pull back to a holomorphic form under $z = w^n$. The pullback is given by $w^{ni}(nw^{n-1}\, dw)^k = nw^{ni+nk-k}\, dw^k$, which is holomorphic iff $ni+nk-k \geq 0$ iff $-i \leq k(1 - 1/n)$. ∎

When $z = 0$ is a *puncture* of the Riemann surface it is like an orbifold point of order infinity, and the result above formally requires a pole of order at most k for holomorphicity. This fits well with the local uniformization $z = f(t) = \exp(2\pi i t)$. Under this map we have

$$f^*((dz/z)^k) = (2\pi i)^k (dt)^k,$$

so a pole of order k is converted into a function which is 'constant and nonzero' as $\operatorname{Im} t \to \infty$ in the uniformizing upper halfplane.

Definition. A rational form $\omega(z)\,dz^k$ is a *modular form* of weight $2k$ on $\mathcal{M}_{0,4} \cong \mathbb{C}$ if it has poles of order at most k, $k/2$ and $2k/3$ at $z = \infty$, $z = 1$ and $z = 0$.

We let M_k denote the finite-dimensinoal space of all modular forms of weight $2k$. The total number of poles of ω as a classical form is $2k$. The allowed orders of zeros and poles, and the dimensions of M_k, are given in Table 1 for small values of k.

k	$\mathbb{Z}/\infty$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	Total	$2k$	$\dim M_k$
0	0	0	0	0	0	1
1	1	$0\frac{2}{3}$	$0\frac{1}{2}$	1	2	0
2	2	$1\frac{1}{3}$	1	4	4	1
3	3	2	$1\frac{1}{2}$	6	6	1
4	4	$2\frac{1}{3}$	2	8	8	1
5	5	$3\frac{1}{3}$	$2\frac{1}{2}$	10	10	1
6	6	4	3	13	12	2
7	7	$4\frac{1}{3}$	$3\frac{1}{2}$	14	14	1
8	8	$5\frac{1}{3}$	4	17	16	2
9	9	6	$4\frac{1}{2}$	19	18	2
10	10	$6\frac{1}{3}$	5	21	20	2
11	11	$7\frac{1}{3}$	$5\frac{1}{2}$	23	22	2
12	12	8	6	26	24	3

Table 1. Allowed poles and spaces of modular forms.

Examples:

(a) There is no meromorphic 1-form on $\widehat{\mathbb{C}}$ with just a single pole, so $\dim M_1 = 0$.

(b) The quadratic differential $F_2 = dz^2/(z(z-1))$ spans M_2. Note that this differential has a simple zero, in the orbifold sense, at $z = 0$. It is covered by the differential on the triply-punctured spheres with double poles of equal residue at each puncture.

(c) The cubic differential $F_3 = dz^3/(z^2(z-1))$ spans M_3. This time there is a zero of order 2 at $z = 1$.

(d) The products F_2^2 and $F_3 F_2$ span M_4 and M_5.

(e) The forms F_2^3 and F_3^2 span M_6, which is two-dimensional.

(f) *The discriminant*

$$D_6 = F_2^3 - F_3^2 = \frac{dz^6}{z^4(z-1)^3} \in M_6$$

vanishes at infinity, but has *no others zeros or poles* (in the orbifold sense).

(g) Ratios of forms of the same weight give all rational functions. Indeed, we have

$$z = F_2^3/D_6,$$

and thus any rational function of degree d can be expressed as a ratio of modular forms of degree $k = 6d$.

19. Cusp forms. We let $M_k^0 \subset M_k$ denote the space of *cusp forms* vanishing at infinity. Every space M_k, $k \geq 2$ contains a form of the type $F_2^i F_3^j$ which is *not* a cusp form; thus $M_k \cong \mathbb{C} \oplus M_k^0$. By inspection, $\dim M_k^0 = 0$ for $k \leq 5$. On the other hand, any cusp form is divisible by D_6. This shows:

Theorem 4.7 *The map $\omega \mapsto D_6 \omega$ gives an isomorphism from M_k to M_{k+6}^0.*

Corollary 4.8 *We have $\dim M_{6n+1} = n$, and $\dim M_{6n+i} = n+1$ for $i = 0, 2, 3, 4, 5$.*

Corollary 4.9 *The forms F_2 and F_3 generate the ring $M = \bigcup M_k$.*

Corollary 4.10 *The forms $F_2^i F_3^j$ with $2i + 3j = k$ form a basis for M_k.*

Proof. By the preceding Corollary these forms span M_k, and the number of them agrees with $\dim M_k$ as computed above. ■

Corollary 4.11 *The ring of modular forms M is isomorphic to the polynomial ring $\mathbb{C}[F_2, F_3]$.*

Proof. The preceding results shows the natural map from the graded ring $\mathbb{C}[F_2, F_3]$ to M is bijective on each graded piece. ■

20. Modular forms: analytic perspective. Classically, a *modular form* of weight $2k$ is a holomorphic form $f = f(z)\, dz^k$ on the upper halfplane which is invariant under the modular group $\mathrm{SL}_2(\mathbb{Z})$, and which is 'holomorphic at infinity'.

    In terms of the *function* $f(z)$, the invariance conditions are equivalent to:

    $$f(z+1) = f(z) \quad \text{and} \quad f(-1/z) = z^{2k} f(z).$$

    The first condition implies we can write

    $$f(z) = \sum_{-\infty}^{\infty} a_n q^n,$$

    where $q(z) = \exp(2\pi i z)$; and the condition that f is 'holomorphic at infinity' means $a_n = 0$ for $n < 0$. In other words, if we regard $f(q)$ as a *function* on $\mathbb{H}/\mathbb{Z} \cong \Delta^*$, it is holomorphically at $q = 0$. If $f(q) = 0$, then f is a *cusp form*.

    Note that $dz = (2\pi i)^{-1} dq/q$. Thus as a *form* on Δ^*, we have $f = (2\pi i)^{-k}(dq/q)^k$, and f has a pole of order at most k at $q = 0$.

    Examples.

    (a) Any holomorphic invariant $f(\Lambda)$ of lattices that is homogeneous of degree $-2k$ and satisfies the right growth conditions defines a modular form of weight $2k$. That is, if $f(t\Lambda) = t^{2k}\Lambda$, then the function

    $$F(\tau) = f(\mathbb{Z} \oplus \tau\mathbb{Z})$$

    satisfies $F(\tau + 1) = F(\tau)$ and

    $$F(-1/\tau) = f(\mathbb{Z} \oplus \tau^{-1})\mathbb{Z}) = f(\tau^{-1}(\mathbb{Z} \oplus \tau\mathbb{Z}) = \tau^{2k} F(\tau).$$

(b) In particular, for $k \geq 2$ the *Eisenstein series*

$$G_k(\tau) = \sum{}' (n + m\tau)^{-2k}$$

are modular forms of weight $2k$. Evidentally G_k converges to $2\mathbb{Z}(2k)$ as $\operatorname{Im}\tau \to \infty$, so these are holomorphic at infinity but not cusp forms.

(c) Similarly for the normalized functions $g_2 = 60G_2$, $g_3 = 140G_3$.

Theorem 4.12 *Under the isomorphism $\mathcal{M}_1 \cong \mathcal{M}_{0,4}$, an analytic modular form is the same as an algebraic modular form.*

Corollary 4.13 *The ring of modular forms is isomorphic to $\mathbb{C}[g_2, g_3]$.*

21. Values of g_2 and g_3. We now note that for $\tau = i$, the zeros of $4\wp^3 - g_2\wp - g_3$ must look like $(-1, 0, 1)$ and thus $g_3(i) = 0$. Similarly for $\tau = \omega$ the zeros are arrayed like the cube roots of unity and hence $g_3(\omega) = 0$.

    To determine the values at infinity, we observe that as $\tau \to \infty$ we have

$$G_2(\tau) \to 2\zeta(4) = \pi^4/45$$

and

$$G_3(\tau) \to 2\zeta(6) = \pi^6/945.$$

This gives the values $g_2(\infty) = 60G_2(\infty) = 4\pi^4/3$ and $g_3(\infty) = 140G_3(\infty) = (4/27)\pi^6$, and thus

$$(g_2^3/g_3^2)(\infty) = 27.$$

22. *The cusp form Δ and the modular function J.* By the preceding calculation, the *discriminant*

$$\Delta(\tau) = g_2^3(\tau) - 27g_3(\tau)^2$$

is a cusp form of weight 12. We have seen such a form is unique up to a scalar multiple, and is nonvanishing everywhere except for a simple zero at infinity.

Theorem 4.14 *We have $J(\tau) = g_2^3(\tau)/\Delta(\tau)$.*

Proof. First note that this is a ratio of forms of weight 12 and hence a modular function, i.e. it is invariant under $\mathrm{SL}_2(\mathbb{Z})$. Since $g_2(\infty) \neq 0$, it has a simple pole at infinity, and thus has degree one on $\mathcal{M}_1$. We also have $J(\omega) = 0$ since $g_2(\omega) = 0$, and $J(i) = 1$ since $g_3(i) = 0$. ■

23. The product expansion: for $q = \exp(2\pi i\tau)$, we have

$$\Delta(q) = q \prod_1^\infty (1 - q^n)^{24}.$$

References

[BD] F. Berteloot and J. Duval. Sur l'hyperbolicité de certain complèmentaires. *Enseign. Math.* **47**(2001), 253–267.

[Bol] B. Bollabas, editor. *Littlewood's Miscellany.* Cambridge University Press, 1986.

[CJ] L. Carleson and P. Jones. On coefficient problems for univalent functions. *Duke Math. J.* **66**(1992), 169–206.